$\boxed{\text{HW 5.1}}$ Let $n = 100$ and $X_i := \begin{cases} 1, & \text{if the } i\text{-th passenger can } \boxed{\text{not}} \text{ swim} \\ 0, & \text{if he/she can.} \end{cases}$

for $i = 1, 2, \dots, n$.

So $S_n := X_1 + \dots + X_n$ is the number of non-swimmers, and the question is $\mathbb{P}(S_n \geq 76)$.

To use the Hoeffding inequality, we use that

- the X_i are independent (by assumption of the exercise)
- $\mathbb{P}(X_i = 1) = \frac{1}{2}$, so $\mathbb{E}X_i = \frac{1}{2}$ and $\mathbb{E}S_n = n \cdot \frac{1}{2} = 50$
- $a_i \leq X_i \leq b_i$, where $a_i = 0$ and $b_i = 1$ work for every i.

So $\mathbb{P}(S_n \geq 76) = \mathbb{P}(S_n \geq \mathbb{E}S_n + t)$ with the choice $t := 26$,

and the Hoeffding inequality says

$$\mathbb{P}(S_n \geq \mathbb{E}S_n + t) \leq \exp\left\{-\frac{2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right\} = \exp\left\{-\frac{2 \cdot 26^2}{\sum_{i=1}^{100}(1-0)^2}\right\} =$$

$$= \exp\left\{-\frac{2 \cdot 26^2}{100 \cdot 1}\right\} = e^{-13.52} \approx 1.34 \cdot 10^{-6}$$

$\boxed{\text{HW 5.2}}$ a) With the notation 0 = „rain", 1 = „shower" and 2 = „cloudburst", the transition probabilities given in the text are

$$P_{00} = \frac{1}{10};\ P_{02} = \frac{6}{10};\ P_{20} = \frac{2}{10};\ P_{22} = \frac{4}{10};\ P_{12} = \frac{5}{10};\ P_{11} = \frac{4}{10}.$$

Putting these in the 3×3 transition matrix P, we get

$$P = \begin{array}{c} \\ 0. \\ 1. \\ 2. \end{array} \begin{array}{c} \begin{array}{ccc} 0. & 1. & 2. \end{array} \\ \left(\begin{array}{c|c|c} \frac{1}{10} & ? & \frac{6}{10} \\ \hline ? & \frac{4}{10} & \frac{5}{10} \\ \hline \frac{2}{10} & ? & \frac{4}{10} \end{array}\right) \end{array}.$$

The missing 3 matrix elements can be found out becaus the sum in every row has to be 1, so

HW 5.2 continued (2/3)

a.) $$P = \begin{pmatrix} \frac{1}{10} & \frac{3}{10} & \frac{6}{10} \\ \frac{1}{10} & \frac{4}{10} & \frac{5}{10} \\ \frac{2}{10} & \frac{4}{10} & \frac{4}{10} \end{pmatrix}$$

b.) ~~P(~~ Let X_i denote the „state" of the weather on the i-th day (of April), so

$$\mathbb{P}(X_1=0,\ X_2=0,\ X_3=0,\ X_4=1,\ X_5=2 \mid X_1=0) = P_{00}\cdot P_{00}\cdot P_{01}\cdot P_{12} =$$

$$= \frac{1}{10}\cdot\frac{1}{10}\cdot\frac{3}{10}\cdot\frac{5}{10} = \frac{15}{10000} = \underline{\underline{0.0015}}$$

c.) $\boxed{\text{Solution 1}}$ – by hand: Starting from state 0 on the 1st of April, we can reach state 1 in 2 days via the following 3 paths:

~~$\mathbb{P}(X_3=1 \mid X_1=0) = \mathbb{P}(001 \mid X_1$~~ 001 ; 011 ; 021

The probabilities of these (assuming that we start from 0) are $P_{00}\cdot P_{01}$; $P_{01}\cdot P_{11}$; $P_{02}\cdot P_{21}$, respectively, so

$$\mathbb{P}(X_3=1 \mid X_1=0) = P_{00}\cdot P_{01} + P_{01}\cdot P_{11} + P_{02}\cdot P_{21} =$$

$$= \frac{1}{10}\cdot\frac{3}{10} + \frac{3}{10}\cdot\frac{4}{10} + \frac{6}{10}\cdot\frac{4}{10} = \frac{3+12+24}{100} = \underline{\underline{0.39}}$$

$\boxed{\text{Solution 2}}$ – using the theory:

$\mathbb{P}(X_3=1 \mid X_1=0)$ is the $(0;1)$ element of the 2-step transition matrix $P^{(2)}$, meaning $\mathbb{P}(X_3=1 \mid X_1=0) = P^{(2)}_{01}$,

but we know that $$P^{(2)} = P^2 = \begin{pmatrix} \frac{1}{10} & \frac{3}{10} & \frac{6}{10} \\ \frac{1}{10} & \frac{4}{10} & \frac{5}{10} \\ \frac{2}{10} & \frac{4}{10} & \frac{4}{10} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{10} & \frac{3}{10} & \frac{6}{10} \\ \frac{1}{10} & \frac{4}{10} & \frac{5}{10} \\ \frac{2}{10} & \frac{4}{10} & \frac{4}{10} \end{pmatrix} = \begin{pmatrix} \dots & \frac{39}{100} & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix},$$

so $$\boxed{\mathbb{P}(X_3=1 \mid X_1=0) = P^{(2)}_{01} = \frac{39}{100}}$$

(We don't need to calculate the other 8 elements of P^2.)

HW 5.2 continued (3/3)

d.) Similarly to question c.), we have

$\mathbb{P}(X_{30}=1 \mid X_1=0)$ is the $(0;1)$ element of the 29-step transition matrix $P^{(29)}=P^{29}$, so

$$\boxed{\mathbb{P}(X_{30}=1 \mid X_1=0)=\left(P^{29}\right)_{01}}$$

Bonus: An explicit computer calculation gives

$$P^{29}\approx\begin{pmatrix} 0.14679 & 0.38532 & 0.46789 \\ 0.14679 & 0.38532 & 0.46789 \\ 0.14679 & 0.38532 & 0.46789 \end{pmatrix},$$

so $\mathbb{P}(X_{30}=1 \mid X_1=0)\approx \underline{\underline{0.38532}}$