Problem Sheet # 2

Multidimensional random variables, the Law of Total Expectation

September 14, 2011

1) We roll two fair dice a blue and a red. We first roll the red die then we roll the blue die as many times as the outcome of the red die. Let Y denote the outcome of the red die and denote by X the *sum* of the outcomes on the blue die.
   a) Find $\mathbf{E}X$ and $\mathbf{Var}X$.
   b) What is the sign of $\operatorname{cov}(X, Y)$?

2) Let $X_1, X_2, \dots$ be i.i.d. $\mathbb{N}$ valued random variables with distribution function F ($F(x) = \mathbf{P}(X \leq x)$). Show that the distribution function of the random variable $Y = \max\{X_1, X_2, \dots, X_n\}$ is $H(x) = F^n(x)$. See hint.[1].

3) A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours. Assume that the miner is at all times equally likely to choose any one of the doors. Find the expected length of time until he reaches safety.

4) In order to help some friends, Harry becomes the east cost sales representative of B& D Software. The software has been favorably reviewed and demand is heavy. Harry sets up a sales booth at the local computer show and takes orders. Each order takes three minutes to fill. While each order is being filled there is a probability p_j that j more customers will arrive and join the line. Assume $p_0 = .4$, $p_1 = 0.4$ and $p_2 = .2$. Harry cannot take a coffee break until a service is completed and no one is waiting in the line. Note that it can be proved that the number of customers waiting for Harry comes to be 0 in finite time with probability 1. What is the expected number of customers he serves before the first coffee break?

5) Assume that a maleware is infecting the computers of an infinite size population of computers. If a computer is infected then by the end of the next day one of the following possibilities occurs. It is detected and deleted with probability p. Consequently, it is not infectious anymore. With probability $(1-p)p$ it is not deleted and does not infect other computer. Further, it is not deleted and it infects $k-1$ new computers having not infected to that date with probability $(1-p)^k p$ for $k = 2, 3, \dots$.
   Let X denote the contribution of one infected machine to the set of infected computers by the end of the next day.
   The distribution of X is $\mathbf{P}(X = k) = (1-p)^k p$ for $k = 0, 1, \dots$. Note that, $\mathbf{E}X = \frac{1-p}{p}$.
   Assume that the contribution of each infected computer is independent of the other infected computers' contributions.
   Let $p = \frac{2}{3}$. Note that if $p = \frac{2}{3}$, it can be proved that the number of infected computers comes to be 0 in finite time with probability 1.
   Find the expected number of infected computers during the the lifetime of the maleware.

6) In a town 40000 families live. The amount of garbage produced by a family in a day is no more than 50 liters, the expectation is 20 liters and the standard deviation is 10 liters.
   The town installs a trash-burning plant. Find the capacity of the plant such that the amount of garbage per day is less than the capacity with probability at least 1%. Apply CLT to estimate the capacity.

7) Harry is playing roulette in a casino. In each game he stakes 1000€in the outcome of "red". By the end of the 100th game his loss is 3000€. Shoud he suspect that the casino is cheating?

[1] Use that $\mathbf{P}(\max\{X_1, \dots, X_n\} \leq x) = \mathbf{P}(X_1 \leq x, \dots, X_n \leq x)$ for any positive integer n and then $X_1, X_2, \dots$ are i.i.d. r.v.'s.