Tools of Modern Probability
exam exercise sheet, 11.01.2023
(working time: 90 minutes)

Every exercise is worth 12 points. Every student should *choose* 5 exercises to solve. If more than 5 are solved, I only take the best 5 into account. The maximum total score is 60. Solve easy exercises if you want a good grade. Solve hard ones if you want me to be proud of you.

1. Describe the asymptotic behaviour of the sequence $A_n := \int_0^1 x^{-nx}\,\mathrm{d}x$ as $n \to \infty$. (By "describing the asymptotic behaviour" I mean: find a sequence a_n given by a nice simple formula, such that $A_n \sim a_n$.)

2. Let the random variables $X_1, X_2, \dots, X_{10}$ be independent and uniform on $[0,1]$ and let μ be the distribution of the product $X_1 X_2 \cdots X_{10}$. Calculate $\int x^3\,\mathrm{d}\mu(x)$.

3. Let λ be Lebesgue measure on $[0,1]$, let $f(x) = \sqrt{x}$ and $\mu = f_*\lambda$. Let the measure ν have density $\rho(x) = x^2$ w.r.t. μ. Calculate $\nu([0,1])$.

4. Let $X, X_1, X_2, \dots$ be random variables such that $|X| \le 1$ and $|X_n| \le 1$ for every n. Prove that if $X_n \Rightarrow X$, then $\mathbb{E}(X_n^2) \to \mathbb{E}(X^2)$, or give a counterexample. Here $\Rightarrow$ denotes weak convergence of random variables.

5. Let λ be Lebesgue measure on $\mathbb{R}$. Let $f_1, f_2, f_3, \dots : \mathbb{R} \to \mathbb{R}$ be non-negative measurable functions, $S_n(x) = \sum_{k=1}^n f_k(x)$ and $S(x) = \sum_{k=1}^\infty f_k(x)$. Prove that $\int_{\mathbb{R}} S_n\,\mathrm{d}\lambda \to \int_{\mathbb{R}} S\,\mathrm{d}\lambda$ as $n \to \infty$, or give a counterexample.

6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where $\Omega = [0,\infty)$, $\mathcal{F}$ is the Borel σ-algebra and $\mathbb{P}$ has density $f(x) = e^{-x}$ w.r.t. Lebesgue measure. Let $X(\omega) = \omega$ and let
$$Y(\omega) = \begin{cases} 1 & \text{if } \omega > 1 \\ 0 & \text{if not} \end{cases}.$$
Calculate
   a.) $\mathbb{E}(Y|X)$
   b.) $\mathbb{E}(X|Y)$

7. We roll a fair die 4 times, call the numbers rolled X_1, X_2, X_3, X_4 and set $Y := X_1 + X_2 + X_3 + X_4$. For $n = 0,1,2,3,4$ let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ (so $\mathcal{F}_0$ is the σ-algebra generated by the empty set). Calculate
$$\mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbb{E}(Y|\mathcal{F}_0)|\mathcal{F}_1)|\mathcal{F}_2)|\mathcal{F}_3)|\mathcal{F}_4)$$

8. Let X, Y be independent random variables with $Exp(1)$ distribution (that is, having density $f(x) = e^{-x}$ on $[0,\infty)$). Calculate $\mathbb{E}(X^2 \,|\, X+Y)$.