a.) $(f_* \mu)([a,b]) = \mu\left(f^{-1}([a,b])\right).$

Since $f(x) \geq 0$ for $\forall x$, $f^{-1}((-\infty, 0)) = \emptyset$,

So only $[a,b] \cap \mathbb{R}^+$ matters.

If $a, b \geq 0$, then $f^{-1}([a,b]) = [\sqrt{a}, \sqrt{b}]$

[because $f(x) = x^2$ is only interesting for $x \in X = [0,1]$].

Now

$$\boxed{(f_* \mu)([a,b]) = \begin{cases} 0 & \text{if } b \leq 0 \\ \mathrm{Leb}([0, \sqrt{b}]) = \sqrt{b} & \text{if } a \leq 0 \leq b \leq 1 \\ \mathrm{Leb}([0,1]) = 1 & \text{if } a \leq 0, b \geq 1 \\ \mathrm{Leb}([\sqrt{a}, \sqrt{b}]) = \sqrt{b} - \sqrt{a} & \text{if } 0 \leq a \leq b \leq 1 \\ \mathrm{Leb}([\sqrt{a}, 1]) = 1 - \sqrt{a} & \text{if } 0 \leq a \leq 1 \leq b \\ \mathrm{Leb}(\emptyset) = 0 & \text{if } a \geq 1 \end{cases}}$$

quite ugly

Of course, $(f_* \mu)([a,b]) = F(b) - F(a)$

where $F(x) := (f_* \mu)((-\infty, x]) = \begin{cases} 0 & , \text{if } x \leq 0 \\ \sqrt{x} & , \text{if } 0 \leq x \leq 1 \\ 1 & , \text{if } x \geq 1 \end{cases}$

a little nicer

b.) The density is $\frac{d}{dx} F(x) = \begin{cases} \frac{1}{2\sqrt{x}} & , \text{if } 0 < x < 1 \\ 0 & , \text{if not} \end{cases} =: g(x)$

any one can check that $\int_a^b g(x)\,dx =$ ✓