Advanced Math. for Electr. Eng. B „Stochastics" half-exam Mogy

— solutions 2015.01.27 —

(1) Let $Z_n$ be the number of processes in the $n$-th generation, so $Z_n$ is a Galton-Watson branching process with $m = \mathbb{E}(\#\text{ of children}) = \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 2 = 1$

and generating function

$$g(z) = \sum_{k=0}^{\infty} \mathbb{P}(k \text{ children}) z^k = \frac{1}{3}\cdot z^0 + \frac{1}{3} z^1 + \frac{1}{3} z^2 = \frac{1+z+z^2}{3}$$

a.) $\mathbb{P}(Z_2 = 0) = r_2$ where $r_0 = 0$

$$r_1 = g(r_0) = g(0) = \frac{1}{3}$$

$$r_2 = g(r_1) = g\left(\frac{1}{3}\right) = \frac{1}{3}\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2\right) = \frac{9+3+1}{27},$$

so $\boxed{\mathbb{P}(Z_2 = 0) = \frac{13}{27}}$

b.) $m = 1$, so the process is critical (and non-degenerate), so $\boxed{\mathbb{P}(\text{extinction}) = 1}$

c.) The total time is $1ms \cdot N$ where $N = \sum_{n=0}^{\infty} Z_n$ is the total number of processes in the tree.

$m = 1$, so $\mathbb{E}N = \infty$, so $\boxed{\mathbb{E}(\text{total time}) = \infty}$

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(2) Let $n=10000$ and let $X_i$ be the consumption of the i-th consumer for $i=1,2,\dots,n$. Let $S_n = X_1+\dots+X_n$ be the total consumption.

We need a large deviation estimate for $\mathbb{P}(S_n \geq 3.5\cdot 10^7)$

The $X_i$ are independent and bounded:

$$0 = a_i \leq X_i \leq b_i = 7360 \text{ for the } 9000 \text{ small consumers,}$$

and $$0 = a_i \leq X_i \leq b_i = 23000 \text{ for the } 1000 \text{ big consumers.}$$

We know that $\mathbb{E}S_n = 3.2\cdot 10^7$, so let $t = 0.3\cdot 10^7$.

This way $k := 3.5\cdot 10^7 = \mathbb{E}S_n + t)$, so

$$\mathbb{P}(S_n \geq k) = \mathbb{P}(S_n \geq \mathbb{E}S_n + t) = ?$$

The Hoeffding inequality says

$$\mathbb{P}(S_n \geq k) = \mathbb{P}(S_n \geq \mathbb{E}S_n + t) \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(b_i-a_i)^2}\right).$$

The denominator is

$$\sum_{i=1}^{n}(b_i-a_i)^2 = 9000\cdot(7360-0)^2 + 1000\cdot(23000-0)^2 = \approx 1.0165\cdot 10^{12}, \text{ so}$$

$$\boxed{\mathbb{P}(S_n \geq k) \leq} \exp\left(-\frac{2\cdot 0.09\cdot 10^{14}}{1.0165\cdot 10^{12}}\right) \approx \exp\left(-\frac{18}{1.0165}\right) \approx e^{-17.7} \approx$$

$$\boxed{\approx 2.04\cdot 10^{-8}}$$

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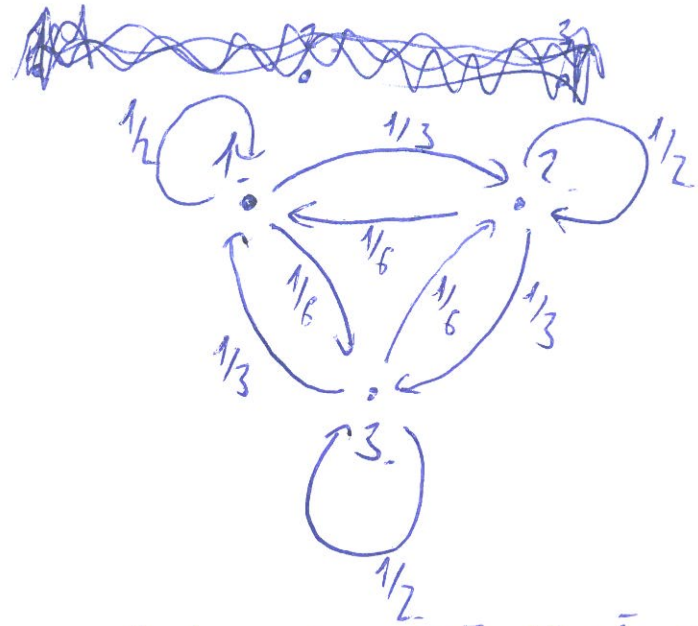
a.)




c.) We need to solve $(P^T - \mathbb{1})\pi^T = 0$, that is

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving this we get $\pi_1 = \pi_2 = \pi_3$, so together with $\pi_1 + \pi_2 + \pi_3 = 1$, the stationary distribution is $\boxed{\pi = \left(\frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}\right)}$

b.) $P(X_2 = 3 \mid X_0 = 1) = (P^2)_{13} = \frac{1}{2}\cdot\frac{1}{6} + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{6}\cdot\frac{1}{2} = \frac{3+4+3}{36} = \boxed{\frac{10}{36}}$

d.) The M.C. is irreducible and aperiodic with a finite state space. $n = 50$ is a long time, so the fundamental theorem of Markov chains says that

$$P(X_n = 3 \mid X_0 = 1) \approx \pi_3 = \boxed{\frac{1}{3}}$$

e.) The M.C. is irreducible with a finite state space, so the ergodic theorem says that

$$\frac{f(X_0) + \dots + f(X_{N-1})}{N} \xrightarrow{N \to \infty} \pi f = \sum_i \pi_i f(i) = \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 4 + \frac{1}{3}\cdot 9 = \boxed{\frac{14}{3}}$$