Advanced Mathematics for Electrical Engineers B
homeworks for the "Stochastics 2" part
fall semester 2014

Every week, the assigned homeworks are worth 1 point in total.

HW 1: (due date: 15.09.2014 – **extended to 24.09.2014**)

HW 1.1 A server computer has 70 users. At any given time, some users are "logged in" to the server, while others are not. On a Monday morning each user wants to log in with probability $\frac{3}{10}$, independently of the other users.

a.) What is the probability that there will be exactly 21 users logged in?

b.) If more that 65 users want to log in, the server becomes overloaded, and will not work correctly. What is the probability that this happens?

HW 1.2 Let the random variable X be binomially distributed with parameters n and p. Use a calculator or a computer to calculate the value $\mathbb{P}(X = 3)$ *numerically*, with 4 digits precision, for the following parameter values:

a.) $n = 10$, $p = \frac{2}{10}$

b.) $n = 100$, $p = \frac{2}{100}$

c.) $n = 1000$, $p = \frac{2}{1000}$

d.) $n = 10000$, $p = \frac{2}{10000}$

e.) Calculate the value of $e^{-2} \cdot \frac{2^3}{3!}$, where e is the Euler number $e \approx 2.71828182845905$.

HW 2: (due date: 01.10.2014)

HW 2.1 We toss a fair coin. If the result is heads, we toss it another 2 times (and then stop). If it is tails, then we toss it another 3 times (and then stop). Let X denote the total number of heads we see (during all the 3 or 4 tosses). Use the law of total expectation to calculate the expectation of X.

HW 2.2 Let the random variable Y have binomial distribution with parameters $n = 10$ and $p = \frac{1}{3}$.

a.) Calculate the generating function of Y. *(Hint: to calcualte the sum in a closed form, use the binomial theorem: $(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.)*

b.) Use the properties of the generating function to calcualte the expectation and the variance of Y.

HW 3: (due date: 08.10.2014)

HW 3.1 We keep rolling a fair die until we first roll a 6. Let X denote the *sum* of the numbers rolled before (and not including) that 6. Calculate

a.) the generating function of X,

b.) the expectation of X.

(Hint: write X as a sum with random number of terms. What is the distribution of the number of terms? ***Warning****: What is the* conditional *distribution of a number rolled* under the condition that it's not a 6*?)*

HW 3.2 Harry is organizing a *pyramid scheme* in his family.
(See http://en.wikipedia.org/wiki/Pyramid_scheme) The participants are not too persistent: every participant keeps trying to recruit new participants until the first failure (i.e. until he is first rejected). The probability of such a failure is p at every recruit attempt, independently of the history of the scheme.

The first participant is Harry, he forms the 0-th generation alone. The first generation consists of those recruited (directly) by Harry. The second generation consists of those recruited (directly) by members of the first generation, and so on.

Let Z_k denote the size of the k-th generation ($k = 0, 1, 2, \ldots$), and let N denote the total number of participants in the scheme (meaning $N = \sum_{k=0}^{\infty} Z_k$).

Answer the questions below

I. for $p = \frac{2}{3}$,

II. for $p = \frac{1}{3}$:

a.) Let X denote the number of participants recruited directly by Harry (so $Z_1 = X$). What is the distribution of X? What is the generating function of X? **Answering this question correctly is crucial for the rest of the exercise.**

b.) What is the generating function of Z_2?

c.) What is the expectation of Z_{10}?

d.) How much is the probability $\mathbb{P}(Z_3 = 0)$?

e.) What is the probability that the scheme dies out (that is, one of the generations will already be empty)? *(You only need to answer this for $p = \frac{2}{3}$. We will learn the solution for $p = \frac{1}{3}$ later.)*

f.) What is the expectation of N?

HW 4: (due date: 15.10.2014)

HW 4.1 Answer question e.) of Exercise 3.2 for $p = \frac{1}{3}$ (as well as $p = \frac{2}{3}$).

HW 4.2 An internet service provider has 10000 (ten thousand) clients. The amount of data downloaded by each client is random, with expectation $100MB$ and standard deviation $200MB$. The random amounts downloaded by the different clients are independent and identically distributed.

Use the central limit theorem to estimate the probability that the total amount of data downloaded by the clients on a certain day exceeds $1020000MB$ (one million and twenty thousand Megabytes).

HW 5: (due date: 22.10.2014)

HW 5.1 On a ship, there are 100 passengers, each of whom can swim with probability $\frac{1}{2}$, independently of the others. The ship starts to sink, but there are only 75 life jackets on board. Use the Hoeffding inequality to estimate the probability that there will not be enought life jackets for every non-swimmer.

HW 5.2 John drives his car to work in London every day. According to his observations, the weather can be of three sorts: *rain*, *shower* or *cloudburst*. Based on his experience, the weather of a certain day allows us to guess the weather of the next day in the following probabilistic sense:

$$\mathbb{P}(\text{rain tomorrow}|\text{rain today}) = 1/10,$$

$$\mathbb{P}(\text{cloudburst tomorrow}|\text{rain today}) = 6/10,$$

$$\mathbb{P}(\text{rain tomorrow}|\text{cloudburst today}) = 2/10,$$

$$\mathbb{P}(\text{cloudburst tomorrow}|\text{cloudburst today}) = 4/10,$$

$$\mathbb{P}(\text{cloudburst tomorrow}|\text{shower today}) = 5/10,$$

$$\mathbb{P}(\text{shower tomorrow}|\text{shower today}) = 4/10.$$

Let us denote the states of the weather by numbers: $0 :=$ "rain", $1 :=$ "shower", $2 :=$ "cloudburst". Let us model the sequence of John's morning observations by a time homogeneous Markov chain.

a.) Write the Markov transition matrix P. (Warning: the transition probabilities above are not in order.)

b.) Assuming that it is raining on the 1-st of April, what is the probability of the observation sequence "00012" (starting with the 1-st of April)?

c.) Assuming that it is raining on the 1-st of April, what is the probability that there is shower on the 3-rd of April?

d.) Assuming that it is raining on the 1-st of April, what is the probability that there is shower on the 30-th of April? *Give a formula only.* ***Bonus exercise:*** *calculate the actual probability with some computer tool capable of multiplying matrices fast.*