Problem Sheet # 4

Markov chains

September 29, 2011

1) In London, the probability that a rainy day follows a sunny day is 50% and the probability that a rainy day follows a rainy day is 30%.
   a) Find the long-run percentage of rainy days.
   b) Assume that today we have a rainy day. Find the probability that we have rainy day on the day after tomorrow. Find the probability that we have rainy day on the third day.

2) We place a die on a table. In each minute we flip the die to one of the neighboring page. The page is selected uniformly randomly. Let X_n denote the number on the upturning page in the nth minute. Is it true that X_n irreducible? Find the transition probability matrix and the stationary distribution.

3) N white and N black balls are distributed in two urns in such a way that each contains N balls. We say that the system is in state i $(i = 0, 1, 2, \dots)$ if the first urn contains i black balls. At each step, we draw one ball from each urn and place the ball drawn from the second urn into the first urn, and conversely with the ball from the first urn. Determine the transition probabilities p_{ij}.

4) Specify the classes of the following Markov chain, find the their period and determine whether or not they are recurrent.

$$P_1 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix},\ P_2 = \begin{bmatrix} 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/2 \end{bmatrix}$$

   Find the graph representation of each P_i.
   a) What is the probability of the finite trajectory 1223 in case 1?
   b) Find $\mathbf{P}(X_2 = 2 \mid X_0 = 1)$ and $\mathbf{P}(X_3 = 3 \mid X_0 = 4)$ in case 2.
   c) Find $\mathbf{P}(X_2 = 2 \mid X_0 = 1)$ and $\mathbf{P}(X_3 = 3 \mid X_0 = 4)$ in case 3.
   d) What is the probability of the finite trajectory 1223 and 1225 in case 1?

   Find the stationary distribution of each P_i.

5) Consider a Markov chain with states $0, 1, 2, \dots$ and with transition probabilities given by $p_{0i} = p_i > 0$, $\sum_{i=0}^{\infty} p_i = 1$, $\sum_{i=0}^{\infty} i p_i < \infty$, $p_{i,i-1} = 1$ $(i \geq 1)$.
   Show that the chain is irreducible, aperiodic, recurrent and positive recurrent, and find the stationary distribution.

6) Consider a random walk with states $0, 1, \dots$ for which

$$p_{i,i+1} = p_i,\ p_{i,i-1} = q_i = 1 - p_i \quad i = 0, 1, \dots$$

   where $p_0 = 1$. Find conditions for the process having stationary distribution.
   Let $p_j = p$ and $q_j = q = 1 - p$ for $j = 0, 1, \dots$. Find the stationary distribution. (We will apply the results for finding the stationary distribution of the queue length in $M/M/1$ system.)

7) Suppose $0 < a, b < 1$ and let

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

   be a transition matrix corresponding to a Markov chain with state space $\{0, 1\}$. Show that

$$P^n = (a+b)^{-1} \left\{ \begin{bmatrix} b & a \\ b & a \end{bmatrix} + (1-a-b)^n \begin{bmatrix} a & -a \\ -b & b \end{bmatrix} \right\}$$

8) A company signs each year a contract for delivery of a certain product with one of three possible subcontractors. The sequence of chosen subcontractors forms a Markov chain with transition probability matrix:

$$\begin{bmatrix} 2/3 & 1/3 & 0 \\ 2/3 & 1/4 & 1/12 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

   a) Find the probability that contractor 2 has been chosen for four successive years.
   b) Find the frequency of choosing the first subcontractor.
   c) The annual cost of the contract with the subcontractor $1, 2, 3$ respectively is \$50,000, \$40,000, \$45,000 respectively. Find the average cost of applying subcontractors.
   d) Compute the probability that, after a long time, the same subcontractor is chosen two years in a row.

9) A professional appliance (e.g. a factory, a large scale computer network, a big block of flats) is controlled by a group of amateurs. From time to time several problems arise that affect the efficiency of the operation of the system. The real experts visit the system and solve the problems perfectly at the end of each week. When they arrive the actual state of the appliance falls into one of the following four categories: 1 - no problem; 2 - small correction is needed; 3 - big correction is needed; 4 - disastrous.

The costs of identifying the state and the correction in these states are: 10 (1), 30 (2), 100 (3), 1000 (4) units.

The states of successive weeks can be modeled by a four state Markov chain since we assume that the next week's state depends only on the previous week's state and the events occurred during the week. On the other hand, the amateur staff is able to handle the problems on a certain random level. The transition probabilities are summarized in the following matrix:

$$\begin{bmatrix} 0,4 & 0,4 & 0,2 & 0 \\ 0,2 & 0,6 & 0,1 & 0,1 \\ 0 & 0,3 & 0,4 & 0,3 \\ 0,5 & 0,4 & 0,1 & 0 \end{bmatrix}$$

(the order of the states are 1,2,3,4).

a) What is the probability that after two weeks the system is without any problem if the time period started with a small correction? (We have not registered the state of one intermediate week.)
b) What is the probability that a week without problems is followed by three weeks without problems?
c) In the long run find the percentage of time that the professional group finds the system in state i, $i = 1, 2, 3, 4$.
d) What is the long run average cost of the application of the professional group?