Exam sheet

Mathematics, Part 1: Probability Theory and Applications

Final Exam

January ???????????????, 2012

Time: 70 minutes

1. (3 points) Define the notion of consistent estimation. Show an example for consistent estimator.

2. (5 points) In 2015 a local internet service provider serves 12000 users. In the peak hours based on their subscriptions and behaviors the users fall into one of the following three categories:

   - beginner: the bandwidth consumption is 100 Mbps in average but no more than 200 Mbps;
   - advanced: the bandwidth consumption is 160 Mbps in average but no more than 280 Mbps;
   - power user: the bandwidth consumption is 250 Mbps in average but no more than 400 Mbps;

   In these groups there are 3500, 6500, 2000 users respectively. Find the minimal bandwidth capacity C such that the probability that the capacity C is not enough is less than 10^{-6}.

3. (6 points) Let $\vec{x} = (x_1, \ldots, x_n)$ be a realization of the i.i.d. sample $X_1, \ldots, X_n$ from exponential distribution with parameter λ, where $\lambda \in (0, \infty)$. Find the maximum likelihood estimation of λ.

4. (11 points) Passengers arrive at a train station. The waiting room has finite capacity of 4 passengers. Let $X(t)$ be the number of passengers in the room at time t. Assume that $X(t), t \geq 0$ is a five state continuous time Markov chain with transition probability matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   Assume that once the number of waiting clients is 0, 1, 2, 3, 4 then the waiting time until the next jump has distribution $Exp(1), Exp(4), Exp(4), Exp(4), Exp(2)$, respectively.

   (a) Find the graph representation of the Markov chain.

   (b) Find the percentage of time that number of waiting passengers is j, for $j = 0, 1, 2, 3, 4$.

   (c) The cost rate of staying in state i is $i + 2$, $i = 0, 1, 2, \ldots, 4$. Find the long-run average cost of the maintenance of the waiting room.