Tools of Modern Probability
exam exercise sheet, 29.05.2019
(working time: 90 minutes)

1. Prove that $\int_0^\infty e^{-\frac{x^2}{2}}\,\mathrm{d}x = \sqrt{\frac{\pi}{2}}$.

2. Let c_n denote the ($n-1$-dimensional) surface volume of the $n-1$-dimensional unit sphere $S_n := \{x \in \mathbb{R}^n \,:\, |x| = 1\}$. Let d_n denote the (n-dimensional) volume of the n-dimensional unit ball $B_n := \{x \in \mathbb{R}^n \,:\, |x| \le 1\}$. Give and prove the relation between c_n and d_n.

3. Let (X, Y) be uniformly distributed on the square $[-1, 1]^2$. Let $Z = X + XY$, and let μ be the distribution of the random variable μ. Calculate $\int_{\mathbb{R}} z^2\,\mathrm{d}\mu(z)$.

4. Give an example of a function $f : [0, 1] \times [0, 1] \to \mathbb{R}$ for which the integrals

$$I_1 := \int_0^1 \int_0^1 f(x, y)\,\mathrm{d}y\,\mathrm{d}x$$
$$I_2 := \int_0^1 \int_0^1 f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

both make sense, but they are not equal, *or* show that there is no such function.

5. Let X, Y, Z be integrable random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Show that if Y and Z both satisfy the definition of the conditional expectation $\mathbb{E}(X|\mathcal{G})$, then $Y = Z$ almost surely.

6. Let X and Y be independent random variables having exponential distribution with parameter 1. Calculate $\mathbb{E}\left(\frac{X}{X+Y}\,\middle|\,X+Y\right)$.

7. Let $0 < R_1 < R_2$ and let $D \subset \mathbb{R}^2$ be the annulus

$$D := \{(x, y) \in \mathbb{R}^2 \,:\, R_1^2 < x^2 + y^2 < R_2^2\},$$

so the boundary of D consist of the circles

$$S_{R_1} := \{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 = R_1^2\},$$
$$S_{R_2} := \{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 = R_2^2\}.$$

find a function $f : \bar{D} \to \mathbb{R}$ for which

$$\begin{cases} \Delta f = 0 & \text{on } D \\ f = 0 & \text{on } S_{R_1} \\ f = 1 & \text{on } S_{R_2} \end{cases}.$$

(Hint: the complex function $z \mapsto \ln z$ is analytic on its domain.)

8. Let H be a Hilbert space with scalar product $\langle .,. \rangle$, let $V \subset H$ be a closed subspace, let $x \in H$ and let $c \in V$ be the point in V which is closest to x. Show that $x - c$ is orthogonal to V (meaning $\langle x - c, v \rangle = 0$ for all $v \in V$).