Advanced Mathematics for Electrical Engineers B
homeworks for the "Stochastics 2" part
fall semester 2012

Every week, the assigned homeworks are worth 1 point in total.

HW 1: (due date: 2012.09.14.)

HW 1.1 70% of students at the technical university are boys, 30% are girls. 20% of the boys and 75% of the girls have long hair. Choosing a *long haired* student at random, what is the probability that we choose a girl?

HW 1.2 We roll a red die and we denote the number rolled by X. After that, we roll X green dice, and denote by Y the *sum* of the numbers rolled on the green dice. What is the expectation of Y?

HW 2: (due date: 2012.09.21.)

HW 2.1 The generating function of a nonnegative integer valued random variable is

$$g(z) = \frac{3}{8} + \frac{3}{8}z + \frac{1}{8}z^2 + \frac{1}{8}z^3$$

. What is the discrete probability distribution (namely the probabilities $\mathbb{P}(X = k)$)? What is the expectation and variance of X?

HW 2.2 We toss a fair coin 3 times, and a biased coin with $\mathbb{P}(heads) = \frac{1}{3}$ also three times. Let Z denote the *total* number of heads seen. Calculate the generating function of Z.

HW 3: (due date: 2012.09.28.)

HW 3.1 We keep rolling a fair die until we first roll a 6. Let X denote the *sum* of the numbers rolled before (and not including) that 6. Calculate

a.) the generating function of X,

b.) the expectation of X,

c.) the variance of X.

(Warning: What is the conditional *distribution of a number rolled* under the condition that it's not a 6*?)*

HW 3.2 Harry is organizing a *pyramid scheme* in his family.
(See http://en.wikipedia.org/wiki/Pyramid_scheme) The participants are not too persistent: every participant keeps trying to recruit new participants until the first failure (i.e. until he is first rejected). The probability of such a failure is p at every recruit attempt, independently of the history of the scheme.

The first participant is Harry, he forms the 0-th generation alone. The first generation consists of those recruited (directly) by Harry. The second generation consists of those recruited (directly) by members of the first generation, and so on.

Let Z_k denote the size of the k-th generation ($k = 0, 1, 2, \ldots$), and let N denote the total number of participants in the scheme (meaning $N = \sum_{k=0}^{\infty} Z_k$).

Answer the questions below

I. for $p = \frac{2}{3}$,

II. for $p = \frac{1}{3}$:

a.) What is the generating function of Z_2?

b.) What is the expectation of Z_{10}?

c.) How much is the probability $\mathbb{P}(Z_3 = 0)$?

d.) What is the probability that the scheme dies out (that is, one of the generations will already be empty)?

e.) What is the expectation of N?

f.) What is the generating function of N?

HW 4: (due date: 2012.10.08.)

HW 4.1 Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables having Bernoulli distribution with parameter $p = \frac{1}{2}$ (meaning $\mathbb{P}(X_i = 1) = p = 1 - \mathbb{P}(X_i = 0) = \frac{1}{2}$). Let $n = 10^6$ and $S_n = X_1 + X_2 + \cdots + X_n$ (so $S_n \sim Bin(n = 10^6; p = \frac{1}{2})$).

a.) If for some $K \in (0; 10^6)$ we approximate the probability $\mathbb{P}(S_n < K)$ using the central limit theorem, at most how big can the error in this estimate be, according to the Berry-Esséen theorem? (Warning: In some sources, the simplest form of the theorem stated is about random variables *with zero expectation.* The Bernoullli distribution is *not like that.*) (The constant C in the Berry-Esséen theorem can be chosen as $C = 0.4784$ (due to a result from 2010).)

b.) Use the Hoeffding inequality to find a bound K, for which we can be sure that

$$\mathbb{P}(S_n > K) \le 10^{-8}.$$

Denote this bound K as K_H.

c.) Calculate the approximate value of the probability $\mathbb{P}(S_n > K_H)$ using the Cramer theorem. Hint: the moment generating function of the Bernoulli distribution with parameter p is $M(\lambda) = 1 - p + pe^{\lambda}$, from which the Cramer rate function is

$$I(x) = x \ln \frac{(1-p)x}{p(1-x)} - \ln \frac{1-p}{1-x}.$$

HW 5: (due date: 2012.10.17.)

HW 5.1 The graph shown in Figure 1 shows the possible one-step transitions (that have positive probability) for a time-homogeneous discrete time Markov chain. Classify the states, grouping in the same class those that communicate with each other. For every class, decide

* if it is essential or inessential,
* if it is recurrent or transient,
* its period, and whether it is periodic or aperiodic.

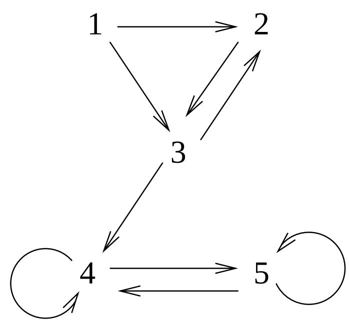

Figure 1: Graph representation of a Markov chain (without probabilities)

HW 5.2 John drives his car to work in London every day. According to his observations, the weather can be of three sorts: *rain*, *shower* or *cloudburst*. Based on his experience, the weather of a certain day allows us to guess the weather of the next day in the following probabilistic sense:

$$\mathbb{P}(\text{rain tomorrow}|\text{rain today}) = 1/10,$$

$$\mathbb{P}(\text{cloudburst tomorrow}|\text{rain today}) = 6/10,$$

$$\mathbb{P}(\text{rain tomorrow}|\text{cloudburst today}) = 2/10,$$

$$\mathbb{P}(\text{cloudburst tomorrow}|\text{cloudburst today}) = 4/10,$$

$$\mathbb{P}(\text{cloudburst tomorrow}|\text{shower today}) = 5/10,$$

$$\mathbb{P}(\text{shower tomorrow}|\text{shower today}) = 4/10.$$

Let us denote the states of the weather by numbers: $0 :=$ "rain", $1 :=$ "shower", $2 :=$ "cloudburst". Let us model the sequence of John's morning observations by a time homogeneous Markov chain.

a.) Write the Markov transition matrix P. (Warning: the transition probabilities above are not in order.)

b.) Assuming that it is raining on the 1-st of April, what is the probability of the observation sequence "00012" (starting with the 1-st of April)?

c.) Assuming that it is raining on the 1-st of April, what is the probability that there is shower on the 3-rd of April?

d.) Assuming that it is raining on the 1-st of April, what is the approximate probability that there is shower on the 30-th of April?

e.) What proportion of the mornings has a shower, on the long run?

f.) If there is rain, John spends 20 minutes driving in a traffic jam, but if there is shower, he spends 30, and if there is a cloudburst, then 70 minutes. How much time does he spend in the morning traffic jam, in daily average, on the long run?