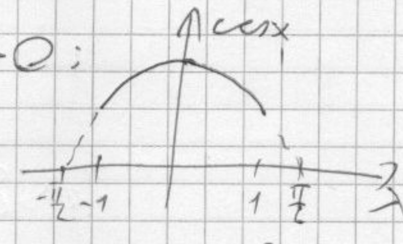
(1) a) If $-1 \le x \le 1$, then $\cos x > 0$:

So let $f(x) = \ln \cos x$.



Then we are looking for the asymptotics of

$$A_n = \int_{-1}^{1} e^{n f(x)}\, dx$$ where $f: [-1,1]$ is C^2 with a

unique global maximum at $x_0 = 0 \in (-1,1)$.

So the Laplace theorem applies:

$$f(x) = \ln \cos x \qquad f(0) = \ln 1 = 0 =: A$$

$$f'(x) = -\frac{\sin x}{\cos x} = -\tan x \qquad f'(0) = 0$$

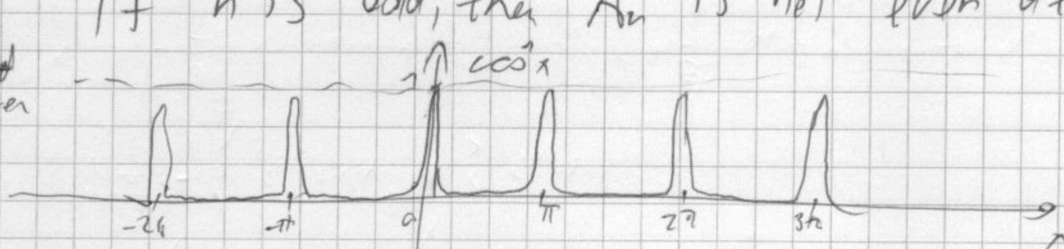
$$f''(x) = -\frac{1}{\cos^2 x} \qquad f''(0) = -1 = -B \text{ when } B = 1$$

So $$A_n \sim e^{nA}\sqrt{\frac{2\pi}{nB}} = e^{n0}\sqrt{\frac{2\pi}{n\cdot 1}} = \sqrt{\frac{2\pi}{n}}$$

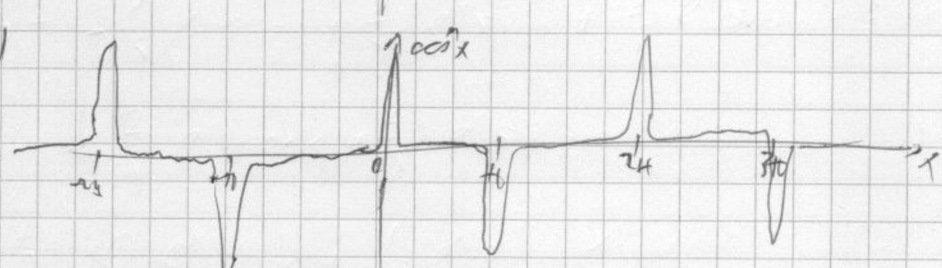
b) If n is even, then $A_n = \infty$.

If n is odd, then A_n is not even defined.

n ~~odd~~ even



n odd



2 Let $g: \mathbb{R} \to \mathbb{R}$ be given by $g(u) = u$.

By the integral substitution theorem

$$\int_{\mathbb{R}} g \, d\nu \overset{\nu = f_* \mu}{=} \int_X g \circ f \, d\mu = \int_{[-1,1]^2} x + y \, d\,\mathrm{Leb}(x,y) =$$

$$\overset{\text{Fubini}}{=} \int_{-1}^{1} \int_{-1}^{1} x + y \, dx \, dy \overset{\text{symmetry}}{=} 2 \int_{-1}^{1} \left[\int_{-1}^{1} x \, dx \right] dy = 0$$

(3) Let $t_n \to t_*$ and let $\varphi_n(x) = e^{it_n x}$, $\varphi_*(x) = e^{it_* x}$.

Then $\varphi_n(x) \xrightarrow{n\to\infty} \varphi_*(x)$ for each $x \in \mathbb{R}$.

Moreover, let $g(x) \equiv 1$.

Then $|\varphi_n(x)| \leq g(x)$ for all x, n

and $\int_{\mathbb{R}} g(x)\, d\mu(x) = \int_{\mathbb{R}} 1\, d\mu(x) = \mu(\mathbb{R}) = 1 < \infty$,

So g is integrable and the <u>dominated convergenc theora</u> ensures that

$$f(t_n) = \int_{\mathbb{R}} \varphi_n\, d\mu \longrightarrow \int_{\mathbb{R}} \varphi_*\, d\mu = f(t_*) \qquad \square$$

(4)

$Lf = \int_A f\,d\mu = \int_X fg\,d\mu = \langle f, g\rangle_{L^2(\mu)}$, where $g = 1\!\!1_A$,

So this <u>would be</u> bounded linear <u>if we knew</u> that $1\!\!1_A \in L^2(\mu)$.

This holds is $\|1\!\!1_A\|^2_{L^2(\mu)} = \int_X 1\!\!1_A^2\,d\mu = \int_A 1\,d\mu = \mu(A) < \infty$.

However, if $\mu(A) = \infty$, then $L(f)$ is not even defined for all $f \in L^2$.

E.g. if $\mu(X) = \infty$ and $A = X$, then

$Lf = \int_X f\,d\mu$ is only defined and finite for $f \in L^1 \cap L^2$

E.g. if $X = \mathbb{R}$, $\mu = Leb$, $f(x) = \begin{cases} \frac{1}{x} & \text{if } x > 1 \\ 0 & \text{if not} \end{cases}$

then clearly $f \in L^2$ but $\int f\,d\mu = \infty$.

Even on f where it's defined Lf is not bounded:

Let $X = \mathbb{R}$, $\mu = Leb$, $f_n = 1\!\!1_{[-n,n]}$, $A = X$

then $\|f_n\|_2 = \sqrt{\int_{\mathbb{R}} f_n^2\,d\mu} = \sqrt{2n}$

$Lf = \int_{\mathbb{R}} f_n\,d\mu = 2n$

so $\frac{Lf}{\|f\|_2} = \sqrt{2n}$, which is not bounded.

(5) <u>Step 1</u>: Assume $X \geq 0$ and define

- $\mu(A) := \mathbb{P}(A)$ for $A \in \mathcal{G}$
- $\nu(A) := \int_A X \, d\mathbb{P}$ for $A \in \mathcal{G}$.

Then ~~ν~~ μ and ν are finite measures on $(X, \mathcal{G})$ with $\nu \ll \mu$, so the Radon-Nikodym theorem says that there is a <u>$\mathcal{G}$-measurable</u> $Y : X \to \mathbb{R}$ s.t. $\nu(A) = \int_A Y \, d\mu$ for all $A \in \mathcal{G}$,

so this Y does the job.

<u>Step 2</u>: In general, if $X = X_+ - X_-$, then construct $Y_+ = \mathbb{E}(X_+ | \mathcal{G})$ and $Y_- := \mathbb{E}(X_- | \mathcal{G})$ using Step 1.

Then $Y := Y_+ - Y_-$ will do the job.

(6) $X = [X + 2Y + 2(2X - Y)]/5$

This is clever, since

$Cov(X + 2Y, 2X - Y) = 2\underbrace{Cov(X,X)}_{1} + 3\underbrace{Cov(X,Y)}_{0} - 2\underbrace{Cov(Y,Y)}_{1} =$

$= 0.$

Since $X + 2Y$ and $2X - Y$ are jointly Gaussian, this means that they are independent.

So $\mathbb{E}(X \mid X + 2Y) =$

$= \frac{1}{5} \mathbb{E}(X + 2Y \mid X + 2Y) + \frac{2}{5} \mathbb{E}(2X - Y \mid X + 2Y) =$

$= \frac{1}{5}(X + 2Y) + \frac{2}{5} \mathbb{E}(2X - Y) = \frac{X + 2Y}{5} + 0 = \underline{\underline{\frac{X + 2Y}{5}}}$

(7) X and Y can only be 0 or 1, so clearly

iff $X+2Y$ is even $\iff X=0$

odd $\iff X=1$

so $\mathbb{E}(X \mid X+2Y) = \begin{cases} 0 & \text{if } X+2Y \text{ is even} \\ 1 & \text{if } X+2Y \text{ is odd} \end{cases}$

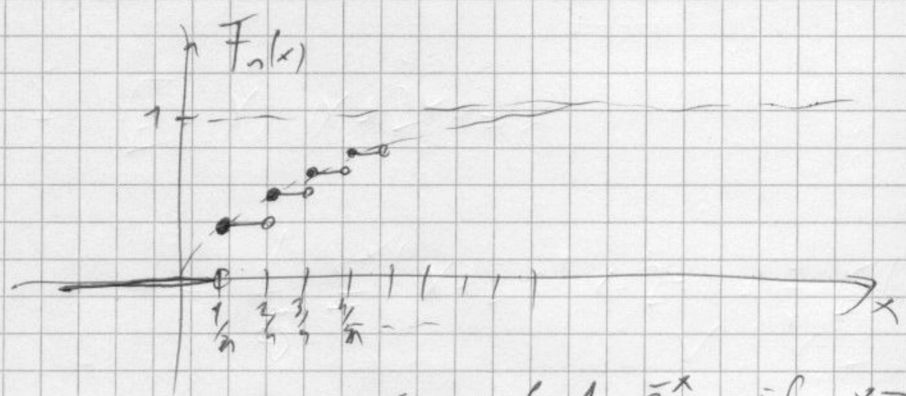
$= \underline{\underline{(X+2Y) \pmod 2}}$

(8) For $x \le 0$ $\mathbb{P}(Y_n \le x) = 0$.

For $x > 0$ $\mathbb{P}(Y_n \le x) = \mathbb{P}\left(\frac{X_n}{n} \le x\right) = \mathbb{P}(X_n \le nx) =$

$$= 1 - \mathbb{P}(X_n > nx) = 1 - \underbrace{q_n^{\lfloor nx \rfloor}}_{\text{no success in the first } \lfloor nx \rfloor \text{ trials}} = 1 - \left(1 - \frac{1}{n}\right)^{\lfloor nx \rfloor}$$

So $F_n(x) = \begin{cases} 1 - \left(1 - \frac{1}{n}\right)^{\lfloor nx \rfloor} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$



So $\lim_{n \to \infty} F_n(x) = \begin{cases} 1 - e^{-x}, & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$

So $\boxed{X_n \Longrightarrow Exp(1)}$