Advanced Mathematics for Electrical Engineers B
homeworks for the "Stochastics 2" part
fall semester 2014

Every week, the assigned homeworks are worth 1 point in total.

HW 1: (due date: 15.09.2014 – **extended to 24.09.2014**)

HW 1.1 A server computer has 70 users. At any given time, some users are "logged in" to the server, while others are not. On a Monday morning each user wants to log in with probability $\frac{3}{10}$, independently of the other users.

a.) What is the probability that there will be exactly 21 users logged in?

b.) If more that 65 users want to log in, the server becomes overloaded, and will not work correctly. What is the probability that this happens?

Solution: Let X denote the number of users logged in (on that Monday morning). You can view each user as "trying" to log in, and having success with probability $p = \frac{3}{10}$, independently of the others. So X is the number of successes out of $n = 70$ independent experiments, each having success probability p. So X has *binomial* distribution with parameters $n = 70$ and $p = \frac{3}{10}$. Notation: $X \sim Bin(70; \frac{3}{10})$. This means that, with the notation $q := 1 - p$,

$$\mathbb{P}(X = k) = \binom{n}{k} p^k q^{n-k} \quad \text{for } k = 0, 1, \ldots, n.$$

In our case ($n = 70$, $p = \frac{3}{10}$)

$$\mathbb{P}(X = k) = \binom{70}{k} \left(\frac{3}{10}\right)^k \left(\frac{7}{10}\right)^{70-k} \qquad \text{for } k = 0, 1, \ldots, 70.$$

So

a.) with $k = 21$ we get $\mathbb{P}(X = 21) = \binom{70}{21} \left(\frac{3}{10}\right)^{21} \left(\frac{7}{10}\right)^{49}$.

b.)

$$\mathbb{P}(X > 65) = \sum_{k=66}^{70} \mathbb{P}(X = k) = \sum_{k=66}^{70} \binom{70}{k} \left(\frac{3}{10}\right)^k \left(\frac{7}{10}\right)^{70-k}.$$

HW 1.2 Let the random variable X be binomially distributed with parameters n and p. Use a calculator or a computer to calculate the value $\mathbb{P}(X = 3)$ *numerically*, with 4 digits precision, for the following parameter values:

a.) $n = 10$, $p = \frac{2}{10}$

b.) $n = 100$, $p = \frac{2}{100}$

c.) $n = 1000$, $p = \frac{2}{1000}$

d.) $n = 10000$, $p = \frac{2}{10000}$

e.) Calculate the value of $e^{-2} \cdot \frac{2^3}{3!}$, where e is the Euler number $e \approx 2.71828182845905$.

Solution: We are calculating

$$\mathbb{P}(X = 3) = \binom{n}{3} p^3 q^{n-3} = \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1} p^3 (1-p)^{n-3}$$

numerically, with four digits precision!

a.) For $n = 10$, $p = \frac{2}{10}$ we get $\frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} \left(\frac{2}{10}\right)^3 \left(\frac{8}{10}\right)^7 = 0.2013$

b.) For $n = 100$, $p = \frac{2}{100}$ we get $\frac{100 \cdot 99 \cdot 98}{3 \cdot 2 \cdot 1} \left(\frac{2}{100}\right)^3 \left(\frac{98}{100}\right)^{97} = 0.1823$

c.) For $n = 1000$, $p = \frac{2}{1000}$ we get $\frac{1000\cdot 999\cdot 998}{3\cdot 2\cdot 1}\left(\frac{2}{1000}\right)^3\left(\frac{998}{1000}\right)^{997} = 0.1806$

d.) For $n = 10000$, $p = \frac{2}{10000}$ we get $\frac{10000\cdot 9999\cdot 9998}{3\cdot 2\cdot 1}\left(\frac{2}{10000}\right)^3\left(\frac{9998}{10000}\right)^{9997} = 0.1805$

e.) With four digits precision, $e^{-2}\cdot\frac{2^3}{3!} = 0.1804$. This illustrates the convergence of binomial probabilities to the Poisson probability – more precisely, the convergence of binomial distributions with $n\cdot p = 2$, as $n\to\infty$, to the Poisson distribution with parameter $\lambda = 2$.

HW 2: (due date: 01.10.2014)

HW 2.1 We toss a fair coin. If the result is heads, we toss it another 2 times (and then stop). If it is tails, then we toss it another 3 times (and then stop). Let X denote the total number of heads we see (during all the 3 or 4 tosses). Use the law of total expectation to calculate the expectation of X.

Solution: Introduce the events A_1: the 1st toss is Heads; A_2: the 1st toss is Tails. These clearly form a partition (meaning that surely exactly one of them occurs), and $\mathbb{P}(A_1) = \mathbb{P}(A_2) = \frac{1}{2}$.

Now observe that for any k, the expected number of "Heads" in k tosses is $k\cdot\frac{1}{2}$. Moreover, if A_1 occurs, then the first toss contributes to X, while if A_2 occurs, then it does not. This means that

$$\mathbb{E}(X|A_1) = 1 + \mathbb{E}(\text{heads in 2 tosses}) = 1 + 2\cdot\frac{1}{2} = 2,$$

$$\mathbb{E}(X|A_2) = 0 + \mathbb{E}(\text{heads in 3 tosses}) = 0 + 3\cdot\frac{1}{2} = \frac{3}{2}.$$

Now the law of total expectation gives

$$\mathbb{E}X = \mathbb{P}(A_1)\mathbb{E}(X|A_1) + \mathbb{P}(A_1)\mathbb{E}(X|A_1) = \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot\frac{3}{2} = \frac{7}{4}.$$

HW 2.2 Let the random variable Y have binomial distribution with parameters $n = 10$ and $p = \frac{1}{3}$.

a.) Calculate the generating function of Y. *(Hint: to calcualte the sum in a closed form, use the binomial theorem: $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$.)*

b.) Use the properties of the generating function to calcualte the expectation and the variance of Y.

Solution:

a.) Y being binomial means $\mathbb{P}(Y = k) = \binom{n}{k}p^k q^{n-k}$ for $k = 0, 1, \ldots, n$ and 0 otherwise. (Here q is just a notation for $1-p$.) So the generating function is

$$g(z) = \sum_{k=0}^{\infty}\mathbb{P}(Y = k)z^k = \sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}z^k = \sum_{k=0}^{n}\binom{n}{k}(pz)^k q^{n-k} = (pz+q)^n.$$

In the last step we used the Binomial Theorem. In our special case $n = 10$, $p = \frac{1}{3}$ and $q = \frac{2}{3}$,

$$g(z) = \left(\frac{1}{3}z + \frac{2}{3}\right)^{10}.$$

b.) Differentiating twice, we get

$$g'(z) = n(pz+q)^{n-1}p,$$

$$g''(z) = n(n-1)(pz+q)^{n-2}p^2.$$

Substituting $z = 1$ and using $p + q = 1$ gives

$$g'(1) = n(p+q)^{n-1}p = np,$$

$$g''(1) = n(n-1)(p+q)^{n-2}p^2 = n^2p^2 - np^2.$$

So, using the properties of the generating function:

$$\mathbb{E}Y = g'(1) = np,$$

$$\mathbf{Var}Y = g''(1) + g'(1) + (g'(1))^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p) = npq.$$

In our special case $n = 10$, $p = \frac{1}{3}$ and $q = \frac{2}{3}$,

$$\mathbb{E}Y = 10 \cdot \frac{1}{3} = \frac{10}{3},$$

$$\mathbf{Var}Y = 10 \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{20}{9}.$$