Problem Sheet - Second practical course

1. The generating function of a nonnegative integer valued random variable is

$$g(z) = \frac{3}{8} + \frac{3}{8}z + \frac{1}{8}z^2 + \frac{1}{8}z^3.$$

   What is the discrete probability distribution (namely the probabilities $P(X = k)$)? What is the expectation and variance of X?

2. We toss a fair coin 3 times, and a biased coin with $P(heads) = \frac{1}{3}$ also three times. Let Z denote the total number of heads seen. Calculate the generating function of Z.

3. Let X be an $\mathbb{N}$ valued random variable. Denote the generating function of X by $G(z)$. Express the generating functions of $Y := X + 1$ and $Z := 2X$ using function $G(z)$.

4. A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours. Assume that the miner is at all times equally likely to choose any one of the doors. Find the generating function of the length of time until he reaches safety. Calculate its expectation using this generating function.

5. Let $X_1, X_2, \ldots$ be i.i.d. $\mathbb{N}$ valued random variables. Furthermore, let N be an $\mathbb{N}$ valued random variable which is independent of the Xs. Let Y be equal to $\sum_{i=1}^{N} X_i$. Show that

   - $E(Y) = E(N)E(X_1)$,
   - the variance $D^2(Y)$ of Y is equal to $D^2(N)(E(X_1))^2 + E(N)D^2(X_1)$.

6. We roll two fair dice a blue and a red. We first roll the red die then we roll the blue die as many times as the outcome of the red die. Let Y denote the outcome of the red die and denote by X the sum of the outcomes on the blue die.

   a) Find $E(X)$ and $Var(X)$.
   b) What is the sign of $Cov(X, Y)$?

7.
   - We keep rolling a fair die until we first roll a 6. Let X denote the number of rolls (including the roll of 6). Find the generating function of X using recursion coming from conditioning on the result of the first roll.
   - We keep rolling a fair die until we first roll two 6's in a row. Let Y denote the number of rolls (including the two rolls of 6 in the end). Find the generating function of Y using recursion coming from conditioning on the result of the next roll after the first roll of 6.

8. We keep rolling a fair die until we first roll a 6. Let X denote the sum of the numbers rolled before (and not including) that 6. Calculate

   - the generating function of X,
   - the expectation of X,
   - the variance of X.

(Warning: What is the conditional distribution of a number rolled under the condition that it is not a 6?)

9. Between 8 and 9 am the number X of requests arriving to the router has Poisson distribution with parameter λ. Each request may come independently of each other from places A and B with probability p and $(1-p)$ respectively. What is the distribution of the number of requests coming from place A?

10. Between 8 and 9 am the number X of requests arriving to the router has Binomial distribution with parameters (n, r). Each request may come independently of each other from places A and B with probability p and $(1-p)$ respectively. What is the distribution of the number of requests coming from place A?