Math A3 English, practice class 2, autumn 2010

2.1 Find the general solution of the following first order linear differential equations.

a.)

$$y' - y = e^{-x}$$

b.)

$$y' - \frac{y}{x} = x^2 + 3x - 2$$

2.2 Consider the differential equation

$$y' - 2y = 3e^t.$$

a.) Give the direction field.

b.) By examining the direction field, describe the behaviour of the solutions for large t.

c.) Find the general solution of the differential equation, and study the behaviour of this solution for large t.

2.3 Find the solution of the following initial value problems:

a.)

$$y' - y = 2te^{2t} \quad ; \quad y(0) = 1$$

b.)

$$ty' + 2y = t^2 - t + 1 \quad ; \quad y(1) = 1/2,\ t > 0$$

c.)

$$ty' + (t+1)y = t \quad ; \quad y(\ln 2) = 1,\ t > 0$$

2.4 The following theorem was told on the lecture:

Theorem 1. *Suppose that both the function $f(t, y)$ and the function $\partial f/\partial y$ are continuous on a rectangle $\alpha < t < \beta, \gamma < y < \delta$, which contains the point (t_0, y_0). Then there exists an interval $(t_0 - h, t_0 + h) \subset (\alpha, \beta)$, in which the solution of the following initial value problem exists and is unique:*

$$y' = f(t, y) \quad ; \quad y(t_0) = y_0.$$

In the following three problems, find those parts of the ty plane, where the conditions of this theorem are satisfied.

(a) $y' = \frac{t-y}{2t+5y}$

(b) $y' = \frac{\ln|ty|}{1-t^2+y^2}$

(c) $y' = (t^2 + y^2)^{3/2}$

2.5 Daniel Bernoulli used mathematical methods to study the spread of smallpox already in 1760. In the following two exercises we will see simple models for this problem. Similar models are used to model the spread of rumours or certain goods.

a.) Suppose that some population can be divided into two parts: those who are infected by a certain disease and are able to infect others, and those who are not yet infected, but may get the disease. The proportion of the latter to the total population will be called x, and the proportion of the former to the total population will be called y. So $x + y = 1$. Suppose that the disease spreads when healthy and infected individuals meet, and the rate $\frac{y}{t}$ of the spread is proportional to the number of these encounters. Suppose that healthy and infected individuals move freely, so the frequency of encounters is proportional to xy. Since $x = 1 - y$, we get that

$$\mathrm{d}\,y/\mathrm{d}\,t = \alpha y(1-y) \quad ; \quad y(0) = y_0,$$

where $\alpha > 0$ is the proportionality constant and $y_0 > 0$ is the initial proportion of infected individuals.

Solve this initial value problem, and thus show that

$$\lim_{t\to\infty} y(t) = 1,$$

that is, the disease infects the entire population.

b.) Certain diseases (e.g. typhus) are usually spread by *carriers*, that is, persons who can pass the disease to others, but the symptoms of the disease are not seen on them. On the other hand, those infected by the carriers become ill, but can no longer pass the disease to others. Let y denote the proportion of carriers to the total population, and let x denote the proportion of healthy persons (who may get infected) to the total population. We make two assumptions:

i) carriers are discovered and removed from the population at a rate $\beta > 0$, meaning that in a time unit, a β proportion of the than existing carriers are removed. On the other hand, no further carriers apeear. That is:

$$\mathrm{d}\,y/\mathrm{d}\,t = -\beta y. \tag{1}$$

ii) The rate of spread of the disease is proportional to xy, because the carriers and the healthy can meet freely. That is:

$$\mathrm{d}\,x/\mathrm{d}\,t = -\alpha xy \tag{2}$$

with some $\alpha > 0$.

Under these assumptions,

b1.) Give y as a function of t by solving (1), assuming that $y(0) = y_0$.

b2.) Give x as a function of t by solving (2), also using the result of the previous point, and assuming that $x(0) = x_0$.

b3.) Give the proportion of those persons in the population, who never get the disease, by calculating the limit

$$\lim_{t\to\infty} x(t).$$