Tools of Modern Probability
exam exercise sheet, 18.12.2018
(working time: 90 minutes)

1. Prove that $\Gamma(s) \sim \sqrt{\frac{2\pi}{s}} \left(\frac{s}{e}\right)^s$ as $s \to \infty$ (Stirling's approximation).

2. Show that if $(X, \mathcal{F}, \mu)$ is a measure space, $f : X \to \mathbb{R}^+$ is measurable and $\nu : \mathcal{F} \to \mathbb{R}$ is defined as $\nu(A) = \int_A f \,\mathrm{d}\mu$, then ν is a measure and $\nu \ll \mu$.

3. Show that if C is a closed and convex subset of a Hilbert space, then it has a shortest element (i.e. there is an $x \in C$ such that $||x|| \leq ||y||$ for every $y \in C$).

4. Show an example of a closed set C in a Hilbert space which does not have a shortest element – i.e. thet set $\{||x|| \,|\, x \in C\}$ does not have a minimum.

5. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\mathcal{G} \subset \mathcal{F}$ is a sub-σ-algebra and $X : \Omega \to \mathbb{R}$ is a *nonnegative* random variable (but may not be integrable). Show that the conditional expectation $Y = \mathbb{E}(X|\mathcal{G})$ exists (i.e. there is a $\mathcal{G}$-measurable Y such that $\int_A Y \,\mathrm{d}\mathbb{P} = \int_A X \,\mathrm{d}\mathbb{P}$ for every $A \in \mathcal{G}$).

6. Show that if $a : \mathbb{N} \to \mathbb{C}$ is in l^2 then $f : [0, 2\pi] \to \mathbb{C}$ defined as $f(x) := \sum_{n=0}^{\infty} a_n e^{inx}$ is in $L^2([0, 2\pi])$ and $||f|| = ||a||$ with appropriate normalization. (In what sence does the series definig f converge)?

7. Let X and Y be independent standard Gaussian random variables. Let $U = X + Y$ and $V = 2X - Y$. Calculate $\mathbb{E}(V|U)$. *(Hint: if W is independent of U, then $\mathbb{E}(W|U) = \mathbb{E}W$. If you choose $\lambda \in \mathbb{R}$ cleverly, then $W := V - \lambda U$ will be independent of U. (Since U and W are jointly Gaussian, to show independence it's enough to check that $Cov(U, W) = 0$.) Then write $V = \lambda U + W$.)*