Master Thesis

# Self-similar sets and measures

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# Contents

# 0 Summary

This Thesis is organized as follows:

- We present the two main results of the Thesis in Subsection 1.1.
- Then in Subsection 1.2 we give a general introduction into the dimensional theory of self-similar sets.
- In Section 2 we give a brief account about the results concerning the Hausdorff measure of the Sierpinski triangle. Then we present our main Theorem, which significantly improves the previous bounds on the Hausdorff measure of the Sierpinski triangle.
- In Section 3 after a general overview about Bernoulli convolutions we extend the work of B. Solomyak, Y. Peres and H. Tóth about Bernoulli convolutions with different probabilities.

The Thesis contains an electric supplement on a compact disk (CD) for Section 2. The content of the CD is also available at:
*http://www.math.bme.hu/˜morap/sierpinski.zip*

# 1 Introduction

## 1.1 Extended abstract

It is well-known that the Hausdorff dimension of the Sierpinski triangle $\Lambda$ is $s = \log 3/\log 2$. However, it is a long standing open problem to compute the $s$-dimensional Hausdorff measure of $\Lambda$ denoted by $\mathcal{H}^s(\Lambda)$. In the literature the best existing estimate is

$$0.670432 \leq \mathcal{H}^s(\Lambda) \leq 0.81794.$$

In Section 2 we improve significantly the lower bound. We also give an upper bound which is weaker than the one above but everybody can check it easily. Namely, we prove that

$$0.77 \leq \mathcal{H}^s(\Lambda) \leq 0.819161232881177$$

holds. We give a general overview about self-similar sets in Section 1.2.

In Section 3 we consider Bernoulli convolutions. Let $p \in (0,1)$ and $\lambda \in (0,1)$, and take the following random sum

$$Y_\lambda^p := \sum_{n=0}^{\infty} \pm \lambda^n,$$

where the signs "$+$" and "$-$" are chosen identically and independently with probability $p$ and $1-p$. Let $\nu_\lambda^p$ be the distribution of $Y_\lambda^p$.

If $\lambda = p^p(1-p)^{1-p}$ then the Hausdorff dimension of $\nu_\lambda^p$ is less than or equal to 1, thus for $\lambda < p^p(1-p)^{1-p}$ the measure $\nu_\lambda^p$ is singular. It is conjectured that for every $p \in (0,1)$ and for almost every $\lambda \in (p^p(1-p)^{1-p}, 1)$ the measure $\nu_\lambda^p$ is absolutely continuous with respect to the Lebesgue measure. It was proved by B. Solomyak and Y. Peres (1998) [14, Corollary 1.4] that this holds for $p \in [1/3, 2/3]$. They [14, Theorem 1.3] also showed that we only have chance for $L^2$-density if $\lambda \geq p^2 + (1-p)^2$ holds. However, Peres and Solomyak left open the corresponding problem for $p \in (0, 1/3)$. The first steps in this case was made by H. Tóth in 2008:

**Theorem 1** (Tóth [13]). *For $p \in (0, 1/3)$ and for almost every*

$$\lambda \in ((1-2p)^{2-\log 41/\log 9}, 1)$$

*the measure $\nu_\lambda^p$ is absolutely continuous with $L^2$-density.*

In Section 3 we improve H. Tóth's result. The motivation of our research was as follows. Recently there have arisen some problems (related to the concentration of medicine in the blood), which require the better understanding of $\nu_\lambda^p$ for $p$ close to zero. Our most important achievement is that our result gives a better tangent for $p \approx 0$.

**Remark 1.** *I want to thank my supervisor, Károly Simon for his support.*

## 1.2 Dimension theory of the self-similar sets

We call $\{f_1, f_2, \ldots, f_m\}$ an *iterated function system (IFS)*, where the functions $f_1, f_2, \ldots, f_m : \mathbb{R}^d \to \mathbb{R}^d$ are contraction mappings. In this Thesis we consider only those cases, where the functions $f_1, f_2, \ldots, f_m$ are contracting similarities. Namely, for $i = 1, 2, \ldots, m$ we have

$$|f_i(x) - f_i(y)| = c_i |x - y|.$$

By a Theorem of Hutchinson [1] there is a unique nonempty compact set $F$, which satisfies

$$\bigcup_{i=1}^{k} f_i(F) = F. \tag{1.1}$$

We say that $F$ is the *attractor* of the IFS. The attractor is often called fractal because of its self-similar property. For example let

$$f_1(x) = \frac{1}{3}x, \quad f_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

In this case the attractor is the triadic Cantor set $C$. See Figure 1.

We say that $\{A_k\}_{k=1}^{\infty}$ is an $F$-cover, if

$$F \subset \bigcup_{k=1}^{\infty} A_k.$$

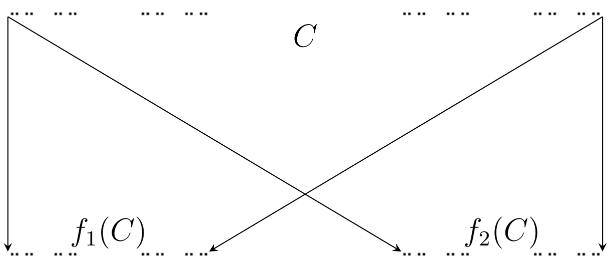


Figure 1: Self-similar property of the Cantor set.

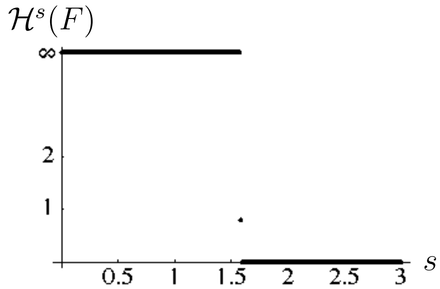


Figure 2: An example for the function $\mathcal{H}^s(F)$.

For every $s \geq 0$ we define the $s$-dimensional Hausdorff measure of $F$.

$$\mathcal{H}^s(F) := \lim_{\delta \to 0} \inf \Big\{ \sum_{k=1}^{\infty} |A_k|^s, \quad \text{where } |A_k| < \delta \text{ and } \{A_k\}_{k=1}^{\infty} \text{ is an } F\text{-cover} \Big\}$$

where $|A_k|$ denotes the diameter of the set $A_k$.

The Hausdorff dimension of the set $F$ is defined as follows:

$$dim_H(F) = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

See Figure 2. We remark that the $s = dim_H(F)$ dimensional Hausdorff measure of $F$ can be zero, positive finite or infinite. Since Hausdorff dimension and measure are basic definitions of fractal theory, there are many methods to obtain a lower or upper bound on these values. Now we present some Lemmas, which will be used in Section 2. It is well known (see [5]) that:

**Lemma 1** (Mass distribution principle). *If there exists a $\mu$ measure on $\mathbb{R}^d$, and there exist $c > 0$ and $s > 0$, such that $0 < \mu(\mathbb{R}^d) < \infty$ and for every $A \subset \mathbb{R}^d$*

$$\mu(A) \leq c|A|^s$$

*holds then $\mathcal{H}^s(F) \geq \mu(F)/c$.*

*Proof.* Let $\{A_k\}_{k=1}^{\infty}$ be an arbitrary $F$-cover. Then we have

$$\sum_k |A_k|^s \geq \sum_k \frac{\mu(A_k)}{c} \geq \frac{\mu(F)}{c}$$

□

For $i = 1, 2, \ldots m$ let $c_i$ be the contraction ratio of $f_i$. We call the solution $s$ of the equation

$$\sum_{i=1}^{m} c_i^s = 1$$

the similarity dimension of the IFS. The value $s$ seems to be the most natural guess to be the Hausdorff dimension. However overlaps in (1.1) can cause that $dim_H(F)$ is strictly smaller than $s$. In 1981 Hutchinson [1] defined a property, which helps to calculate $dim_H(F)$ in some basic cases.

**Definition 1.** $f_i : \mathbb{R}^n \to \mathbb{R}^n$ $(i = 1, 2, \ldots, m)$ *satisfies the open set condition (OSC) if there exists a nonempty open set $U$, such that for $i \neq j$ we have $f_i(U) \cap f_j(U) = \emptyset$ and for all $i$, $f_i(U) \subset U$ holds.*

Another Theorem of Hutchinson asserts that OSC implies that the Hausdorff dimension is equal to the similarity dimension $s$:

**Theorem 2** (Hutchinson [1]). *If $f_i$ $(i = 1, 2, \ldots, m)$ satisfies the open set condition, then the Hausdorff dimension of the attractor $F$ is equal to the similarity dimension ($dim_H(F) = s$). Moreover the $\mathcal{H}^s(F)$ is positive and finite.*

In Section 2 we will need the following Lemma:

**Lemma 2** (Zhou [2]). *Let $F$ be a self similar set satisfying the open set condition and $s = dim_H(F)$, then for any measurable set $U$, we have*

$$\mathcal{H}^s(F \cap U) \leq |U|^s.$$

*Sketch of proof.* We use the following fact without proving: if OSC holds, then we have

$$\mathcal{H}^s(\Lambda) = \inf \big\{ \sum_{k=1}^{\infty} |A_k|^s, \quad \text{where } \{A_k\}_{k=1}^{\infty} \text{ is a } \Lambda\text{-cover} \big\}.$$

Let $\varepsilon > 0$ be arbitrary. By definition of $\mathcal{H}^s(F \setminus U)$ there exist a set $\{B_k\}_{k=1}^{\infty}$ which is an $F \setminus U$-cover and

$$\sum_{i=1}^{\infty} |B_i|^s \leq \mathcal{H}^s(F \setminus U) + \varepsilon.$$

The set $U$ can be covered by $U$, so the set $\{U, B_1, B_2, \dots, \}$ is an $F$-cover. This implies that

$$\mathcal{H}^s(F) \leq \sum_{i=1}^{\infty} |B_i|^s + |U|^s \leq \mathcal{H}^s(F \setminus U) + |U|^s + \varepsilon.$$

Since $\mathcal{H}^s(F) = \mathcal{H}^s(F \cap U) + \mathcal{H}^s(F \setminus U)$, therefore

$$\mathcal{H}^s(F \cap U) \leq |U|^s + \varepsilon.$$

Because $\varepsilon$ was chosen arbitrary, we have

$$\mathcal{H}^s(F \cap U) \leq |U|^s.$$

□

**Notation 1.** *To compute the Hausdorff dimension and measure we often need some stationary measure, which is actually the push-down measure of a Bernoulli measure defined on a symbolic space. More precisely, let us define the $n$-cylinder of the set $F$ (with respect to $f_1, \dots, f_m$):*

$$F_{i_1, i_2, \dots, i_n} := f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_n}(F),$$

*Let assume that OSC holds. We denote* $\Sigma = \{1, 2, \ldots, m\}^{\mathbb{N}}$ *the symbolic space and we introduce the infinite product measure* $\nu := \{c_1^s, c_2^s, \ldots, c_m^s\}^{\mathbb{N}}$*. The coding of* $F$ *is given by the most natural projection* $\Pi : \Sigma \to F$*:*

$$\Pi(i_1, i_2, \ldots) = \lim_{n\to\infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(0).$$

*The most natural measure of* $F$ *is the push-down measure* $\mu = \Pi_* \nu$*. For an* $A \subset F$ *we have*

$$\mu(A) = \nu(\Pi^{-1}(A)),$$

*and for an* $n$*-cylinder of the set* $F$*:*

$$\mu(F_{i_1, i_2, \ldots, i_n}) = c_{i_1}^s c_{i_2}^s \ldots c_{i_n}^s.$$

# 2 Estimate of the Hausdorff measure of the Sierpinski Triangle

## 2.1 Introduction

In this Section we consider the Sierpinski triangle $\Lambda$. This is constructed as follows: take an equilateral triangle of side length equal to one, remove the inverted equilateral triangle of half length having the same center, then repeat this process for the remaining triangles infinitely many times as showed on Figures 3, 4.

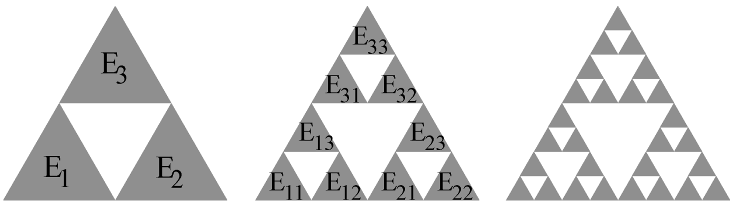


Figure 3: The triangles at the 1st, 2nd and the 3rd level

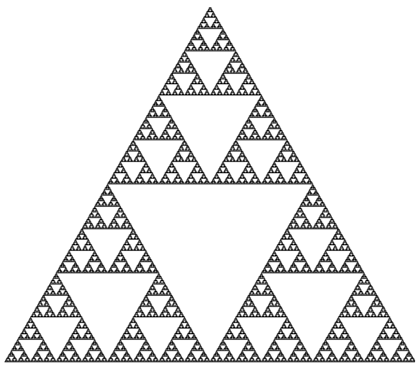

Figure 4: The Sierpinski triangle

We assumed that the diameter of the Sierpinski triangle is equal to 1. If the Sierpinski triangle is rescaled in such a way that its diameter is equal to $t$ then the lower and upper bounds should be multiplied by $t^{\log 3/\log 2}$.

The Sierpinski triangle is one of the most famous fractals, and the Hausdorff dimension and measure are the most important characteristics of a fractal sets. The Sierpinski triangle is defined by an iterated function system, which satisfies the open set condition (OSC). Thus it follows from Theorem 2 that the Hausdorff dimension is equal to $s = \log 3/\log 2$, and the $s$-dimensional Hausdorff measure $\mathcal{H}^s(\Lambda)$ of $\Lambda$ is positive and finite. Since the Sierpinski triangle has an important role in many applications, it would be desirable to get a better understanding of its size. Therefore in the last two decades there have been a considerable attention paid to the computation of the $s$-dimensional Hausdorff measure of the Sierpinski triangle:

In 1987 Marion [7] showed that 0.9508 is an upper bound. In 1997 this was improved to 0.915, and later to 0.89 by Z. Zhou [8], [9]. In 2000 Z. Zhou and Li Feng proved that $\mathcal{H}^s(\Lambda) \leq 0.83078$ in [10]. The best upper bound is 0.81794, which was given by Wang Heyu and Wang Xinghua [11] in 1999 (in Chinese) with a computer algorithm.

In 2002 B. Jia, Z. Zhou and Z. Zhu [4] showed that 0.5 is a lower bound on the $s$-dimensional Hausdorff measure of the Sierpinski triangle. In 2004 R. Houjun and W. Weiyi [12] improved it to 0.5631. Finally, in 2006 B. Jia, Z. Zhou and Z. Zhu [6] proved that 0.670432 is a lower bound.

The main result of this Section is that $\mathcal{H}^s(\Lambda) \geq 0.77$.

The difficulty comes from geometry. Recall that the $s$-dimensional Hausdorff measure of $\Lambda$ is defined by

$$\mathcal{H}^s(\Lambda) = \lim_{\delta \to 0} \inf \Big\{ \sum_{k=1}^{\infty} |A_k|^s, \text{ where } |A_k| < \delta \text{ and } \{A_k\}_{k=1}^{\infty} \text{ is a } \Lambda\text{-cover} \Big\}. \tag{2.1}$$

When we estimate the Hausdorff measure we need to understand what is the most economical (in the sense of (2.1)) system of covers. Our most natural guess for this system is the covers by the level $n$ triangles (the equilateral triangles on Figure 3). However, this system of covers would result that the $s$-dimensional Hausdorff measure of $\Lambda$ was equal to 1. On the other hand it is known that $\mathcal{H}^s(\Lambda) < 0.81794$. Therefore the best system of covers cannot possibly be the trivial one and this makes the problem difficult. To improve the existing best estimate on $\mathcal{H}^s(\Lambda)$ we use a Theorem of B. Jia. [3]. To state this Theorem we need to introduce some definitions.

It is well known (see [5]) that

$$\Lambda = \bigcup_{i=1}^{3} S_i(\Lambda), \tag{2.2}$$

where

$$S_1(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right),$$

$$S_2(x,y) = \left(\frac{1}{2} + \frac{1}{2}x, 0 + \frac{1}{2}y\right),$$

$$S_3(x,y) = \left(\frac{1}{4} + \frac{1}{2}x, \frac{\sqrt{3}}{4} + \frac{1}{2}y\right).$$

Let $E$ be the equilateral triangle of side length one with vertices: $(0,0)$, $(1,0)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Now we define the level $n$ triangles

$$E_{i_1 \dots i_n} := S_{i_1 \dots i_n}(E) = S_{i_1} \circ \cdots \circ S_{i_n}(E)$$

for all $(i_1 \dots i_n) \in \{1,2,3\}^n$. For $i_1, \dots, i_n$ we also define the level $n$-cylinders as we did in Notation 1:

$$F_{i_1 \dots i_n} := S_{i_1 \dots i_n}(\Lambda) = S_{i_1} \circ \cdots \circ S_{i_n}(\Lambda).$$

Let $\mu$ be the uniform distribution measure on the Sierpinski triangle that is for all $n$ and for all $i_1 \dots i_n$

$$\mu(E_{i_1 \dots i_n}) = \mu(F_{i_1 \dots i_n}) = \frac{1}{3^n}.$$

After B. Jia we introduce the sequence

$$a_n = \min \frac{|\bigcup_{j=1}^{k_n} \Delta_j^{(n)}|^s}{k_n / 3^n} = \min \frac{|\bigcup_{j=1}^{k_n} \Delta_j^{(n)}|^s}{\mu(\bigcup_{j=1}^{k_n} \Delta_j^{(n)})}, \tag{2.3}$$

where the minimum is taken for all non-empty sets of distinct level $n$ triangles $\left\{\Delta_1^{(n)}, \dots, \Delta_{k_n}^{(n)}\right\}$. It is easy to see that $a_n$ is non-increasing (see [3]). Further B. Jia showed ([3]) that $a_n$ is an upper bound on the Hausdorff measure of the Sierpinski triangle, and he also gave a lower bound using $a_n$ :

**Theorem 3** (B. Jia)**.** *The Hausdorff measure of the Sierpinski triangle satisfies:*

$$a_n e^{-\frac{16\sqrt{3}}{3}\cdot s\cdot\left(\frac{1}{2}\right)^n} \le \mathcal{H}^s(\Lambda) \le a_n \tag{2.4}$$

**Corollary 1.** *Theorem 3 implies that $a_n$ tends to $\mathcal{H}^s(\Lambda)$.*

Unfortunately there seems to be no way to compute $a_n$ for $n \ge 6$. B. Jia [3] calculated $a_1$ and $a_2$. We can calculate $a_3, a_4, a_5$, but by (2.4) it results only that $\mathcal{H}^s(\Lambda) > 0.54$, which is not an improvement on the already existing lower bound. So instead of this direct approach we give a lower bound on $a_n$ for every $n$. By using Corollary 1 this lower bound is also a lower bound on $\mathcal{H}^s(\Lambda)$. Using some complicated algorithm described in Subsections 2.5, 2.6 we point out that

$$a_n \ge 0.77$$

for all $n \in \mathbb{N}$. With Corollary 1 this implies that

$$\mathcal{H}^s(\Lambda) \ge 0.77.$$

We remind the reader that the best existing lower bond in the literature [6] was given in 2006: $\mathcal{H}^s(\Lambda) \ge 0.670432$.

Using the second inequality of Theorem 3, in Subsection 2.3 an upper bound is given on $\mathcal{H}^s(\Lambda)$ as follows: we provide a carefully selected collection of level 30 triangles $\left\{\Delta_1^{(30)}, \dots, \Delta_{k_{30}}^{(30)}\right\}$. This collection results an upper bound on $a_{30}$ which in return gives the upper bound $\mathcal{H}^s(\Lambda) \le 0.819161232881177$. In 1999 two Chinese mathematicians [11] published an upper bound which is better than this but their paper was published in Chinese giving in this way limited opportunity to check if their algorithm was correct.

## 2.2 Proof of Corollary 1

The proof of Theorem 3 contains many calculations. Since we only use that Corollary 1 holds, we present a proof of two Propositions of B. Jia here. The Corollary 1 is an immediately consequence of these Propositions.

**Proposition 1** (B. Jia)**.** *For $n > 0$ $\mathcal{H}^s(F) \le a_n$ holds.*

*Proof.* Fix $n > 1$ integer. Let $\{\Delta_j^{(n)}\}_{j=1}^{k_n}$ be a non-empty sets of distinct level $n$ triangles. The set $U = \cup_{j=1}^{k_n} \Delta_j^{(n)}$ is measurable, so by Lemma 2 we have

$$\mu(U)\mathcal{H}^s(E) = \mathcal{H}^s(E \cap U) \le |U|^s.$$

Therefore for all $U = \cup_{j=1}^{k_n} \Delta_j^{(n)}$ we obtain

$$\mathcal{H}^s(E) \le \frac{|U|^s}{\mu(U)}.$$

Taking the minimum of $|U|^s/\mu(U)$ over the possible sets we get $a_n$ by definition. Thus for all $n > 0$ we have

$$\mathcal{H}^s(E) \le a_n.$$

□

**Proposition 2** (B. Jia)**.** *If for $\forall n > 1$ $a_n \ge A$ holds, then $\mathcal{H}^s(F) \ge A$.*

*Proof.* For any open set $V$, let

$$G_n = \bigcup_{F_{i_1,i_2,\ldots,i_n} \subset V} F_{i_1,i_2,\ldots,i_n}.$$

Then we have $G_n \subset G_{n+1}$, and $\cup_{n=1}^{\infty} G_n = \Lambda \cap V$. The support of $\mu$ is $\Lambda$, thus

$$\mu(V) = \mu(\Lambda \cap V) = \lim_{n\to\infty} \mu \left( \bigcup_{F_{i_1,i_2,\ldots,i_n} \subset V} F_{i_1,i_2,\ldots,i_n} \right) \le \lim_{n\to\infty} \frac{|V|^s}{a_n} \le \frac{1}{A}|V|^s$$

holds. Measurable sets can be approximated with open sets, and using the last inequality and the Mass distribution principle (Lemma 1) we have $\mathcal{H}^s(F) \ge A$. □

## 2.3 Upper bound

In the definition of $a_n$ (2.3) the minimum is taken for all non-empty sets of distinct $n$-cylinder triangles. We provide a collection of $n$-cylinder triangles for all $n$, which gives an upper bound on $a_n$ by definition, and an upper bound on $\mathcal{H}^s(\Lambda)$ by Proposition 1.

Take the following 6 points:

$$\{(1/4,0),(3/4,0),(1/8,\sqrt{3}/8),(3/8,3\sqrt{3}/8),(5/8,3\sqrt{3}/8),(7/8,\sqrt{3}/8)\}.$$

Let $D_1, D_2, \dots, D_6$ be the closed discs centered at these six points with radius 0.75. We write $D := D_1 \cap D_2 \cap \cdots \cap D_6$. Take all those level $n$ triangles, which are contained in $D$ (see Figure 5 for an example). It is easy to see that the maximum distance between the chosen triangles will be exactly 0.75. Let us denote

$$c_n = \frac{0.75^{\log 3/\log 2}}{k_n/3^n},$$

where $k_n$ is the number of the chosen level $n$ triangles, which are in the region of intersection of the six discs.

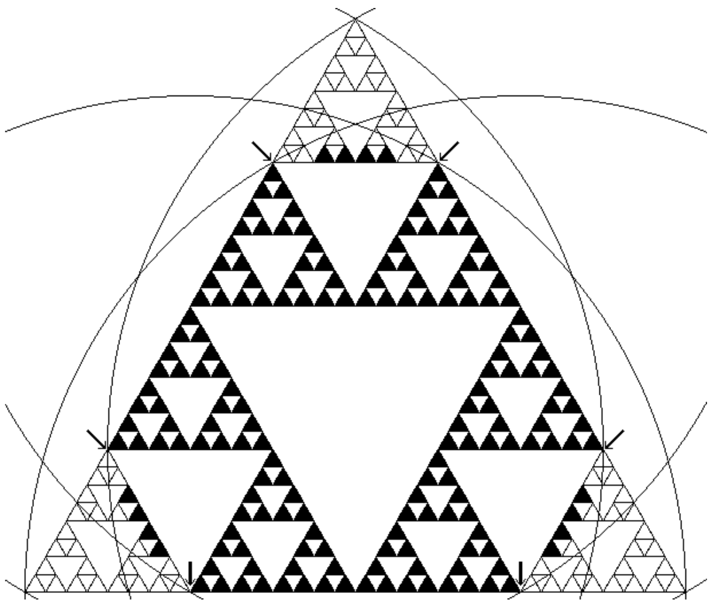

Figure 5: The black triangles are the chosen 174 level 5 triangles, so $c_5 = \frac{0.75^{\log 3/\log 2}}{174/3^5}$. Six arrows show the six given points.

The values for the $c_n$ for small $n$ are given by the following table:

| $n$ | Number of chosen triangles ($k_n$) | $k_n/3^n$ | $\frac{0.75^{\log 3/\log 2}}{k_n/3^n}$ |
|---|---|---|---|
| 2 | 6 | 0.666666666666667 | 0.950753749115186 |
| 3 | 18 | 0.666666666666667 | 0.950753749115186 |
| 4 | 54 | 0.666666666666667 | 0.950753749115186 |
| 5 | 174 | 0.716049382716049 | 0.885184525038276 |
| 6 | 546 | 0.748971193415638 | 0.846275315146484 |
| ... | | | |
| 28 | 17701192624554 | 0.773761997421774 | 0.819161234146210 |
| 29 | 53103577928148 | 0.773761998215679 | 0.819161233305724 |
| 30 | 159310733867010 | 0.773761998616697 | 0.819161232881177 |

Therefore using Theorem 3 we obtain that

$$c_{30} \geq a_{30} \geq \mathcal{H}^s(\Lambda)$$

holds. This implies:

**Theorem 4.** *The Hausdorff measure of the Sierpinski triangle is less than 0.819161232881177.*

One can show we cannot get a better upper bound on the $s$-dimension Hausdorff measure of the Sierpinski triangle than 0.819161232089868.

## 2.4 Lower bound, basic idea

For the convenience of the reader after giving the necessary definitions we are going to present a strongly simplified rough version of the idea of the algorithm. In Subsection 2.6 we will present the algorithm itself.

**Definition 2.** *Let $g > h$ be positive integers, and let $\Delta_1^{(g)}, \Delta_2^{(g)}, \dots \Delta_k^{(g)}$ be a set of distinct level $g$ triangles, $\Delta_1^{(h)}, \Delta_2^{(h)}, \dots \Delta_l^{(h)}$ be a set of distinct level $h$ triangles. We say that the set $\{\Delta_i^{(g)}\}_{i=1}^k$ is a* descendant *of the set $\{\Delta_j^{(h)}\}_{j=i}^l$ and we write $\{\Delta_j^{(h)}\}_{j=i}^l \xrightarrow{desc} \{\Delta_i^{(g)}\}_{i=1}^k$, if* both *of the following conditions hold:*

- *For all* $i \in \{1, 2, \dots k\}$ *there is a* $j$*, such that* $\Delta_i^{(g)} \subset \Delta_j^{(h)}$.
- *For all* $j \in \{1, 2, \dots l\}$ *there is at least one* $i$*, such that* $\Delta_i^{(g)} \subset \Delta_j^{(h)}$.

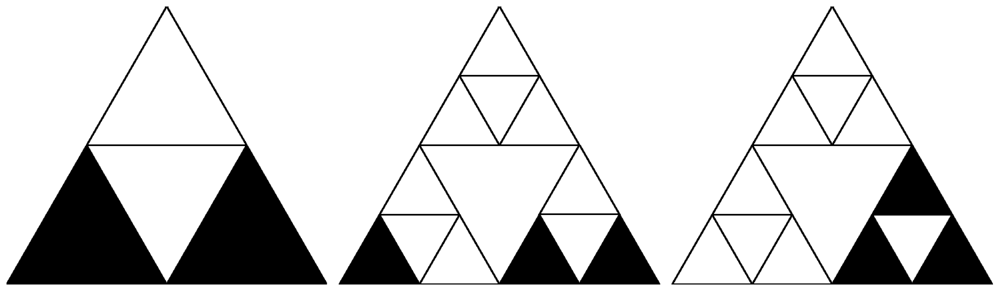

Figure 6: The set in the middle is a descendant of the left one, but the set on the right is *NOT* a descendant of the left one.

See Figure 6 for an example. This relation naturally defines a tree $\mathcal{T}$ for which the equilateral triangle $E$ is the root. The set of level $n$ nodes is equal to the set of all (non-empty) union of level $n$ triangles. A level $n$ node $\{\Delta_j^{(n)}\}_{j=1}^{l}$ is connected to a level $(n+1)$ node $\{\Delta_i^{(n+1)}\}_{i=1}^{k}$ if $\{\Delta_i^{(n+1)}\}_{i=1}^{k}$ is a descendant of $\{\Delta_j^{(n)}\}_{j=1}^{l}$. Figure 7 shows the top of the tree.

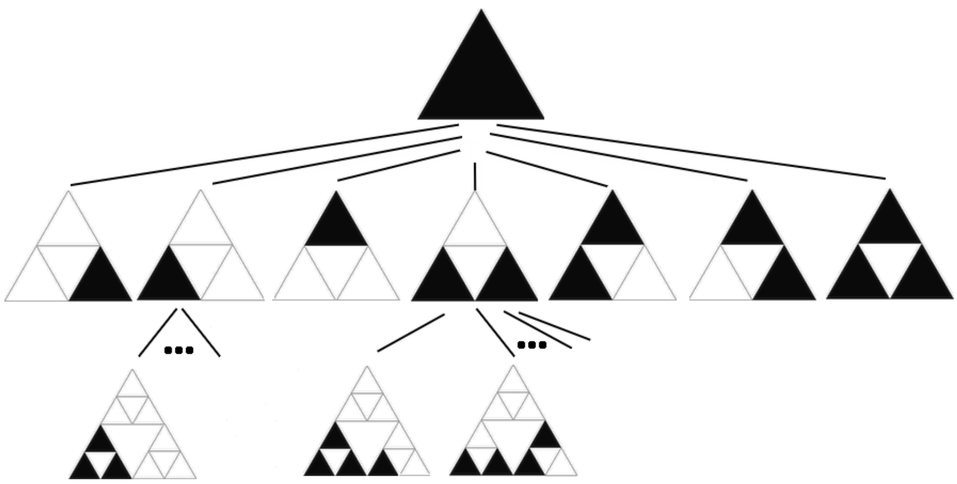

Figure 7: The top of the tree $\mathcal{T}$.

Let $v=\{\Delta_i^{(n)}\}_{i=1}^k$ be a level $n$ node. Then we write $v^0=n$ and we denote $\mathcal{T}_v$ the sub tree of $\mathcal{T}$ having $v$ as root. ($\mathcal{T}_v$ consists of $v$ and all those nodes $w$, which are descendant of $v$.) Let $E_v:=\cup_{i=1}^k\Delta_i^{(n)}$.

We define

$$a_v:=\frac{|E_v|^s}{k/3^n}=\frac{|E_v|}{\mu(E_v)}. \tag{2.5}$$

Our purpose is to give a lower bound on $a_n$ (defined in (2.3)) for sufficiently large $n$, so we obtain a lower bound on its limit, $\mathcal{H}^s(\Lambda)$. It comes directly from the definitions that

$$a_n=\min_{v^\circ=n}a_v. \tag{2.6}$$

By using $a_n\downarrow\mathcal{H}^s(\Lambda)$ and taking infimum on both sides on $n$ we obtain

$$\inf_{v\in\mathcal{T}}a_v=\lim_{n\to\infty}a_n=\mathcal{H}^s(\Lambda). \tag{2.7}$$

Let $v=\{\Delta_i^{(n)}\}_{i=1}^k$. We write

$$b_v:=\max_{1\le i,j\le k}\ \min_{\mathbf{x}\in\Delta_i^{(n)},\mathbf{y}\in\Delta_j^{(n)}}\frac{|\mathbf{x}-\mathbf{y}|^s}{k/3^n}. \tag{2.8}$$

Observe that for these $\mathbf{x}$, $\mathbf{y}$ we have

$$|\mathbf{x}-\mathbf{y}|=\max_{1\le i,j\le k}\operatorname{dist}(\Delta_i,\Delta_j).$$

**Lemma 3.** *The value $b_v$ is a lower bound for $a_w$ whenever $v\xrightarrow{desc}w$ holds. Namely,*

$$b_v\le\inf_{w\in\mathcal{T}_v}a_w.$$

*Proof.* For $v=\{\Delta_i^{(n)}\}_{i=1}^k$ let $w=\{\Delta_t^{(g)}\}_{t=1}^l\in\mathcal{T}_v$ be arbitrary. To give a lower bound on $a_w$ first we give a lower bound on the diameter of $E_w$, then we give an upper bound on $\mu(E_w)$. We consider $\Delta_i^{(n)}$ and $\Delta_j^{(n)}$ for some $1\le i,j\le k$. $w$ is a descendant of $v$, so $E_w\cap\Delta_i^{(n)}$ and $E_w\cap\Delta_j^{(n)}$ are non-empty (see Figure 8 for example). Thus the diameter of $E_w$ is at least

$$|E_w|>\min_{\mathbf{x}\in\Delta_i^{(n)},\mathbf{y}\in\Delta_j^{(n)}}|\mathbf{x}-\mathbf{y}|=\operatorname{dist}(\Delta_i^{(n)},\Delta_j^{(n)}).$$

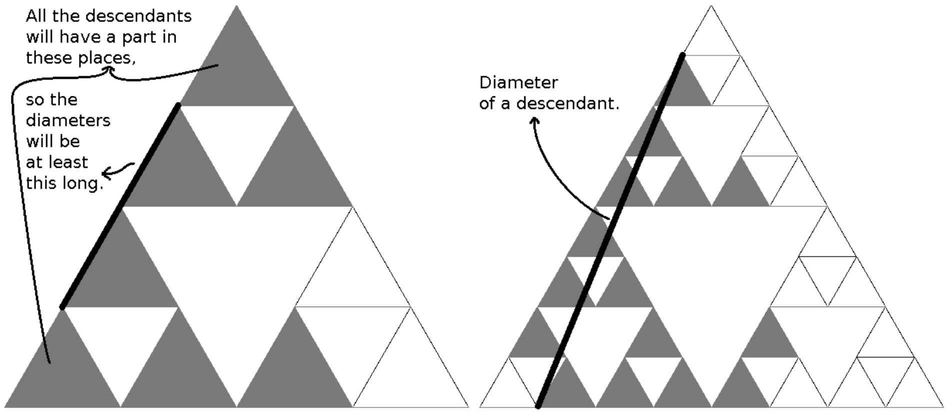


Figure 8: The set on the right is a descendant of the left one.

This inequality holds for all $1 \le i, j \le k$, so we can take the maximum over these pairs:

$$|E_w| > \max_{1 \le i,j \le k} \min_{\mathbf{x} \in \Delta_i^{(n)}, \mathbf{y} \in \Delta_j^{(n)}} |\mathbf{x} - \mathbf{y}|.$$

Because $v \xrightarrow{\text{desc}} w$, we have $E_w \subset E_v$. This yields

$$l/3^g = \mu(E_w) \le \mu(E_v) = k/3^n,$$

therefore

$$b_v = \max_{1 \le i,j \le k} \min_{\mathbf{x} \in \Delta_i^{(n)}, \mathbf{y} \in \Delta_j^{(n)}} \frac{|\mathbf{x} - \mathbf{y}|^s}{k/3^n} \le \frac{|E_w|^s}{l/3^g} = a_w.$$

□

First of all we give a lower bound only on a subtree defined by a finite set of nodes $A$. We define the set $\mathcal{T}_A$ as follows: $v \in \mathcal{T}_A$ if $v \in A$, or $v$ is a descendant of a node $w$, which is in $A$. Namely,

$$\mathcal{T}_A = \bigcup_{w \in A} \mathcal{T}_w.$$

We write

$$B_A = \min_{v \in A} b_v.$$

We can apply the previous Lemma for every node in the set $A$, thus we have

$$B_A \leq \inf_{v \in \mathcal{T}_A} a_v. \tag{2.9}$$

We are going to apply this inequality for the so called cross-sections. These are some subsets $C \subset \mathcal{T}$ of the nodes such that the lower bound on $a_v, v \in \mathcal{T}_C$ is a lower on the Hausdorff measure of the Sierpinski triangle. To make this definition precise first we define the set of the parents of $C$ called $P_C$ as

$$P_C = \{v \mid v \stackrel{\text{desc}}{\longrightarrow} w, w \in C\},$$

namely $P_C$ is the set of nodes, which have a descendant in $C$.

**Definition 3.** *We call a finite set $C \subset \mathcal{T}$ a* cross-section*, if there exists a function $\varphi$, $\varphi : \mathcal{T} \setminus (\mathcal{T}_C \cup P_C) \to \mathcal{T}_C \cup P_C$ such that for every node $v \in \mathcal{T} \setminus (\mathcal{T}_C \cup P_C)$ we have*

$$a_{\varphi(v)} \leq a_v,$$

*and*

$$\mu(E_{\varphi(v)}) \geq 3\mu(E_v).$$

Let $v$ be a level $n$ node. For a $k > n$ we write $\Gamma_k(v)$ for that level $k$ descendant of $v$ which has maximal $\mu$ measure. That is

$$\Gamma_k(v) = \{E_{i_1,\dots i_k} \mid E_{i_1,\dots i_k} \subset E_v\}.$$

See Figure 9. We remark that

$$a_v = a_{\Gamma_k(v)}. \tag{2.10}$$

Namely, $|E_v| = |E_{\Gamma_k(v)}|$ and $\mu(E_v) = \mu(E_{\Gamma_k(v)})$ hold.

**Fact 1.** *Let $H$ be an arbitrary subset of $\mathcal{T}$. Then for every $k \geq 0$ we have*

$$\inf_{v \in \mathcal{T}_H} a_v = \inf_{v \in \mathcal{T}_H \bigcap \{w \mid w^\circ \geq k\}} a_v. \tag{2.11}$$

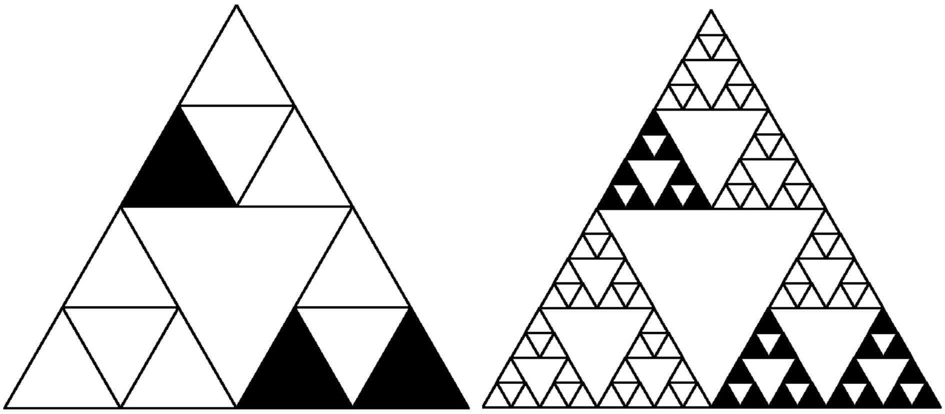

Figure 9: The node $v$ on the left is a level 2 node, the node on the right is the node $\Gamma_4(v)$.

*Proof.* It is enough to verify that

$$\inf_{v\in\mathcal{T}_H} a_v \geq \inf_{v\in\mathcal{T}_H\bigcap\{w\,|\,w^\circ\geq k\}} a_v$$

holds. To do so, let $u \in \mathcal{T}_H \setminus \{w\,|\,w^\circ \geq k\}$ be arbitrary. By using $\Gamma_k(u) \in \mathcal{T}_H \bigcap \{w\,|\,w^\circ \geq k\}$ and (2.10) we have

$$a_u = a_{\Gamma_k(u)} \geq \inf_{v\in\mathcal{T}_H\bigcap\{w\,|\,w^\circ\geq k\}} a_v,$$

which completes the proof. □

For a cross-section $C$ we define

$$B_C = \min_{v\in C} b_v.$$

**Lemma 4.** *For every cross-section $C$ we have*

$$\inf_{v\in\mathcal{T}} a_v = \inf_{v\in\mathcal{T}_C} a_v, \tag{2.12}$$

*and*

$$B_C \leq \inf_n a_n = \mathcal{H}^s(\Lambda). \tag{2.13}$$

*Proof.* It is easy to see that (2.13) is an immediately consequence of (2.12). Namely, using (2.7) and (2.9) we have

$$B_C \le \inf_{v \in \mathcal{T}_C} a_v = \inf_{v \in \mathcal{T}} a_v = \inf_n a_n = \mathcal{H}^s(\Lambda)$$

Let $M_C$ be the maximum level of the nodes which are contained in the set $C$. We define

$$K_{M_C} = \left\{ v \,|\, v^0 \ge M_C \right\}.$$

Using Fact 1 we have

$$\inf_{v \in \mathcal{T}} a_v = \inf_{v \in \mathcal{T} \cap K_{M_C}} a_v, \quad \text{and} \quad \inf_{v \in \mathcal{T}_C} a_v = \inf_{v \in \mathcal{T}_C \cap K_{M_C}} a_v.$$

Thus to prove (2.12) it is enough to verify that

$$\inf_{v \in \mathcal{T} \cap K_{M_C}} a_v = \inf_{v \in \mathcal{T}_C \cap K_{M_C}} a_v \tag{2.14}$$

holds.

We fix a $v \in K_{M_C} \setminus \mathcal{T}_C$. To verify (2.14) we will show that there exists a node $t \in K_{M_C} \cap \mathcal{T}_C$ such that

$$a_v \ge a_t \tag{2.15}$$

holds. Since $K_{M_C} \cap P_C = \emptyset$, thus $v \in \mathcal{T} \setminus (\mathcal{T}_C \cup P_C)$. $C$ is a cross-section, by definition there exists $\varphi$ such that $\varphi(v) \in \mathcal{T}_C \cup P_C$. If $\varphi(v) \in K_{M_C}$, then $\varphi(v) \in \mathcal{T}_C$ as well, so (2.15) follows from choosing $t = \varphi(v)$ and by using

$$a_v \ge a_{\varphi(v)}.$$

If $\varphi(v) \in P_C$, then let us consider $\Gamma_{M_C}(\varphi(v))$. If $\Gamma_{M_C}(\varphi(v)) \in \mathcal{T}_C$ then $t := \Gamma_{M_C}(\varphi(v))$ yields (2.15). If $\Gamma_{M_C}(\varphi(v)) \notin \mathcal{T}_C$ then by (2.10) and by the definition of $\varphi$ and $\Gamma_{M_C}$ we have

$$\begin{aligned} a_v \ge a_{\varphi(v)} &= a_{\Gamma_{M_C}(\varphi(v))} \\ \mu(E_{\Gamma_{M_C}(\varphi(v))}) &\ge 3\mu(E_v) \\ \Gamma_{M_C}(\varphi(v)) &\in K_{M_C} \setminus \mathcal{T}_C \end{aligned} \tag{2.16}$$

So, we can repeat the same for the node $w_1 := \Gamma_{M_C}(\varphi(v))$ instead of $v$. If $\Gamma_{M_C}(\varphi(w_1)) \in \mathcal{T}_C$ then we are ready as we saw above. If not then (2.16) holds for $w_1$ instead of $v$. Note that this follows that $0 < 9\mu(E_v) \leq \mu(E_{\Gamma_{M_C}(\varphi(w_1))}) \leq 1$. This shows that there must exists a finite $N$ such that $\Gamma_{M_C}(\varphi(w_N)) \in \mathcal{T}_C$, where $w_{k+1} := \Gamma_{M_C}(\varphi(w_k))$. This completes the proof of (2.15)

□

Take the following set:

$$C_0 = \{v \,|\, v^0 = 2, v \in \mathcal{T}_{\{E_1,E_2\}} \cup \mathcal{T}_{\{E_1,E_2\}} \cup \mathcal{T}_{\{E_1,E_3\}} \cup \mathcal{T}_{\{E_2,E_3\}},$$
$$v \neq \{E_{1,2}, E_{2,1}\}, v \neq \{E_{1,3}, E_{3,1}\}, v \neq \{E_{2,3}, E_{3,2}\}\} \cup \{\{E_1, E_2, E_3\}\}. \quad (2.17)$$

See Figure 3 for labelling. There are $7 \cdot 7 = 49$ descendants of the node $\{E_1, E_2\}$ at level 2. Counting the same for $\{E_1, E_3\}$ and $\{E_2, E_3\}$ we have $3 \cdot 49 = 147$ nodes. Let us remove the nodes $\{E_{1,2}, E_{2,1}\}$,$\{E_{1,3}, E_{3,1}\}$,$\{E_{2,3}, E_{2,3}\}$, and take the node $\{E_1, E_2, E_3\}$, so we get the set $C_0$. Thus $C_0$ consists of $147 - 3 + 1 = 145$ nodes.

**Proposition 3.** *The set $C_0$ is a cross-section.*

*Proof.* Note that

$$\mathcal{T} \setminus (\mathcal{T}_{C_0} \cup P_{C_0}) = \mathcal{T}_{E_1} \cup \mathcal{T}_{E_2} \cup \mathcal{T}_{E_3} \cup \mathcal{T}_{\{E_{1,2},E_{2,1}\}} \cup \mathcal{T}_{\{E_{1,3},E_{3,1}\}} \cup \mathcal{T}_{\{E_{2,3},E_{2,3}\}}.$$

To define the function $\varphi$ in the Definition 3, first we define an auxiliary function $\psi : \mathcal{T} \setminus (\mathcal{T}_{C_0} \cup P_{C_0}) \to \mathcal{T}$. (See Figure 10 and Figure 11.)

- For $v \in \mathcal{T}_{E_j}$, where $j = 1, 2, 3$ let

$$\psi(v) := \{E_{i_2,i_3,\ldots,i_n} \,|\, E_{j,i_2,i_3,\ldots,i_n} \in v\},$$

- for $v \in \mathcal{T}_{E_{1,2},E_{2,1}}$ let

$$\psi(v) := \{E_{1,i_1,i_2,\ldots,i_n} \,|\, E_{1,2,i_1,i_2,\ldots,i_n} \in v\} \cup \{E_{2,i_1,i_2,\ldots,i_n} \,|\, E_{2,1,i_1,i_2,\ldots,i_n} \in v\},$$

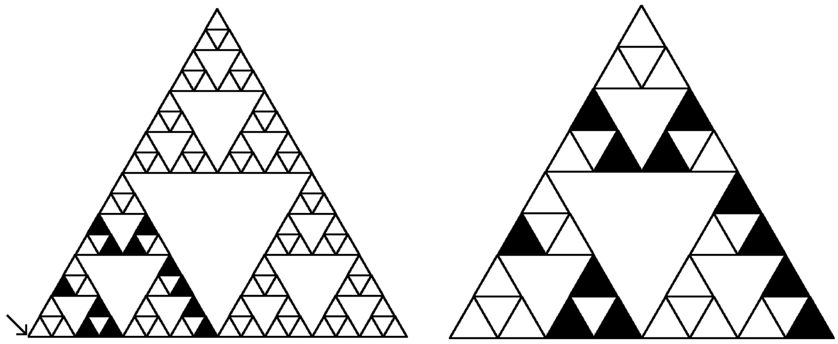

Figure 10: The node $v$ on the left is a descendant of the node $\{E_1\}$, the node on the right is $\psi(v)$. The arrow shows the fix point of rescaling.

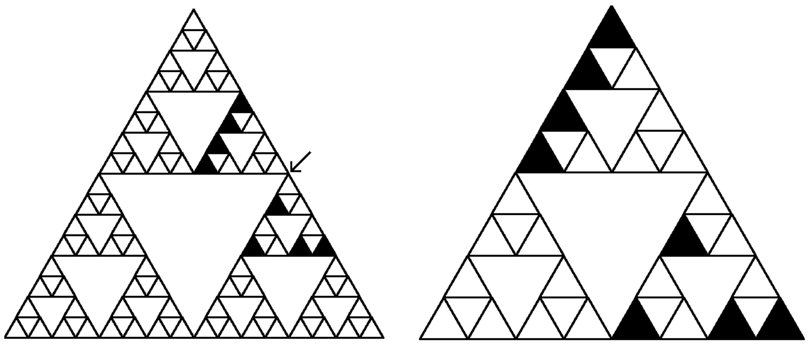

Figure 11: The node $v$ on the left is a descendant of the node $\{E_{2,3}, E_{3,2}\}$, the node on the right is $\psi(v)$. The arrow shows the fix point of rescaling.

- for $v \in \mathcal{T}_{E_{1,3},E_{3,1}}$ let

$$\psi(v) := \{E_{1,i_1,i_2,\ldots,i_n} \mid E_{1,3,i_1,i_2,\ldots,i_n} \in v\} \cup \{E_{3,i_1,i_2,\ldots,i_n} \mid E_{3,1,i_1,i_2,\ldots,i_n} \in v\},$$

- for $v \in \mathcal{T}_{E_{2,3},E_{3,2}}$ let

$$\psi(v) := \{E_{2,i_1,i_2,\ldots,i_n} \mid E_{2,3,i_1,i_2,\ldots,i_n} \in v\} \cup \{E_{3,i_1,i_2,\ldots,i_n} \mid E_{3,1,i_1,i_2,\ldots,i_n} \in v\}.$$

Clearly,

$$|E_{\psi(v)}| = 2|E_v|$$

and

$$\mu(E_{\psi(v)}) = 3\mu(E_v).$$

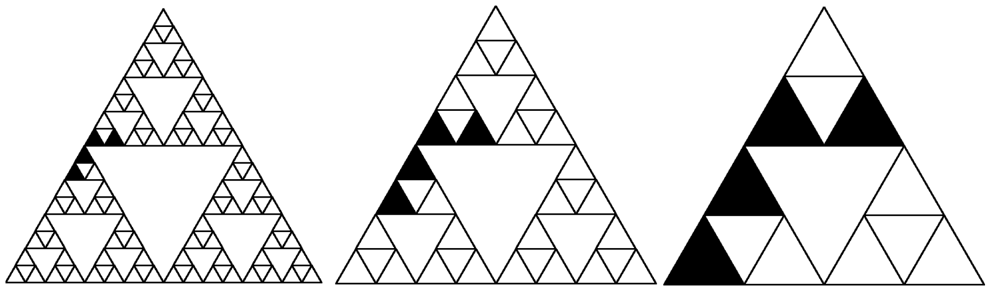

Figure 12: The node $v$ on the left is a descendant of the node $\{E_{1,3}, E_{3,1}\}$. The node in the middle is $\Psi(v)$. The node on the right is $\varphi(v) = \Psi(\Psi(v))$.

Thus we have

$$a_{\psi(v)} = a_v.$$

This follows that for every $v \in \mathcal{T} \setminus (\mathcal{T}_{C_0} \cup P_{C_0})$ there exists an $N$ such that $C_0$ is a cross-section with the function

$$\varphi(v) := \psi^N(v) = \underbrace{\psi \circ \cdots \circ \psi}_{N}(v) \in \mathcal{T}_{C_0} \cup P_{C_0}.$$

(See Figure 12.)

□

For the convenience of the reader we present a simplified algorithm for choosing cross-sections $C_n$ in the next Subsection. Finally, in Subsection 2.6 we improve this algorithm significantly by using symmetries and a convexity argument.

## 2.5 Algorithm

Our purpose is to choose cross-sections $C_n$ in such a way that $B_{C_n}$ gets as large as possible, but a computer can check it in acceptable length of time. It is a natural idea to choose a starting cross-section, and modify it in hope to get a better lower bound. For $n = 0$ take the set $C_0$ defined in (2.17). For every $n$ in the $n$-th step find a node $v \in C_n$ where

$$b_v = \min_{w \in C_n} b_w.$$

To obtain $C_{n+1}$ from $C_n$ we throw away $v$ from $C_n$ and we add to $C_n$ all the next level descendants of $v$. It follows from the definition of $b_v$ that $B_{C_{n+1}} \geq B_{C_n}$.

The following algorithm consists of three steps. It gives a lower bound on the $s$-dimensional Hausdorff measure of the Sierpinski triangle every time it reaches Step 2. It will run forever, but during its running it will give better and better lower bounds.

**Algorithm 1.**

*Step 1. Start with the set $C_0$ from the previous Subsection. Let $n := 0$.*

*Step 2. Find $\min_{v \in C_n} b_v$. Below we prove that $C_n$ is a cross-section. So, it follows from Lemma 4 that we have*

$$\min_{v \in C_n} b_v \leq \mathcal{H}^s(\Lambda). \tag{2.18}$$

*Step 3. Find a node $v \in C_n$ for which $b_v = \min_{w \in C_n} b_w$ (if such a $v$ is not unique, then choose any of them). Let us suppose $v$ is level $m$ node. We define $S_n$ as the set of all of those level $m+1$ nodes, which are descendants of the node $v$. That is*

$$S_n := \{w \mid w^0 = m+1,\, v \overset{desc}{\longrightarrow} w\},$$

*Let*

$$C_{n+1} := S_n \cup C_n \setminus \{v\}.$$

*Increase $n$ by 1. Go to Step 2.*

Above we used the fact that $C_n$ is a cross-section for every $n$. This is so because we have already seen that $C_0$ is a cross-section and

$$\mathcal{T}_{C_{n+1}} \cup P_{C_{n+1}} = \mathcal{T}_{C_n} \cup P_{C_n}$$

holds for all $n$.

## 2.6 Making the algorithm faster

Our aim here is to improve the algorithm presented in the previous Subsection. To do so, for every $n$ we define a cross section $Q_n$. Namely, let $Q_0 := C_0$. Assume that $Q_n$ is already defined. To define $Q_{n+1}$ first we define a certain set of nodes $D_n \subset Q_n$ as it is detailed later in the Subsection. It is important that the set $D_n$ is much smaller than $Q_n$. We choose a $v \in D_n$ for which

$$b_v = \min_{w \in D_n} b_w. \tag{2.19}$$

Then the special choice of $D_n$ will guarantee that

$$b_v \leq \mathcal{H}^s(\Lambda). \tag{2.20}$$

To get $Q_{n+1}$ we replace $v$ (defined in (2.19)) with its next level descendants.

To define $D_n$ we need to introduce the notion of the convexity of a node. We remark that $D_n$ will consist only of convex nodes.

**Definition 4.** *Let $v$ be a level $n$ node. We write*

$$conv(v) = \{E_{i_1,i_2,\dots,i_n} \mid E_{i_1,i_2,\dots,i_n} \text{ is contained in the convex hull of } E_v\}.$$

See Figure 13 for an example. We call a node $v$ *convex*, if $v = conv(v)$, otherwise we call it non-convex.

**Lemma 5.** *Let $v$ be a non-convex level $n$ node. If $v'$ is a level $m$ descendant of the node $v$, and $\Theta \in conv(v) \setminus v$ is a level $n$ triangle, then the closed convex hull of $E_{v'}$ intersects $\Theta$.*

*Proof.* We assume that $v = \{\Delta_i\}_{i=1}^k$. By definition of convexity we have

$$\Theta \subset \left\{ \sum_{i=1}^k \alpha_i \mathbf{x_i} \mid \mathbf{x_i} \in \Delta_i, \alpha_i \geq 0, \sum_{i=1}^k \alpha_i = 1 \right\}.$$

For $i = 1, 2, \dots, k$ let $\mathbf{t_i} \in \Delta_i$ be arbitrary points. To verify the assertion of the Lemma it is enough to show that

$$\Theta \cap \left\{ \sum_{i=1}^k \alpha_i \mathbf{t_i} \mid \alpha_i \geq 0, \sum_{i=1}^k \alpha_i = 1 \right\} \neq \emptyset \tag{2.21}$$

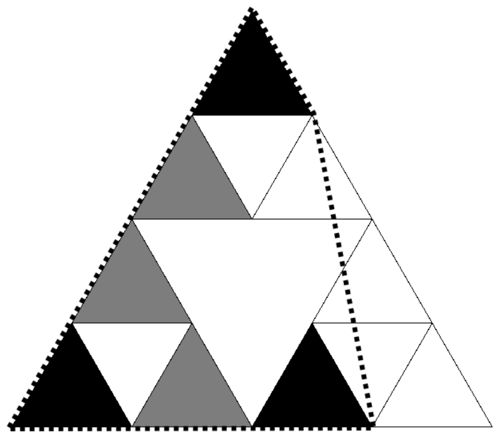

Figure 13: The node $v$ consists of the black triangles. The convex hull of $E_v$ is showed with dashed lines. The black and the gray triangles together form the node $conv(v)$.

holds for every choice of $\mathbf{t_1}, \ldots, \mathbf{t_k}$. We prove it by contradiction. Let us suppose there exist $\mathbf{t_1}, \ldots, \mathbf{t_k}$ such that (2.21) does not hold. Then there exists a line $e$, such that $e$ separates $\Theta$ and the convex hull of $\mathbf{t_1}, \ldots, \mathbf{t_k}$. Let $\mathbf{a}$ be one of the normal unit vectors of $e$. Put $r := \mathbf{z} \cdot \mathbf{a}$, where $\mathbf{z} \in e$ arbitrary, and dot means the scalar product. Let us define

$$q := \max_{\mathbf{x},\mathbf{y} \in \Theta} (\mathbf{x} - \mathbf{y}) \cdot \mathbf{a}. \tag{2.22}$$

Without loss of generality we may assume that

$$\max \left\{ \left( \sum_{i=1}^{k} \alpha_i \mathbf{t_i} \right) \cdot \mathbf{a} \,|\, \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1 \right\} < r$$

and

$$\min_{\mathbf{x} \in \Theta} \mathbf{x} \cdot \mathbf{a} > r$$

hold, otherwise take $-\mathbf{a}$ instead of $\mathbf{a}$. The last inequality and (2.22) implies that

$$\max_{\mathbf{x} \in \Theta} \mathbf{x} \cdot \mathbf{a} > q + r,$$

let us denote $\mathbf{x_0}$ where the maximum is attained. Since $\Theta \in conv(v) \setminus v$, thus for $i = 1, 2, \ldots, k$ there exist $\mathbf{u_i} \in \Delta_i$ and $\beta_i \geq 0$ such that $\sum_{i=1}^{k} \beta_i = 1$ and

$$\mathbf{x_0} = \sum_{i=1}^{k} \beta_i \mathbf{u_i}$$

hold. Using the fact all level $n$ triangles are translations of each othes and by using (2.22), for $i = 1, 2, \ldots, k$ we have

$$(\mathbf{u_i} - \mathbf{t_i}) \cdot \mathbf{a} \leq q.$$

Observe that

$$q + r < \mathbf{x_0} \cdot \mathbf{a} = \left( \sum_{i=1}^{k} \beta_i \mathbf{t_i} \right) \cdot \mathbf{a} + \left( \sum_{i=1}^{k} \beta_i (\mathbf{u_i} - \mathbf{t_i}) \right) \cdot \mathbf{a} < r + q,$$

which is a contradiction, and completes the proof.

□

The next Lemma shows that for any descendant $w$ of a non-convex node $v$ the value of $a_w$ can be at most slightly bigger than some of the same level descendants of $conv(v)$. We will need this to verify (2.20).

**Lemma 6.** *Let $v$ be a non-convex level $n$ node and let $m > n$ be arbitrary. If $v \xrightarrow{desc} v'$, $v'^{\circ} = m$, then there exists a node $w'$, $w'^{\circ} = m$, $conv(v) \xrightarrow{desc} w'$, such that*

$$|E_{w'}| \leq |E_{v'}| + \frac{2}{2^m} \tag{2.23}$$

*and*

$$E_{v'} \subset E_{w'}$$

*hold.*

*Proof.* We write $v = \{\Delta_i\}_{i=1}^{k}$, and $conv(v) \setminus v = \{\Theta_j\}_{j=1}^{l}$. Let us define the polygon $H$ as the closed convex hull of $E_{v'}$. We proved in Lemma 5 that $\Theta_j$ intersects $H$ for $1 \leq j \leq l$. If the polygon $H$ intersects a triangle $\Theta_j$, then for all $j$ there exists at least one level $m$ triangle $\Theta'_j \subset \Theta_j$, such that $\Theta'_j$

intersects $H$ as well. We write $\mathbf{t_j}$ for a point where $H$ intersects the triangle $\Theta'_j$. Let

$$w' = v' \cup \{\Theta'_j\}_{j=1}^{l}.$$

Let $\mathbf{q_0}, \mathbf{w_0} \in E_{w'}$ be some points where the maximum

$$|E_{w'}| = \max_{\mathbf{q},\mathbf{w} \in E_{w'}} |\mathbf{q} - \mathbf{w}|,$$

is attained. If $\mathbf{q_0}, \mathbf{w_0} \in E_{v'}$ then we have $|E_{w'}| = |\mathbf{q_0} - \mathbf{w_0}| \leq |E_{v'}|$, thus the inequality (2.23) holds. If one of them is not in $E_{v'}$, let say $\mathbf{q_0}$, then there exists a $j$ such that $\mathbf{q_0} \in \Theta'_j$ and $\mathbf{w_0} \in E_{v'}$. Using triangle inequality we have

$$|E_{w'}| = |\mathbf{q_0} - \mathbf{w_0}| \leq |\mathbf{q_0} - \mathbf{t_j}| + |\mathbf{t_j} - \mathbf{w_0}| \leq |\Theta'_j| + |E_{v'}| = \frac{1}{2^m} + |E_{v'}|$$

because $\mathbf{q_0}, \mathbf{t_j} \in \Theta'_j$. If both $\mathbf{q_0}$ and $\mathbf{w_0}$ are not in $E_{v'}$, then using triangle inequality twice we have $|E_{w'}| \leq \frac{2}{2^m} + |E_{v'}|$.

□

The following Lemma helps us to reduce the number of cases to be checked in an analogous way to the previous Lemma.

**Lemma 7.** *Let $v = \{\Delta_i\}_{i=1}^{k}$ be a level $n$ node, $\Delta = E_{i_1,\dots,i_n}$ be a level $n$ triangle such that $\Delta \notin v$. Further, let $x$ be one of the vertices of the triangle $\Delta$. We write $D(x, r)$ for the closed disc centered at $x$ with radius $r$. If*

$$E_v \subseteq D(x, \max_{1 \leq i,j \leq k} dist(\Delta_i, \Delta_j))$$

*holds then for all level $m$ descendant $v'$ of the node $v$ there exists a level $m$ triangle $\Delta' \subset \Delta$, such that*

$$|E_{v'} \cup \Delta'| \leq |E_{v'}| + \frac{1}{2^m}$$

*Proof.* Let $\Delta'$ be that level $m$ triangle, which has $x$ as one of its vertices, and $\Delta' \subset \Delta$. As we saw in the proof of Lemma 3, $\max_{1 \leq i,j \leq k} \mathrm{dist}(\Delta_i, \Delta_j))$ is a lower bound on $|E_{v'}|$. Furthermore,

$$|E_{v'} \cup \Delta'| = \max_{\mathbf{q},\mathbf{w} \in E_{v'} \cup \Delta'} |\mathbf{q} - \mathbf{w}|.$$

Let $\mathbf{q_0}, \mathbf{w_0}$ be those points where this maximum is attained. Either $\mathbf{q_0}, \mathbf{w_0} \in E_{v'}$, or one of the points, let us say $\mathbf{q_0} \in \Delta'$, and $\mathbf{w_0} \in E_{v'}$. By using $|\Delta'| = 1/2^m$ and triangle inequality both cases implies the statement.

□

The following Theorem will show (with $D = D_n$) how the sequence of sets $\{D_n\}_{n=0}^{\infty}$ mentioned in the introduction of this Subsection gives us a lower bound on the Hausdorff measure $\mathcal{H}^s(\Lambda)$. Then after this Theorem we will construct $\{D_n\}_{n=0}^{\infty}$.

**Theorem 5.** *Let $Q \subset \mathcal{T}$ be a cross-section. We choose an arbitrary $D \subset Q$ which satisfies the following assumption:*

*For all $v \in Q \setminus D$ there exists a node $w \in \mathcal{T}_Q \cup P_Q$, such that*

- $w^\circ = v^\circ$,
- $E_v \subsetneq E_w$,
- *for* $v \overset{desc}{\longrightarrow} v'$ *there exists a* $w \overset{desc}{\longrightarrow} w'$ *with* $v'^\circ = w'^\circ =: m$ *such that*

$$E_{v'} \subset E_{w'} \text{ and } |E_{w'}| \leq |E_{v'}| + \frac{2}{2^m}.$$

*Then*

$$B_D = \min_{t \in D} b_t \leq \mathcal{H}^s(\Lambda).$$

*Proof.* Let us denote the finitely many elements of $Q \setminus D$ by:

$$Q \setminus D = \{v_1, v_2, \ldots, v_k\}.$$

Let $\varepsilon > 0$ be arbitrary. Choose an $M > \max_{v \in Q} v^\circ$ which also satisfies

$$\left(1 + \frac{2}{\delta 2^M}\right)^{k \cdot s} < 1 + \varepsilon,$$

where

$$\delta := \inf_{v \in \tau_{C_0}} |E_v|.$$

We remind the reader that $C_0$ was defined in (2.17). It is easy to see that $\delta > 0$. Recall that $K_M = \{v \mid v^0 \geq M\}$. Fix an arbitrary $v'_0 \in K_M \cap \mathcal{T}_Q$. To prove the assertion of the Theorem, it is enough to show that

$$a_{v'_0} \geq \frac{B_D}{1+\varepsilon}. \tag{2.24}$$

Namely,

$$\mathcal{H}^s(\Lambda) = \inf_{v \in \mathcal{T}} a_v = \inf_{v \in \mathcal{T}_Q} a_v = \inf_{v \in K_M \cap \mathcal{T}_Q} a_v \geq \frac{B_D}{1+\varepsilon},$$

here we used first (2.7) then (2.12) and at the third equality we used Fact 1.

Now we define by mathematical induction a finite (at least one and at most $k$ elements) sequence of nodes

$$v'_0, v'_1, \dots, v'_l,$$

where $v'_l \in \mathcal{T}_D$ and $v'_0, v'_1, \dots, v'_{l-1} \in \mathcal{T}_Q \setminus \mathcal{T}_D$. Namely, assume that we have already defined $v'_n$ for an $n \geq 0$. If $v'_n \in \mathcal{T}_D$, then let $v'_l = v'_n$ be the last element of the sequence. Otherwise $v'_n \in \mathcal{T}_Q \setminus \mathcal{T}_D$, so there exists a node $v_{i_n} \in \{v_1, v_2, \dots, v_k\}$ such that $v_{i_n} \overset{\text{desc}}{\longrightarrow} v'_n$. By the assumptions of the Theorem there exist nodes $w_{i_n}$ and $w'_{i_n}$, such that $w_{i_n} \overset{\text{desc}}{\longrightarrow} w'_{i_n}$, $v'^{\circ}_n = w'^{\circ}_{i_n}$, $E_{v'_n} \subset E_{w'_{i_n}}$ and $|E_{w'_{i_n}}| \leq |E_{v'_n}| + \frac{2}{2^m}$. Now we let $v'_{n+1} := w'_{i_n}$. From this it immediately follows that

$$\frac{|E_{v'_n}|}{|E_{v'_{n+1}}|} \geq \frac{|E_{v'_n}|}{|E_{v'_n}| + \frac{2}{2^M}} \geq \frac{\delta}{\delta + \frac{2}{2^M}} = \frac{1}{1 + \frac{2}{\delta 2^M}}.$$

So, using that $\mu\left(E_{v'_n}\right) \leq \mu\left(E_{v'_{n+1}}\right)$ we obtain that

$$a_{v'_n} = \frac{|E_{v'_n}|^s}{\mu(E_{v'_n})} \geq \frac{|E_{v'_{n+1}}|^s}{\mu(E_{v'_{n+1}})(1 + \frac{2}{\delta 2^M})^s} = \frac{a_{v'_{n+1}}}{(1 + \frac{2}{\delta 2^M})^s}. \tag{2.25}$$

Note that for $n = 0, 1, 2, \dots, l-1$ we have $E_{v_{i_n}} \subsetneq E_{w_{i_n}}$, $v_{i_n} \overset{\text{desc}}{\longrightarrow} v'_n$, $w_{i_n} \overset{\text{desc}}{\longrightarrow} v'_{n+1}$ and $E_{v'_1} \subsetneq E_{v'_2} \subsetneq \cdots \subsetneq E_{v'_l}$. This yields that $v_{i_0}, v_{i_1}, \dots, v_{i_{l-1}}$ are all different. This follows that $l \leq k$ holds and $v'_l \in \mathcal{T}_D$. By applying (2.25) $l$ times we get

$$a_{v'_0} \geq a_{v'_l} \bigg/ \left(1 + \frac{2}{\delta 2^M}\right)^{l \cdot s} \geq B_D \bigg/ \left(1 + \frac{2}{\delta 2^M}\right)^{k \cdot s} \geq \frac{B_D}{1+\varepsilon},$$

which gives (2.24) and completes the proof. □

In the following we present the Algorithm. We remark that the starting set can be reduced by using symmetry. We will consider it at the end of this Subsection.

**Algorithm 2.**

*Step 1.* *Let* $Q_0 := C_0$ *(which was defined in (2.17)).*

*Step 2.* *Let*

$$D_0 = \{v \,|\, v \in C_0, v \text{ is convex}\}.$$

*Let* $n := 0$.

*Step 3.* *Find* $\min_{v\in D_n} b_v$. *Below we prove that*

$$\min_{v\in D_n} b_v \leq \mathcal{H}^s(\Lambda) \tag{2.26}$$

*holds.*

*Step 4.* *Find a node* $v \in D_n$ *for which* $b_v = \min_{w\in D_n} b_w$ *(if such a* $v$ *is not unique, then choose any of them). Let* $U_n$ *be the set of non-convex descendants of* $v$ *in one generation. That is*

$$U_n := \{w \,|\, w^\circ = v^\circ + 1,\ v \overset{desc}{\longrightarrow} w, w \text{ is non-convex } \}.$$

$$\begin{aligned} V_n := \{w \,|\, w^\circ = v^\circ + 1,\ \exists \text{ a level } v^\circ + 1 \text{ triangle } \Delta \notin w, \\ \text{such that the conditions of Lemma 7 holds} \\ \text{by replacing } n \text{ with } w^\circ \text{ and } v \text{ with } w \text{ in Lemma } 7.\} \end{aligned} \tag{2.27}$$

*Moreover, we define*

$$W_n := \{w \,|\, w^\circ = v^\circ + 1,\ v \overset{desc}{\longrightarrow} w\} \setminus (U_n \cup V_n).$$

*Note that the set* $U_n \cup V_n \cup W_n$ *contains all of those nodes which are descendants of the node* $v$ *in one generation. Let*

$$D_{n+1} := W_n \cup (D_n \setminus \{v\}).$$

*Increase* $n$ *by* 1. *Go to Step* 3.

The only thing remained to be done is to verify (2.26). To do so, we will use Theorem 5. Let us fix $n$, and consider the set

$$Q_n = D_n \cup (C_0 \setminus D_0) \cup \bigcup_{k=0}^{n-1} U_k \cup V_k.$$

In the following we will check the assumptions of Theorem 5 by replacing $Q$ with $Q_n$ and $D$ with $D_n$.

It is easy to see that $Q_n$ is a cross-section, because

$$\mathcal{T}_{Q_n} \cup P_{Q_n} = \mathcal{T}_{C_0} \cup P_{C_0}.$$

For $v \in Q_n \backslash D_n$ there exists an $i = 0, 1, 2, \ldots, n-1$ such that an $v \in U_i \cup V_i$, or $v \in C_0 \setminus D_0$. If $v \in U_i$ or $v \in C_0 \setminus D_0$, then $v$ is non-convex. Let $w = conv(v)$. We have $v^\circ = w^\circ$, and $E_v \subsetneq E_w$. Let $v \xrightarrow{\text{desc}} v'$ be arbitrary. By using Lemma 6 for $v$ and $m = v'^\circ$, there exist $w'$, $w'^\circ = v'^\circ = m$, such that

$$E_{v'} \subset E_{w'} \quad \text{and} \quad |E_{w'}| \leq |E_{v'}| + \frac{2}{2^m}.$$

If $v \in V_i$, then the conditions of Lemma 7 holds for $v$, $n = v^\circ$ and for a level $n$ triangle $\Delta \notin v$. Let $w = v \cup \{\Delta\}$. We have $v^\circ = w^\circ$, and $E_v \subsetneq E_w$. Let $v \xrightarrow{\text{desc}} v'$ be arbitrary. By using Lemma 7 there exists a level $m := v'^\circ$ triangle $\Delta'$, such that

$$|E_{v'} \cup \Delta'| \leq |E_{v'}| + \frac{1}{2^m}.$$

By choosing $w' = v' \cup \{\Delta'\}$, we obtain that

$$E_{v'} \subset E_{w'} \text{ and } |E_{w'}| \leq |E_{v'}| + \frac{2}{2^m}.$$

So, by using Theorem 5 we get

$$B_{D_n} \leq \mathcal{H}^s(\Lambda)$$

which completes the proof of (2.26).

By symmetry we can assume that for every level 4 descendants $v$ of the node $\{E_1, E_2, E_3\}$ we have

$$\#(v \cap (\mathcal{T}_{E_{11}} \cup \mathcal{T}_{E_{12}} \cup \mathcal{T}_{E_{13}})) \leq \#(v \cap (\mathcal{T}_{E_{21}}, \mathcal{T}_{E_{22}}, \mathcal{T}_{E_{23}})) \leq$$
$$\leq \#(v \cap (\mathcal{T}_{E_{31}}, \mathcal{T}_{E_{32}}, \mathcal{T}_{E_{33}})).$$

To reduce the usage of the computer memory we modify the Algorithm 2. First we fix a constant $Z$. We store only those nodes, which are necessary to prove that a fixed constant $Z$ is a lower bound on the Hausdorff measure of the Sierpinski triangle. Let

$$\overline{D}_n = \{v \,|\, v \in D_n, b_v \leq Z\}.$$

During the modified Algorithm we store $\overline{D}_n$ instead of $D_n$. We use $D_n$ to find a node $v \in D_n$ such that $b_v = \min_{w \in D_n} b_w$. If $\overline{D}_n$ is the empty set, then

$$Z < \min_{v \in D_n} b_v \leq \mathcal{H}^s(\Lambda),$$

otherwise we have

$$\min_{v \in \overline{D}_n} b_v = \min_{v \in D_n} b_v.$$

If this modified Algorithm reaches a state where $\overline{D}_n = \emptyset$, then by using inequality (2.26) we have

$$Z < \mathcal{H}^s(\Lambda).$$

## 2.7 Running results

I wrote the program in *C++* language. For $Z = 0.73$ the program runs for half an hour, for $Z = 0.77$ it runs for a 4 days. The best result, what I managed to reach, is 0.77.

The program is available as an electric supplement on a compact disk and at my homepage:
*http://www.math.bme.hu/˜morap/sierpinski.zip*

# 3 Infinite Bernoulli convolutions with probabilities $p \in (0, \frac{1}{3})$

## 3.1 Introduction

Let $p \in (0,1)$ and $\lambda \in (0,1)$, and take the following random sum

$$Y_\lambda^p := \sum_{n=0}^{\infty} \pm \lambda^n,$$

where the signs " + " and " − " are chosen identically and independently with probability $p$ and $1-p$. Let $\nu_\lambda^p$ be the distribution of $Y_\lambda^p$. That is for every set $E$ we have $\nu_\lambda^p(E) := \mathbf{P}(Y_\lambda^p \in E)$. The measure $\nu_\lambda^p$ has an important self-similar property. Namely, for every set $E$ we have

$$\nu_\lambda^p(E) = p \cdot \nu_\lambda^p(S_1^{-1}(E)) + (1-p) \cdot \nu_\lambda^p(S_2^{-1}(E)),$$

where $S_1(x) = \lambda x + 1$ and $S_2(x) = \lambda x - 1$. Since $\nu_\lambda^p$ is an infinite convolutions product of $\frac{1}{2}(\delta_{-\lambda^n} + \delta_{\lambda^n})$, it is called infinite Bernoulli convolutions.

In this Subsection we give a brief summary of the results about infinite Bernoulli convolutions in the unbiased ($p = 1/2$) case. A more detailed overview can be found in [16]. It was asked by Pál Erdős in 1930's whether the measure $\nu_\lambda^{1/2}$ is absolutely continuous or singular with respect to the Lebesgue measure. In 1935 Jessen an Wintner proved a so called "the law of pure types" statement:

**Theorem 6** (Jessen and Wintner 1935)**.** *The measure $\nu_\lambda^{1/2}$ is either absolutely continuous or purely singular, depending on $\lambda$.*

In 1935 Kershner and Wintner showed that $\nu_\lambda^{1/2}$ is singular if $\lambda \in [0, 1/2)$, because it is supported on a Cantor set, which has zero Lebesgue measure. For $\lambda \in (1/2, 1)$ the first result was made by Erdős. To claim his statement we need the definition of the Pisot numbers.

**Definition 5.** *We call a algebraic number $\theta > 1$ Pisot if all the other roots of the minimal polynomial of $\theta$ are less than 1 in modulus.*

**Theorem 7** (Erdős 1939)**.** *If $\lambda \neq \frac{1}{2}$ and $\lambda$ is a reciprocal of a Pisot number then $\nu_\lambda^{1/2}$ is singular.*

The first result about the absolute continuity was also achieved by Erdős.

**Theorem 8** (Erdős 1940)**.** *There exist a constant $a_0 < 1$, where $a_0$ is rather close to 1, such that for almost every $\lambda \in (a_, 1)$ the measure $\nu_\lambda^{1/2}$ is absolutely continuous. Moreover, there exists a sequence $a_k \to 1$, such that for almost every $\lambda \in (a_k, 1)$ the measure $\nu_\lambda^{1/2}$ has a density in $C^k(\mathbb{R})$.*

In 1995 K. Simon and M. Pollicott [17] made progress about the "$\{1, 2, 3\}$-problem" (see [18]). The main idea was to define transversality for power series. Transversality seems to be the key for solving many problems related to absolute continuity and dimensional theory. Using the definition B. Solomyak proved the following Theorem:

**Theorem 9** (B. Solomyak 1995 [19])**.**

1. *For almost every $\lambda \in (\frac{1}{2}, 2^{-1/2})$ the measure $\nu_\lambda^{1/2}$ is absolutely continuous with a density in $L^2(\mathbb{R})$.*
2. *For almost every $\lambda \in (2^{-1/2}, 1)$ the measure $\nu_\lambda^{1/2}$ is absolutely continuous with a density in $C(\mathbb{R})$.*

A simpler proof of this Theorem was found by B. Solomyak and Y. Peres [20][Section 4 and 5].

## 3.2 Biased case, $p \in (0, 1)$

One of the generalisations of the Bernoulli convolutions is to let $p \in (0, 1)$ be arbitrary. If $\lambda = p^p(1-p)^{1-p}$ then using some basic estimates (see [14][Theorem 1.3, (a)] for example) the Hausdorff dimension of $\nu_\lambda^p$ is less than or equal to 1, thus for $\lambda < p^p(1-p)^{1-p}$ the measure $\nu_\lambda^p$ is singular. It is conjectured that for every $p \in (0, 1)$ and for almost every $\lambda \in (p^p(1-p)^{1-p}, 1)$ the measure $\nu_\lambda^p$ is absolutely continuous with respect to the Lebesgue measure. In 1998 B. Solomyak and Y. Peres proved [14, Corollary 1.4] that this holds for $p \in [1/3, 2/3]$:

**Theorem 10** (Peres, Solomyak [14]). *For $p \in [1/3, 2/3]$ the measure $\nu_\lambda^p$ is absolutely continuous for a.e. $\lambda \in [p^p(1-p)^{1-p}, 1)$, and has $L^2$-density for a.e. $\lambda \in [p^2 + (1-p)^2, 1)$.*

Solomyak and Peres [14, Theorem 1.3] also showed that we only have chance for $L^2$-density if $\lambda \geq p^2 + (1-p)^2$ holds. However, Peres and Solomyak left open the corresponding problem for $p \in (0, 1/3)$. The first steps in this case was made by H. Tóth in 2008:

**Theorem 11** (Tóth [13]). *For $p \in (0, 1/3)$ and for almost every*

$$\lambda \in ((1-2p)^{2-\log 41/\log 9}, 1)$$

*the measure $\nu_\lambda^p$ is absolutely continuous with $L^2$-density.*

In the rest of this Section we improve H. Tóth's result. We remark that both H. Tóth's and our proof use Theorem 10. Following [13] we introduce the function $h(p)$:

$$h(p) := p^2 + (1-p)^2.$$

For every positive integer $T$ and $p \in (1/3, 1/2)$ let

$$g_T(p) := \frac{1-(1-2p)^{1/T}}{2},$$

which function was defined by H. Tóth for $T = 2$. Since $g_T(p)$ is monotone increasing, its inverse is well defined for $g_T(1/3) < p < 1/2$:

$$g_T^{-1}(p) = \frac{1-(1-2p)^T}{2}.$$

The following function has appeared in [13] in a different form:

$$f(p) = (h(g_T^{-1}(p)))^{1/T} \quad \text{where } T = \min(K = 2^L, L \in \mathbb{N}, \, g_K(1/3) \leq p)$$

H. Tóth showed that for all $p \in (0, 1/3)$ and for almost every $\lambda \in (f(p), 1)$ the measure $\nu_\lambda^p$ is absolutely continuous. She also introduced the function $F(p) = (1-2p)^{2-\log 41/\log 9}$, which has a closed form, and for $p \in (0, 1/3)$, $F(p)$ is larger than $f(p)$.

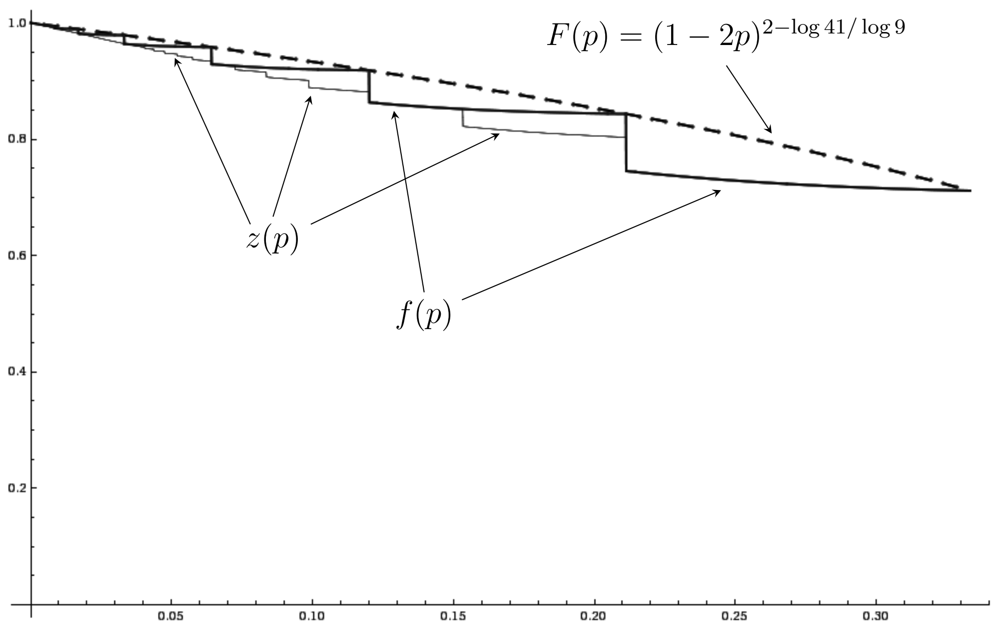


Figure 14: The functions $f(p)$, $F(p)$ and $z(p)$ for $p \in (0, 1/3)$.

For $p \in (0, 1/3)$ we define $z(p)$ the following way:

$$z(p) = (h(g_T^{-1}(p)))^{1/T} \quad \text{where } T = \min(K \in \mathbb{N},\, g_K(1/3) \leq p).$$

See Figure 14 and Figure 15. In this Section we prove the following Theorem:

**Theorem 12.** *For $p \in (0, 1/3)$ and almost every $\lambda \in (z(p), 1)$ the measure $\nu_\lambda^p$ is absolutely continuous with respect to the Lebesgue measure with $L^2$-density.*

The motivation of our research was as follows. Recently there have arisen some problems (related to the concentration of medicine in the blood), which require the better understanding of $\nu_\lambda^p$ for $p$ close to zero. Our most important achievement is that Theorem 12 gives a better tangent for $p \approx 0$. On the other hand we remark that $z(p) = f(p)$ for some intervals. H. Tóth showed that $F'(0) \approx -0.619761$ and $F(p) \leq 1 - 0.6p$ for $p \in (0, 1/3)$. We will prove that

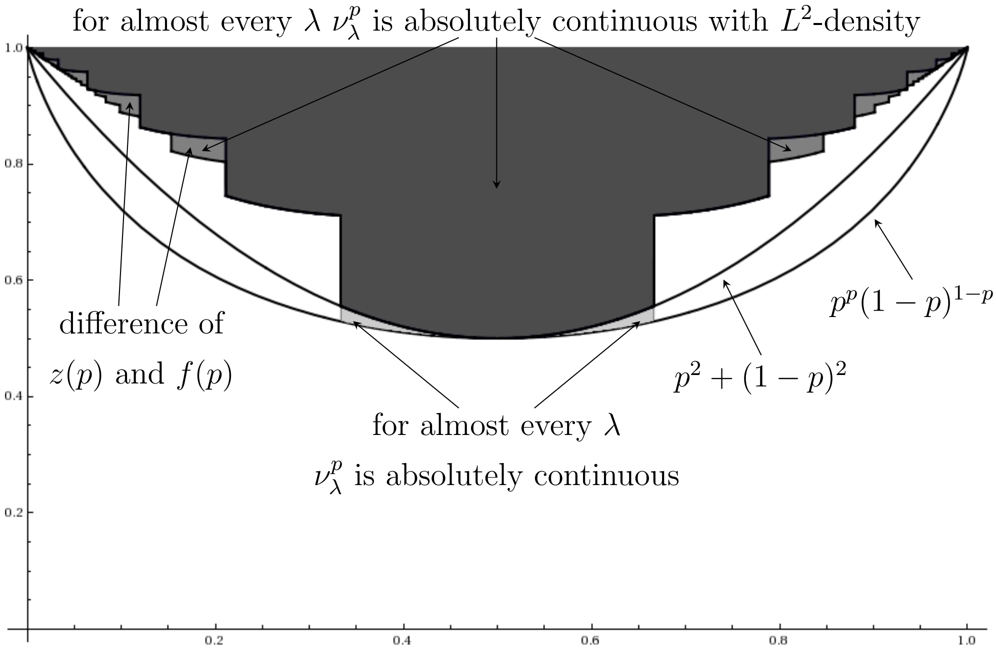


Figure 15: Absolute continuity of $\nu_\lambda^p$.

**Lemma 8.**

$$\limsup_{p->0+} \frac{z(p)-1}{p} \leq \frac{-2\log(9/5)}{\log(3)} \approx -1.07005$$

### 3.3 Basic idea

In this Subsection we prove Theorem 12 via some lemmas. In the next Subsection we continue with the proofs of the lemmas used.

It follows from Theorem 10 that for $p \in (1/3, 1/2)$ and for almost every $\lambda \in (p^2+(1-p)^2, 1)$, $\nu_\lambda^p$ is absolutely continuous with $L^2$-density. Fix $p$ and $\lambda$ such that $\nu_\lambda^p$ is absolutely continuous with $L^2$-density. Using Plancherel's Theorem for the Fourier transform we have $\hat{\nu}_\lambda^p \in L^2$. First we prove that

**Lemma 9.** *For $p \in (0,1)$, $\lambda \in (0,1)$ and for $T > 1$ if $\hat{\nu}_\lambda^p(\xi) \in L^2$ then $\hat{\nu}_\lambda^{g_T(p)}(\xi) \in L^{2T}$.*

Further, we claim that

**Lemma 10.** *For $p \in (0,1)$, $\lambda \in (0,1)$ and for $T > 1$ integer if $\hat{\nu}_{\lambda}^{g_T(p)}(\xi) \in L^{2T}$ then $\hat{\nu}_{\lambda^{1/T}}^{g_T(p)}(\xi) \in L^2$.*

Using Plancherel's Theorem again $\nu_{\lambda^{1/T}}^{g_T(p)}(\xi) \in L^2$ is a consequence of $\hat{\nu}_{\lambda^{1/T}}^{g_T(p)}(\xi) \in L^2$.

*Proof of Theorem 12.* Using Lemma 9, Lemma 10 and Plancherel's Theorem twice we obtain that for all $T > 1$ integer, for all $\overline{p} \in (1/3, 1/2)$ and for almost every $\overline{\lambda} \in (h(\overline{p}), 1)$ we have $\nu_{\overline{\lambda}^{1/T}}^{g_T(\overline{p})} \in L^2$.

Let $p \in (0, 1/3)$ be arbitrary. It follows from the definition of $g_K$ that $g_K(1/3) \downarrow 0$. Let

$$T := \min(K \in \mathbb{N},\ g_K(1/3) \leq p).$$

For $\overline{p} \in [1/3, 1/2)$ the function $g_T(\overline{p})$ is monotone increasing and we have $g_T(1/2) = 1/2$. Since $g_T(1/3) \leq p$, thus there exists $\overline{p} \in [1/3, 1/2)$ such that $p = g_T(\overline{p})$ holds. For almost every $\overline{\lambda} \in (h(\overline{p}), 1)$ we have $\nu_{\overline{\lambda}^{1/T}}^{g_T(\overline{p})} \in L^2$. The function $x \to x^{1/T}$ is monotone increasing, thus for almost every

$$\lambda := \overline{\lambda}^{1/T} \in (h(g_T^{-1}(p))^{1/T}, 1)$$

we have $\nu_{\lambda}^{g_T(\overline{p})} \in L^2$. □

## 3.4 Proof of the Lemmas

Instead of $Y_\lambda^p$ we consider the following random sum:

$$Z_\lambda^p = \sum_{n=0}^{\infty} a_n \cdot \lambda^n,$$

where $a_n$ independently equals to 0 or 1 with probability $p$ and $1-p$. Let $\psi_\lambda^p$ be the distribution of $Z_\lambda^p$. We have

$$Y_\lambda^p \stackrel{d}{=} \frac{1}{1-\lambda} - 2 \cdot Z_\lambda^p,$$

thus the measures $\nu_\lambda^p$ and $\psi_\lambda^p$ are absolutely continuous with $L^2$-density at the same time. We repeat the steps of the proof of the main result of [13] to obtain a convenient form for the Fourier transform of the measure $\psi_\lambda^p$. The measure $\psi_\lambda^p$ is the infinite convolution product of $p \cdot \delta_0 + (1-p) \cdot \delta_{\lambda^n}$. Thus for $\xi \in \mathbb{R}$ we have

$$\hat{\psi}_\lambda^p(\xi) = \prod_{n=0}^{\infty} \left(p + (1-p) \cdot \mathbf{e}^{-i \cdot \lambda^n \cdot \xi}\right).$$

Using trigonometric identities we get the following equation for the absolute value of $\hat{\psi}_\lambda^p(\xi)$:

$$|\hat{\psi}_\lambda^p(\xi)| = \prod_{n=0}^{\infty} \sqrt{1 - 4p(1-p)\sin^2\left(\frac{\lambda^n \cdot \xi}{2}\right)}. \tag{3.1}$$

*Proof of Lemma 9.* Let $T > 1$. To prove that $\hat{\nu}_\lambda^{g_T(p)} \in L^{2T}$ is a consequence of $\hat{\nu}_\lambda^p \in L^2$, it is enough to show that

$$\prod_{n=0}^{\infty} \sqrt{1 - 4p(1-p)\sin^2\left(\frac{\lambda^n \cdot \xi}{2}\right)} \geq \prod_{n=0}^{\infty} \sqrt{1 - 4g_T(p)(1-g_T(p))\sin^2\left(\frac{\lambda^n \cdot \xi}{2}\right)}^T$$

holds. To do so, we give a bound on each term. Namely, we prove that for all $0 \leq \delta \leq 1$ and $p \in (0, 1/2)$ we have

$$\sqrt{1 - 4p(1-p)\delta} \geq \sqrt{1 - 4g_T(p)(1-g_T(p))\delta}^T.$$

Since both sides are positive, we show that the inequality holds for their square. Let $h(p, \delta, T)$ be the difference of these two squares. Using the definition of $g_T(p)$ we have

$$h(p, \delta, T) := 1 - 4p(1-p)\delta - \left(1 - 4\frac{1 - (1-2p)^{1/T}}{2}\left(1 - \frac{1 - (1-2p)^{1/T}}{2}\right)\delta\right)^T$$

which leads to

$$h(p, \delta, T) = 1 - 4p(1-p)\delta - \left(1 - \left(1 - (1-2p)^{\frac{2}{T}}\right)\delta\right)^T.$$

Our purpose is to show that $h(p, \delta, T) \geq 0$ holds. For $p = 0$ it is trivial. In the following we prove that for $p \in (0, 1/2)$ the derivative with respect to $p$ is positive. Namely,

$$h'_p(p, \delta, T) \geq 0$$

holds, which completes the proof of the Lemma.

$$h'_p(p,\delta,T) = 4\delta\left(-(1-2p) + (1-2p)^{\frac{2}{T}-1}(1+((1-2p)^{\frac{2}{T}}-1)\delta)^{T-1}\right)$$

If $\delta = 0$ then $h'_p(p,\delta,T) = 0$. Otherwise we can divide by $4\delta$, thus we have to show that

$$-(1-2p) + (1-2p)^{\frac{2}{T}-1}(1+((1-2p)^{\frac{2}{T}}-1)\delta)^{T-1} \geq 0 \tag{3.2}$$

holds. By using that $(1-2p) \leq 1$ it is easy to see that the left side of the equation 3.2 is monotone decreasing in $\delta$. For $\delta = 1$ we have

$$-(1-2p) + (1-2p)^{\frac{2}{T}-1}(1-2p)^{\frac{2}{T}(T-1)} = 0.$$

This yields that the equation 3.2 holds for $0 < \delta < 1$, and it completes the proof. □

*Proof of Lemma 10.* Let $T > 1$ integer and $\hat{\nu}_\lambda^{g_T(p)}(\xi) \in L^{2T}$. Using integration by substitutions it is easy to see that

$$\hat{\nu}_\lambda^{g_T(p)}(\xi\lambda^{1/T}), \hat{\nu}_\lambda^{g_T(p)}(\xi\lambda^{2/T}), \ldots, \hat{\nu}_\lambda^{g_T(p)}(\xi\lambda^{(T-1)/T}) \in L^{2T} \tag{3.3}$$

as well. Since the terms of (3.1) are positive and less than 1, the infinite product of $|\hat{\psi}_{\lambda^{1/T}}^{g_T(p)}(\xi)|$ is absolute convergent. Thus the terms of can be rearranged in the following way:

$$\hat{\psi}_{\lambda^{1/T}}^{g_T(p)}(\xi) = \hat{\psi}_\lambda^{g_T(p)}(\xi) \cdot \hat{\psi}_\lambda^{g_T(p)}(\xi\lambda^{1/T}) \cdot \cdots \cdot \hat{\psi}_\lambda^{g_T(p)}(\xi\lambda^{(T-1)/T}).$$

Using (3.3) and Hölder inequality we have $\hat{\psi}_{\lambda^{1/T}}^{g_T(p)}(\xi) \in L^2$.

Note that this principal was used by Solomyak in [15] for $p = 1/2$, and by Tóth in [13] for $T = 2$. □

*Proof of Lemma 8.* Fix $p \in (0, 1/3)$, and let

$$T = \min(K \in \mathbb{N}, g_K(1/3) \leq p). \tag{3.4}$$

Since $g_T(1/3) \leq p$, we have $1/3 \leq g_T^{-1}(p)$. Thus

$$z(p) = (h(g_T^{-1}(p)))^{1/T} \leq (h(1/3))^{1/T} = \left(\frac{5}{9}\right)^{1/T},$$

because the function $h$ is monotone decreasing on $(1/3, 1/2)$. Since $T$ was minimal, we get $g_{T-1}(1/3) > p$. Since $z(p) - 1 < 0$, we obtain that

$$\frac{z(p)-1}{p} \leq \frac{\left(\frac{5}{9}\right)^{1/T} - 1}{g_{T-1}(1/3)} = \frac{\left(\frac{5}{9}\right)^{1/T} - 1}{\frac{1-(1/3)^{1/(T-1)}}{2}}.$$

If $p \downarrow 0$ and $T$ is defined by (3.4), then $T \to \infty$. Using L'Hospital's rule we have

$$\lim_{T\to\infty} \frac{\left(\frac{5}{9}\right)^{1/T} - 1}{\frac{1-(1/3)^{1/(T-1)}}{2}} = \frac{-2\log(9/5)}{\log 3},$$

which completes the proof. □

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