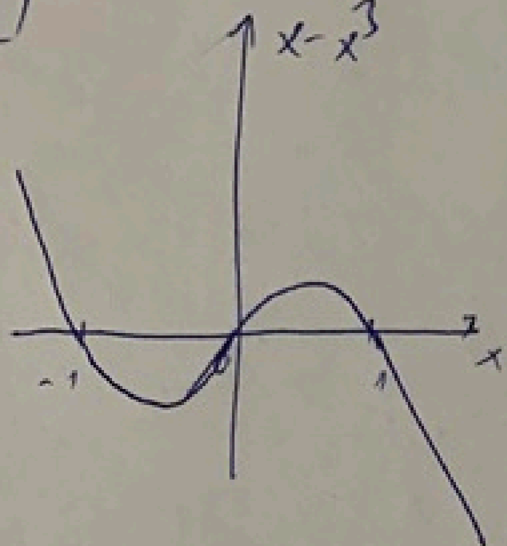
$\boxed{1.}$



a) $x-x^3$ is positive on $(0,1)$, so

$$A_n = \int_0^1 e^{n f(x)}\,dx \quad \text{with}$$

$f(x) := \ln(x-x^3)$ (at least for $0<x<1$)

This f is twice differentiable with a unique global maximum: $f'(x) = \frac{1-3x^2}{x-x^3} = 0 \iff x = \frac{1}{\sqrt{3}} =: x_0$, the unique global max. place

$$f''(x) = \frac{-6x(x-x^3)-(1-3x^2)^2}{(x-x^3)^2} = \frac{6x^4-6x^2-(1-3x^2)^2}{x^2(1-x^2)^2}$$

so $\boxed{A := f(x_0) = \ln\left(\frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}}\right) = \ln\left(\frac{2\sqrt{3}}{9}\right)}$

$$\boxed{-B := f''(x_0) = \frac{6\cdot\frac{1}{9} - 6\cdot\frac{1}{3} - 0^2}{\frac{1}{3}\left(1-\frac{1}{3}\right)^2} = \frac{\frac{2}{3} - \frac{6}{3}}{\frac{1}{3}\cdot\frac{4}{9}} = \frac{-\frac{4}{3}}{\frac{4}{3\cdot 9}} = -9}$$

$B = -f''(x_0) > 0$, so the maximum is non-degenerate

$\Rightarrow$ by the Laplace method

$$A_n \sim e^{nA}\sqrt{\frac{2\pi}{nB}} = \left(\frac{2\sqrt{3}}{9}\right)^n \sqrt{\frac{2\pi}{9n}}$$

b) For symmetry reasons $B_n = \begin{cases} 2A_n & \text{if } n \text{ is } \cancel{\text{odd}}\ \text{even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$, that is

$$\boxed{B_n = (1+(-1)^n)A_n \sim [1+(-1)^n]\left(\frac{2\sqrt{3}}{9}\right)^n\sqrt{\frac{2\pi}{9n}}}$$

2.

f is spherically symmetric: $f(\underline{x}) = \begin{cases} |\underline{x}|^2 & \text{if } |\underline{x}| \le 1 \\ 0 & \text{if not} \end{cases} =: \tilde{f}(r) \quad r = |\underline{x}|$

so by the high-dimensional polar coordinate substitution theorem

$$\int_{\mathbb{R}^n} f(\underline{x})\,d\underline{x} = \int_0^\infty c_n r^{n-1} \tilde{f}(r)\,dr = c_n \int_0^1 \underbrace{r^{n-1} \cdot r^2}_{r^{n+1}}\,dr = c_n \left[\frac{r^{n+2}}{n+2}\right]_0^1 =$$

$$= \frac{c_n}{n+2} = \frac{2\pi^{n/2}}{(n+2)\Gamma(\frac{n}{2})}$$

3.

$$\int f\,d\mu \overset{\mu = f_*\nu}{=\!=} \int f\,d f_*\nu \underset{\text{theorem}}{\overset{\text{integral substitution}}{=\!=}} \int f \circ f\,d\nu =$$

$$\overset{d\nu = f\,d\chi}{=\!=} \int (f \circ f) \cdot f\,d\chi \underset{\text{measure on } \mathbb{Z}}{\overset{\chi \text{ is counting}}{=\!=}}$$

$$= \sum_{k \in \mathbb{Z}} f(f(k)) \cdot f(k) = \sum_{k \in \mathbb{Z}} (k^2)^2 \cdot k^2 = \sum_{k \in \mathbb{Z}} k^6 = \infty$$

4. NOT ~~ab~~ true since $\varphi(x) := x$ is <u>not</u> bounded.

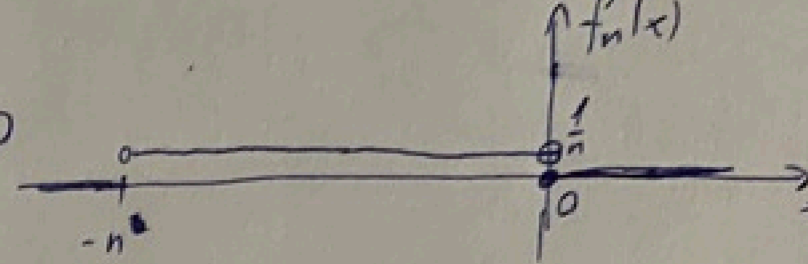
Example: Let $X_n = \begin{cases} 0 & \text{with prob. } 1 - \frac{1}{n} \\ n & \text{with prob. } 1/n \end{cases}$; $X \equiv 0$

Then $X_n \Rightarrow X$, but $\mathbb{E}X_n = 1 \not\to 0 = \mathbb{E}X$

5

<u>NOT true</u> in general: e.g. let

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } -n < x < 0 \\ 0 & \text{if not} \end{cases}$$



$$f(x) \equiv 0$$

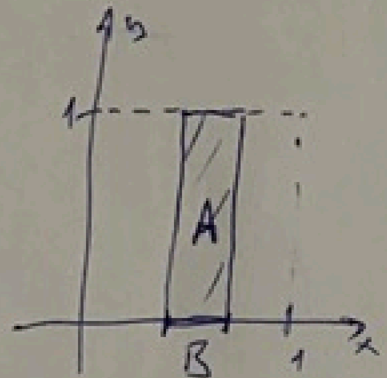
Then $f_n(x) \to f(x)$ for every x, but

e.g. for $y = 0$ $\quad \int_{-\infty}^{y} f_n(x)dx = 1 \not\to 0 = \int_{-\infty}^{y} f(x)dx$

<u>Remark</u>: The statement is true under the extra assumption that $\int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)\,dx$ for $\forall n$:

See the homework „weak convergence and densities": convergence of the densities to a ~~densi~~ probability density implies weak convergence.

$\boxed{6}$ SOLUTION 1



$\mathcal{G}$ consists of sets $A = B \times [0,1]$ where $B \subset [0,1]$ is Borel.

Correspondingly, $\mathcal{G}$-measurable functions $Z : \Omega = \{(x,y) \mid 0 \le x, y \le 1\} \to \mathbb{R}$ depend on x only:

$$Z := \mathbb{E}(Y \mid \mathcal{G}) = \cancel{Z(x)}\ Z(x,y) = f(x).$$

The other requirement is $\iint_{B\times[0,1]} Z \, d\mathbb{P} = \iint_{B\times[0,1]} Y \, d\mathbb{P}$ for every Borel set $B \subset [0,1]$,

so for $B = [a,b] \subset [0,1]$ this means

$$\int_a^b \int_0^1 \cancel{Z} f(x) \, \cancel{dy\,dx}\, d\mathbb{P}(x,y) = \int_a^b \int_0^1 y \, d\mathbb{P}(x,y)$$

a.) If $\mathbb{P} = \text{Leb}$, then

$$LHS = \int_a^b \int_0^1 f(x) \, d\mathbb{P}(x,y) = \int_a^b \int_0^1 f(x) \, dy \, dx = \int_a^b f(x) \, dx$$

$$RHS = \int_a^b \int_0^1 y \, d\mathbb{P}(x,y) = \int_a^b \underbrace{\int_0^1 y \, dy}_{1/2} \, dx = \int_a^b \frac{1}{2} \, dx$$

so $\boxed{f(x) \equiv \frac{1}{2}}$ will do.

$$\boxed{\mathbb{E}(Y \mid \mathcal{G}) = \frac{1}{2}}$$

b.) If $\mathbb{P}$ has density g, then

$$LHS = \int_a^b \int_0^1 f(x) \, d\mathbb{P}(x,y) = \int_a^b \int_0^1 f(x) g(x,y) \, dy \, dx = \int_a^b f(x) \left[\int_0^1 g(x,y) \, dy \right] dx$$

$$RHS = \int_a^b \int_0^1 y \, d\mathbb{P}(x,y) = \int_a^b \left[\int_0^1 \cancel{f(x)}\, y \, g(x,y) \, dy \right] dx$$

so these are equal for $\forall\, a < b$ iff

6 Solution 1 continued

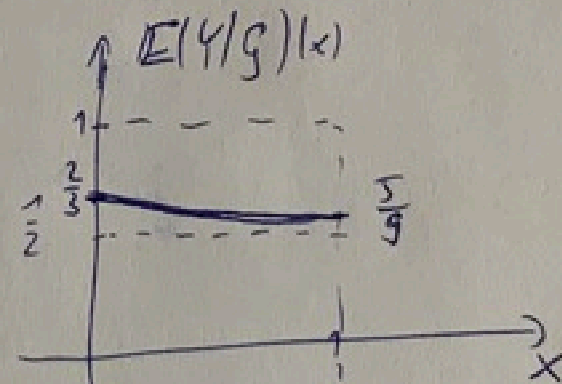
$$\boxed{f(x) = \frac{\int_0^1 y\, g(x,y)\,dy}{\int_0^1 g(x,y)\,dy}}$$

In our particular case $g(x,y) = x+y$, so

$$\left.\begin{aligned} \int_0^1 g(x,y)\,dy &= \int_0^1 x+y\,dy = x + \frac{1}{2} \\ \int_0^1 y\, g(x,y)\,dy &= \int_0^1 y(x+y)\,dy = \int_0^1 xy + y^2\,dy = \frac{x}{2} + \frac{1}{3} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow f(x) = \frac{\frac{x}{2}+\frac{1}{3}}{x+\frac{1}{2}} = \frac{3x+2}{3(2x+1)} = \frac{1}{2} + \frac{1/12}{x+\frac{1}{2}}$$

so $\boxed{\mathbb{E}(Y|\mathcal{G}) = \frac{1}{2} + \frac{1/12}{x+\frac{1}{2}}}$



6 SOLUTION 2

Since X and Y are coordinate projections, g is the joint density of (X,Y). So we know from class that

a.) X and Y are independent $\Rightarrow \mathbb{E}(Y|\mathcal{G}) = \mathbb{E}(Y|X) = \mathbb{E}Y = \frac{1}{2}$

b.) $\mathbb{E}(Y|\mathcal{G}) = \mathbb{E}(Y|X) = f(X)$ with $f(x) = \dfrac{\int_{\mathbb{R}} y\, g(x,y)\,dy}{\int_{\mathbb{R}} g(x,y)\,dy} = \ldots = \frac{1}{2} + \dfrac{1/12}{x+\frac{1}{2}}$

$$\Rightarrow \boxed{\mathbb{E}(Y|\mathcal{G}) = \frac{1}{2} + \frac{1/12}{X+\frac{1}{2}}}$$

$\boxed{7}$ Z^3 determines Z uniquely, so it also determines Z^2 uniquely, so Z^2 is $\sigma(Z^3)$-measurable

$$\Rightarrow \mathbb{E}(Z^2 \mid Z^3) = \mathbb{E}(Z^2 \mid \sigma(Z^3)) = Z^2$$

$\boxed{8}$ The covariance matrix C is degenerate with a zero eigenvalue, meaning that (X,Y) is actually concentrated on a lower dimensional subspace of $\mathbb{R}^2$.

Indeed, the correlation coefficient is

$$r(X,Y) = \frac{Cov(X,Y)}{DX\,DY} = \frac{1}{\sqrt{1}\cdot\sqrt{1}} = 1 \quad \text{(and } DX = DY\text{)}$$

So $Y = X + c$ with some constant $c \in \mathbb{R}$

$$\Rightarrow \mathbb{E}(X \mid X+Y) = \mathbb{E}(X \mid 2X+c) \overset{X \in \sigma(2X+c)}{=} X$$