Tools of Modern Probability
exam exercise sheet, 08.01.2018
(working time: 90 minutes)

1. Describe the asymptotic behaviour of the sequence $I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n x \,\mathrm{d}x$.

2. Calculate $\int_0^\infty x^\alpha e^{-x} \,\mathrm{d}x$ for nonnegative integer and half-integer values of α.

3. Prove that $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ as $n \to \infty$ (Stirling's approximation).

4. Define the charactersitic function ψ_X of a random variable X. Define the characteristic function ψ_ν of a probability distribution ν. Show that if X has distribution ν then $\psi_X = \psi_\nu$.

5. Give an example of a measure space $(X, \mathcal{F}, \mu)$ and a sequence of measurable functions $f_n : X \to \mathbb{R}$ such that f_n converges to some f everywhere, $\lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu$ and $\int_X f \,\mathrm{d}\mu$ exist but $\int_X f \,\mathrm{d}\mu > \lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu$. *(Hint: think of Fatou's lemma.)*

6. Let μ and ν be finite measures on the measurable space $(X, \mathcal{F})$ and let $\rho = \mu + \nu$. For functions $f \in L^2(\rho)$ let $\phi(f) = \int_X f \,\mathrm{d}\nu$. Show that ϕ is a bounded linear functional on $L^2(\rho)$.

7. We roll two fair dice. Let X and Y be the two numbers rolled. Calculate $\mathbb{E}(X+Y|X-Y)$.

8. Give a solution of the Laplace equation $\Delta f = 0$ on the domain $D = \{(x,y) \in \mathbb{R}^2 \,|\, x, y > 0\}$ with boundary conditions $f(x,0) = 0$, $f(0,y) = 1$ (for $x, y > 0$). Are there other solutions?