We already knew from c.) that $\varphi_X'(t) = \mathbb{E}(iXe^{itX})$,

so let's check that this is continuous.

let $t_n \to t \in \mathbb{R}$, then let $f_n, f : \Omega \to \mathbb{R}$ be

$$f_n(\omega) := i X(\omega) e^{i t_n X(\omega)} \xrightarrow{n \to \infty} i X(\omega) e^{itX(\omega)} =: f(\omega)$$

so $\varphi_X'(t_n) = \int_\Omega f_n \, d\mathbb{P}$ and $\varphi_X'(t) = \int_\Omega f \, d\mathbb{P}$.

Now the dominated convergence thm ensures that

$\int_\Omega f_n \, d\mathbb{P} \longrightarrow \int_\Omega f \, d\mathbb{P}$, because $|f_n(\omega)| = |X(\omega)|$

So $g(\omega) := |X(\omega)|$ is a dominating fn for $\forall n$,

and it is integrable: $\int_\Omega g \, d\mathbb{P} = \int_\Omega |X| \, d\mathbb{P} = E|X| < \infty$

by assumption. $\square$