① Z_n is a Galton-Watson branching process with $Z_0 = 1$

and 1-step offspring distribution

k	0	1	2
$P(X=k)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

,

which has generating function

$$g(z) = \frac{1}{2}\cdot z^0 + \frac{1}{4}\cdot z^1 + \frac{1}{4}\cdot z^2 = \frac{2+z+z^2}{4}$$

and expectation $m = \mathbb{E}X = \frac{1}{2}\cdot 0 + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 2 = \frac{3}{4}$.

a.) $\mathbb{P}(Z_3 = 0) = r_3$, where the sequence r_n satisfies

$r_0 = 0$ and $r_{n+1} = g(r_n)$, so $r_1 = g(0) = \frac{1}{2}$

$$r_2 = g\left(\frac{1}{2}\right) = \frac{2+\frac{1}{2}+\frac{1}{4}}{4} = \frac{11}{16}$$

$$\boxed{\mathbb{P}(Z_3 = 0) = \frac{809}{1024} \approx 0.79} \Longleftarrow \quad r_3 = g\left(\frac{11}{16}\right) = \frac{2+\frac{11}{16}+\frac{121}{256}}{4} = \frac{809}{1024}$$

b.) $\mathbb{E}Z_6 = m^6 = \underline{\underline{\left(\frac{3}{4}\right)^6 \approx 0.178}}$

c.) $m = \frac{3}{4} < 1$, the process is subcritical $\Rightarrow$

$$\boxed{\mathbb{P}(\text{proc. completes}) = \mathbb{P}(\text{extinction}) = 1}$$

d.) $m < 1$, so $\underline{\underline{\mathbb{E}N}} = \frac{1}{1-m} = \frac{1}{1-\frac{3}{4}} = \underline{\underline{4}}$

2 Sorry, there's a typo: ~~Cramer~~ → Hoeffding

Let $X_i = \begin{cases} 1, & \text{if the } i\text{-th run gives a correct answer} \\ 0, & \text{if not.} \end{cases}$

$n = 100$

So $S_n = X_1 + \dots + X_n$ is the # of correct answers,

and the question is

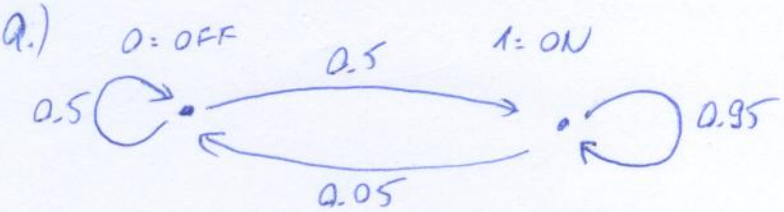
$$\mathbb{P}(\text{wrong end-result}) = \mathbb{P}\left(S_n \le \frac{n}{2}\right) \le ?$$

X_i are i.i.d. $\sim B(p)$ with $p = 0.9$, so $a_i \le X_i \le b_i$ with $a_i = 0$, $b_i = 1$ for $\forall i$. So, to apply the Hoeffding inequality, $E S_n = n \mathbb{E} X_i = np = 90$,

$$\sum_{i=1}^{n} (b_i - a_i)^2 = n \cdot (1-0)^2 = n = 100, \text{ so}$$

$$\boxed{\mathbb{P}\left(S_n \le \frac{n}{2}\right) = \mathbb{P}(S_n \le 50) \overset{t:=40}{=\!=} \mathbb{P}(S_n \le \mathbb{E} S_n - t) \le \\ \overset{\text{Hoeffding}}{\le} \exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right) = \exp\left(-\frac{2 \cdot 40^2}{100}\right) = e^{-32}}$$

③ a.)



b.) $S = \{OFF, ON\} = \{0, 1\}$, $P = \begin{pmatrix} 0.5 & 0.5 \\ 0.05 & 0.95 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{20} & \frac{19}{20} \end{pmatrix}$

c.) We know that $X_0 = 1$, so

$$\mathbb{P}(1110010) = P_{11} P_{11} P_{10} P_{00} P_{01} P_{10} = \frac{19}{20} \cdot \frac{19}{20} \cdot \frac{1}{20} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{20} = \frac{361}{640000}$$

$$\approx 0.000564$$

d.) $\mathbb{P}(X_4 = 1) \underset{\text{that } X_0 = 1}{\overset{\text{we know}}{=}} \mathbb{P}(X_4 = 1 \mid X_0 = 1) = (P^4)_{11}$. To calculate this,

$$P^2 = P \cdot P = \frac{1}{20} \cdot \begin{pmatrix} 10 & 10 \\ 1 & 19 \end{pmatrix} \cdot \frac{1}{20} \cdot \begin{pmatrix} 10 & 10 \\ 1 & 19 \end{pmatrix} = \frac{1}{400} \begin{pmatrix} 110 & 290 \\ 29 & 371 \end{pmatrix}$$

$$\Rightarrow P^4 = P^2 \cdot P^2 = \frac{1}{400} \cdot \begin{pmatrix} 110 & 290 \\ 29 & 371 \end{pmatrix} \cdot \frac{1}{400} \cdot \begin{pmatrix} 110 & 290 \\ 29 & 371 \end{pmatrix} =$$

$$= \frac{1}{400^2} \cdot \begin{pmatrix} \cdots & \cdots \\ \cdots & 29 \cdot 290 + 371^2 \end{pmatrix} = \frac{29 \cdot 290 + 371^2}{400^2} \approx$$

$$\approx$$

e.) $n = 100$ is a long time, and the Markov chain is irreducible and aperiodic, so the fundamental theorem of Markov chains says that $\mathbb{P}(X_n = 1) \approx \Pi_1$ where $\underline{\Pi} = (\Pi_0\ \Pi_1)$ is the only stationary distribution. To find $\underline{\Pi}$, we solve

3) e.) continued

$$(\pi_0 \quad \pi_1)\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{20} & \frac{19}{20} \end{pmatrix} = (\pi_0 \quad \pi_1)$$, which means that

$$\begin{cases} \frac{1}{2}\pi_0 + \frac{1}{20}\pi_1 = \pi_0 \\ \frac{1}{2}\pi_0 + \frac{19}{20}\pi_1 = \pi_1 \end{cases} \iff \begin{cases} -\frac{1}{2}\pi_0 + \frac{1}{20}\pi_1 = 0 \iff \pi_1 = 10\pi_0 \\ \cancel{\frac{1}{2}\pi_0 - \frac{1}{20}\pi_1 = 0} \end{cases}$$

so $\underline{\pi} \parallel (1 \quad 10)$, but $\pi_0 + \pi_1 = 1$, so $\underline{\pi} = \left(\frac{1}{11} \quad \frac{10}{11}\right)$

$$\Rightarrow \mathbb{P}(X_{100} = 1) \approx \pi_1 = \frac{10}{11} \approx 0.909$$