Adv. Math. for Elect. Eng. B „Stoch part, 2014 fall   Plegy

HW3

3.1 $X = \sum_{k=1}^{N} Y_k$, where $N :=$ # of rolls until (but not incl.) the 1st 6

$Y_k :=$ k-th roll (which is not 6)

So $N \sim \text{Pessbeen}(p = \frac{1}{6})$ (we are „counting failures")

$Y_k \sim Uni(\{1,2,3,4,5\})$, meaning

i	1	2	3	4	5
$P(Y_k = i)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

a.) $\Rightarrow$ $g_N(z) = \sum_{k=0}^{\infty} q^k p z^k = \frac{p}{1-qz}$ | X = sum w. random # of terms $\Longrightarrow$

$g_Y(z) = \frac{z+z^2+z^3+z^4+z^5}{5}$

$$\Rightarrow \boxed{g_X(z) = g_N(g_Y(z)) = \frac{p}{1-q\frac{z+z^2+z^3+z^4+z^5}{5}} \overset{p=\frac{1}{6},\ q=\frac{5}{6}}{=} \frac{1/6}{1-\frac{5}{6}\frac{z+z^2+z^3+z^4+z^5}{5}} = \frac{1}{6-(z+z^2+z^3+z^4+z^5)}}$$

b.) $EN = \frac{1}{p} - 1 = 6 - 1 = 5$, $EY_k = \frac{1+2+3+4+5}{5} = 3$

$$\Rightarrow \boxed{EX = EN\, EY_k = 5 \cdot 3 = 15}$$

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HW3.2

Z_n is a Galton-Watson branching process w. 1-step offspring distribution X

a.) X = # of „failures to reject the offer" before the 1st success $\Rightarrow X \sim \text{PessGeom}(p)$,

so $\mathbb{P}(X=k) = q^k p$ for $k = 0,1,2,\dots$

and $g(z) = p\sum_{k=0}^{\infty} q^k z^k = \frac{p}{1-qz}$;

$m = \mathbb{E}X = \frac{1}{p} - 1$

(I) $p = \frac{2}{3}$, so a.) $g(z) = \frac{2/3}{1-\frac{1}{3}z} = \frac{2}{3-z}$, $m = \frac{3}{2} - 1 = \frac{1}{2}$

b.) $g_2(z) = g(g(z)) = \frac{2}{3-\frac{2}{3-z}} = \frac{2(3-z)}{9-3z-2} = \frac{6-2z}{7-3z}$

c.) $\mathbb{E}Z_{10} = m_{10} = m^{10} = \left(\frac{1}{2}\right)^{10} \approx 0.001$

d.) $\mathbb{P}(Z_3 = 0) = r_3$, where $r_0 = 0$

$r_1 = g(r_0) = g(0) = \frac{2}{3-0} = \frac{2}{3}$

$r_2 = g(r_1) = g\left(\frac{2}{3}\right) = \frac{2}{3-\frac{2}{3}} = \frac{6}{7}$

$\boxed{r_3 = g(r_2) = g\left(\frac{6}{7}\right) = \frac{2}{3-\frac{6}{7}} = \frac{14}{15}}$

e.) $m < 1$, so $\mathbb{P}(\text{extinction}) = 1$

f.) $m < 1$ so $\mathbb{E}N = \frac{1}{1-m} = \frac{1}{1-\frac{1}{2}} = 2$

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HW3.2 continued

(11) for $p = \frac{1}{3}$

a.) $g(z) = \dfrac{1/3}{1-\frac{2}{3}z} = \dfrac{1}{3-2z}$, $m = 3-1 = 2$

b.) $g_2(z) = g(g(z)) = \dfrac{1}{3-2\frac{1}{3-2z}} = \dfrac{3-2z}{9-6z-2} = \dfrac{3-2z}{7-6z}$

c.) $\mathbb{E} Z_{10} = m_{10} = m^{10} = \underline{1024}$

d.) $\mathbb{P}(Z_3 = 0) = r_3$, where $r_0 = 0$

$$r_1 = g(r_0) = g(0) = \frac{1}{3-2\cdot 0} = \frac{1}{3}$$

$$r_2 = g(r_1) = g\left(\tfrac{1}{3}\right) = \frac{1}{3-2\cdot\frac{1}{3}} = \frac{3}{7}$$

$$\boxed{r_3 = g(r_2) = g\left(\tfrac{3}{7}\right) = \frac{1}{3-2\cdot\frac{3}{7}} = \frac{7}{15}}$$

~~d.)~~ e) $m > 1$, so $\mathbb{P}(\text{extinction})$ is the solution in $[0,1)$ of the equation $z = g(z)$. In our case

$$z = \frac{1}{3-2z} \quad / \cdot (3-2z)$$

$$3z - 2z = 1 \quad / + 2z^2 - 3z$$

$$0 = 2z^2 - 3z + 1$$

$$\Rightarrow z = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4} = \begin{cases} 1 \\ \boxed{\frac{1}{2}} \end{cases}$$

$$\boxed{\mathbb{P}(\text{extinction}) = \frac{1}{2}}$$

f.) $m > 1$, so $\mathbb{E}N = \infty$.

HW4.1

$g(z) = \frac{1}{3-2z}$ and $m = 2$ for a branching process.

$m > 1$, so $\mathbb{P}(\text{extinction})$ is the solution in $[0,1)$ of the equation $z = g(z)$. In our case

$$z = \frac{1}{3-2z} \Rightarrow \ldots \Rightarrow z = \begin{cases} 1 \\ \boxed{\frac{1}{2}} \end{cases}$$

$$\boxed{\mathbb{P}(\text{extinction}) = \frac{1}{2}}$$

HW4.2

$n := 10000$

X_i = the amount of data (in MB) downloaded by the ith client. The X_i are i.i.d. $\mu = \mathbb{E}X_i = 100$,

$\sigma = DX_i = \sqrt{\mathrm{Var}\, X_i} = 200$

$S_n := X_1 + \ldots + X_n$. Set $K = 1.020.000$

By the C.L.T. $S_n \approx \mathcal{N}(nm, n\sigma^2) = \mathcal{N}(nm, (\sqrt{n}\sigma)^2)$

so $\mathbb{P}(S_n > K) \approx 1 - \phi\left(\frac{K - nm}{\sqrt{n}\sigma}\right) =$

$$= 1 - \phi\left(\frac{1020.000 - 1000.000}{\sqrt{10000} \cdot 200}\right) =$$

$$= 1 - \phi\left(\frac{20.000}{100 \cdot 200}\right) = 1 - \phi(1) \approx 1 - 0.8413 =$$

$$= \underline{\underline{0.1587}}$$