Midterm exam 1 – solutions

30 October 2013. 16:00

Advanced Mathematics for Electrical Engineers B, Stochastics part

Working time: 60 minutes. Every exercise is worth 6.67 points.

1. At a bank counter, serving a costumer takes exactly one minute. During that minute, the number of newly arriving costumers can be 0, 1 or 2, all with probability $\frac{1}{3}$, independently of what happened before. Just before time 0 the queue is empty, but at time 0 the first costumer (Robert) arrives.

   (a) What is the probability that the queue will again become empty some (any) time in the future?

   (b) What is the expected number of costumers served before the queue gets empty again?

   (Hint: While Robert is being served, a random number of new customers arrive. Call them the "first generation".)

   Solution: Let Z_n denote the number of costumers in the n-th generation, where Robert alone is the 0-th generation, and those arriving while the n-th generation is being served form the $(n+1)$th generation for $n = 0, 1, 2, \ldots$. So Z_n is a Galton-Watson branching process with one-step offspring distribution

k	0	1	2
$\mathbb{P}(k \text{ offspring})$	1/3	1/3	1/3

   This has expectation $m = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 = 1$, so the process is critical. This impies that

   (a) The probability of extinction is 1.

   (b) The expected value of the total population size is ∞.

2. Joe decides to keep rolling a fair die until he manages 1000 times to roll 6. (His successes, of course, don't have to be consecutive.) Use your favourite large deviation theorem to estimate the probability that he succeeds in at most 5000 rolls.

   (Help: the Cramer rate function of the Bernoulli distribution with parameter p is

   $$I(x) = x \ln\left(\frac{x}{1-x}\frac{1-p}{p}\right) + \ln\left(\frac{1-x}{1-p}\right)$$

   (if $0 < x < 1$). The Cramer rate function of the (optimistic) geometric distribution with parameter p is

   $$I(x) = x \ln\left(\frac{x-1}{x}\frac{1}{1-p}\right) + \ln\left(\frac{1}{p}\frac{1-p}{x-1}\right)$$

   (if $x > 1$).)

   Solution 1/a, 1/b: We can also formulate the question in the following way: If Joe rolls a fair die 5000 times, what's the probability that at least 1000 of the rolls will give 6?

   So set $n = 5000$ and for $k = 1, 2, \ldots, n$ let $X_k = 1$ if the k-th roll is a 6, and $X_k = 0$ if not. So the X_k are independent and have Bernoulli distirbution with parameter $p = \frac{1}{6}$. $S_n := X_1 + \cdots + X_n$ is the number of 6-es rolled, and the question is $\mathbb{P}(S_n \geq 1000)$.

a.) The easiest solution is given by the Hoeffding inequality. $\mathbb{E}S_n = np = \frac{5000}{6}$, so set $t = 1000 - S_n = \frac{1000}{6}$, $a_k = 0$ and $b_k = 1$ for $k = 1, 2, \ldots, n$. The Hoeffding inequality says

$$\mathbb{P}(S_n \geq \mathbb{E}S_n + t) \leq \exp\left\{ -\frac{2t^2}{\sum\limits_{k=1}^{n}(b_k - a_k)^2} \right\} = \exp\left\{ -\frac{2 \cdot \left(\frac{1000}{6}\right)^2}{5000} \right\} \approx e^{-11.1} \approx 0.000015.$$

b.) We can get a more accurate estimate using the Cramer theorem. This is also easy - once we know the rate function from the hint. We set $m = \mathbb{E}X_k = p = \frac{1}{6}$, and we are interested in $\mathbb{P}(S_n \geq 1000) = \mathbb{P}(\frac{S_n}{n} \geq \frac{1}{5}) = \mathbb{P}(\frac{S_n}{n} \in [a, b))$ where $a = \frac{1}{5}$ and $b = \infty$. Since $m < a$, the Cramer theorem gives

$$\mathbb{P}\left(\frac{S_n}{n} \in [a, b)\right) \lessapprox e^{-n \cdot I(a)} = e^{-5000 \cdot I(\frac{1}{5})} \approx e^{-19} \approx 0.0000000054.$$

(We have calcualted the rate function for the Bernoulli distribution from the hint with $p = \frac{1}{6}$ and $x = \frac{1}{5}$.)

Solution 2: We can also look at the question directly: set $n = 1000$ and let $Y_1, Y_2, \ldots, Y_n$ be the number of rolls he needs to get the 1st, 2nd, …, 1000th occurrence of 6 (always counted from the previous occurrence). So the Y_k are independent and (optimistic) geometrically distributed with parameter $p = \frac{1}{6}$, and $S_n := Y_1 + \cdots + Y_n$ is the total number of rolls he needs. We are interested in $\mathbb{P}(S_n \leq 5000)$.

Notice that this time the Y_k are *not bounded*, so the Hoeffding inequality can not be used. However, the Cramer theorem still works. To use it, we write the question as $\mathbb{P}(S_n \leq 5000) = \mathbb{P}(\frac{S_n}{n} \leq 5) = \mathbb{P}(\frac{S_n}{n} \in (a, b])$ where $a = -\infty$ and $b = 5$. Set $m = \mathbb{E}Y_k = \frac{1}{p} = 6$. Since $b < m$, the Cramer theorem gives

$$\mathbb{P}\left(\frac{S_n}{n} \in (a, b]\right) \lessapprox e^{-n \cdot I(b)} = e^{-1000 \cdot I(5)} \approx e^{-19} \approx 0.0000000054.$$

(We have calcualted the rate function for the geomertic distribution from the hint with $p = \frac{1}{6}$ and $x = 5$.)

Remark: Any attempt to estimate the probability in question using the central limit theorem is wrong. The CLT is not *a large deviation theorem, and is not at all suitable for estimating such small probabilities of such extreme values of the average.*

3. We put a standard die on the table, and play the following game: In every step, we "flip" the die to one of the 4 faces neighbouring the bottom face, choosing randomly and uniformly from the 4 possibilities, independently of the past. We let X_n denote the number seen on the top after n steps (for $n = 0, 1, 2, \ldots$). This X_n is (of course) a Markov chain. We start with 6 being on top, so $X_0 = 6$.

   Remark: on a standard die, the sum of the two numbers on any two opposite faces is always 7.

   a.) Draw the transition graph of the Markov chain.

   b.) Give the transition probability matrix of the Markov chain.

   c.) What is the probability that $X_3 = 6$?

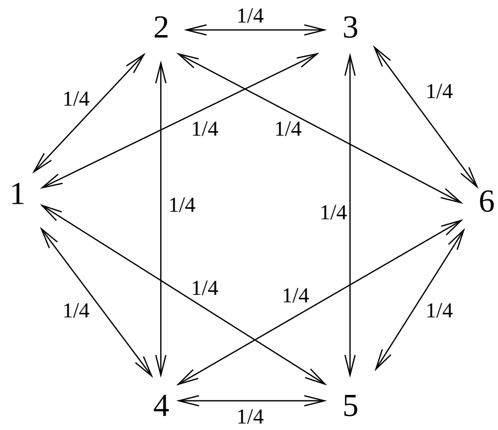
   d.) What is the stationary distribution of the Markov chain? *(Hint: it's possible to guess the answer and then check that your guess is correct.)*

e.) What is the approximate probability that $X_{100} = 3$?

f.) What is the average of the numbers on top on the long run?

Solution:

a.) The tranisiton graph is



b.) The transition matrix is

$$P = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{pmatrix}$$

c.) One possible soultion: To get back from 6 to 6 in 3 steps:

- The first step can be anything.
- The second step should not be back to 6 and should not be to the opposite face 1, but the other 2 choices are OK - so we make it with probability $\frac{2}{4} = \frac{1}{2}$.
- The last step should be back to 6, which is 1 out of the 4 possibilities, so we make it with (conditional) probability $\frac{1}{4}$ (conditioned on the success of the previous move).

All together the probability of finding our way back in 3 steps is $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$.

d.) One can guess, based on the symmetry of the system, that all states will have equal stationary weights – that is, the distribution $\pi = (\frac{1}{6}; \frac{1}{6}; \frac{1}{6}; \frac{1}{6}; \frac{1}{6}; \frac{1}{6})$ is stationary. The fact that it is indeed so can be proven by checking that the matrix equation $\pi P = \pi$ is satisfied.

e.) Since 100 is a lot of steps, and our Markov chain is irreducible and aperiodic, the fundamental theorem of Markov chains says that $\mathbb{P}(X_{100} = 6 | X_0 = 6) \approx \pi_6 = \frac{1}{6}$.

f.) We are interested in the (long term) time average of the observable $f(i) = i$, or the function

$$f = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}.$$

The ergodic theorem says that the time average is (converges to)

$$\bar{f} = \pi f = \sum_{i=1}^{6} \pi_i f(i) = \frac{1+2+3+4+5+6}{6} = 3.5$$