Midterm exam 1

29 October 2014. 18:00

Advanced Mathematics for Electrical Engineers B, Stochastics part

Working time: 70 minutes. Every exercise is worth 6.67 points.

1. In a randomized computer algorithm there is initially a single process. Before this process exits, it initiates a random number of new ("child") processes (possibly zero): two processes with probability $\frac{1}{4}$, one with probability $\frac{1}{4}$, and zero with probability $\frac{1}{2}$. These new processes work exactly the same way as the original: each of them, in turn, also initiates a random number of child processes, with the same distribution, independently of what happened before. Let us call the initial process alone the "zeroth generation" of processes. Call its children the "first generation", the children of these the "second generation", and so on. Let Z_n denote the size of the nth generation. The algorithm completes successfully if every (sub)process in all generations has exited.

   (a) How much is the probability $\mathbb{P}(Z_3 = 0)$?

   (b) How much is $\mathbb{E}Z_6$?

   (c) What is the probability that the algorithm ever completes successfully?

   (d) Let $N = \sum_{n=0}^{\infty} Z_n$ denote the total number of processes in the entire history of the algorithm. How much is $\mathbb{E}N$?

2. A computer program is designed to decide a "yes-no question" (concerning its input) with some probabilistic method. As a result, it gives the correct answer with 90% probability, but it answers wrong with probability 10%. To increase realiability, the user decides to run the program independently 100 times, and do a "majority vote": he will believe the answer which comes out more often.

   Use the Cramer large deviation inequality to estimate the probability that this procedure gives the wrong answer (or there is no decision because of a draw between "yes" and "no").

   (Hint: define random variables $X_1, X_2, \dots, X_n$ cleverly so that $X_1 + X_2 + \dots + X_n$ is the number of runs with a correct answer.)

3. Each day, a device in a factory can be in two states: operational (ON) or out of order (OFF). If it is ON on a certain day, then with 95% probability it will also be ON the next day as well, but it goes out of order (OFF) with 5% probability, independently of what happened before. On the other hand, if it is OFF, then it is repaired (goes ON) for the next day with 50% probability, and remains OFF with another 50% probability, independently of what happened before. On day 0 the device is ON. Let X_n denote the state of the device on the nth day: we write 1 for ON and 0 for OFF. X_n is a (discrete time) Markov chain.

   a.) Draw the transition graph of the Markov chain.

   b.) Give the state space and the transition probability matrix of the Markov chain.

   c.) What is the probability that we observe the sequence of states 1110010 (starting from day 0)?

   d.) What is the probability that the device is ON on the 4th day?

   e.) Approximate the probability $\mathbb{P}(X_{100} = 1)$.