Midterm exam 1

30 October 2013. 16:00

Advanced Mathematics for Electrical Engineers B, Stochastics part

Working time: 60 minutes. Every exercise is worth 6.67 points.

1. At a bank counter, serving a costumer takes exactly one minute. During that minute, the number of newly arriving costumers can be 0, 1 or 2, all with probability $\frac{1}{3}$, independently of what happened before. Just before time 0 the queue is empty, but at time 0 the first costumer (Robert) arrives.

   (a) What is the probability that the queue will again become empty some (any) time in the future?

   (b) What is the expected number of costumers served before the queue gets empty again?

   (Hint: While Robert is being served, a random number of new customers arrive. Call them the "first generation".)

2. Joe decides to keep rolling a fair die until he manages 1000 times to roll 6. (His successes, of course, don't have to be consecutive.) Use your favourite large deviation theorem to estimate the probability that he succeeds in at most 5000 rolls.

   (Help: the Cramer rate function of the Bernoulli distribution with parameter p is

   $$I(x) = x \ln\left(\frac{x}{1-x}\frac{1-p}{p}\right) + \ln\left(\frac{1-x}{1-p}\right)$$

   (if $0 < x < 1$). The Cramer rate function of the (optimistic) geometric distribution with parameter p is

   $$I(x) = x \ln\left(\frac{x-1}{x}\frac{1}{1-p}\right) + \ln\left(\frac{1}{p}\frac{1-p}{x-1}\right)$$

   (if $x > 1$).)

3. We put a standard die on the table, and play the following game: In every step, we "flip" the die to one of the 4 faces neighbouring the bottom face, choosing randomly and uniformly from the 4 possibilities, independently of the past. We let X_n denote the number seen on the top after n steps (for $n = 0, 1, 2, \ldots$). This X_n is (of course) a Markov chain. We start with 6 being on top, so $X_0 = 6$.

   Remark: on a standard die, the sum of the two numbers on any two opposite faces is always 7.

   a.) Draw the transition graph of the Markov chain.

   b.) Give the transition probability matrix of the Markov chain.

   c.) What is the probability that $X_3 = 6$?

   d.) What is the stationary distribution of the Markov chain? *(Hint: it's possible to guess the answer and then check that your guess is correct.)*

   e.) What is the approximate probability that $X_{100} = 3$?

   f.) What is the average of the numbers on top on the long run?