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EMPIRICAL LOG-OPTIMAL PORTFOLIO SELECTION

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We show that the empirical log-optimal portfolio performs asymptotically under certain conditions as well as the optimal one.

1. Introduction

Let $\mathbf{X} \in \mathbb{R}^m$ denote a random stock market return vector, where X_i is the value of one unit investment in stock i at the end of the trading day. We require that $X_i \geq 0$ for $i = 1, 2, \ldots, m$, that is, an investor can not loose more than the invested capital. Let $\mathbf{b}$, $b_i \geq 0$, $\sum\limits_{i=1}^{m} b_i = 1$, denote a portfolio, that is, an allocation of investor's capital across the investment alternatives. Let B denote the set of such portfolios. Thus b_i is the proportion of current capital invested in stock i. The resulting wealth is $S = \sum\limits_{i=1}^{m} b_i X_i = \mathbf{bX}$. This is the wealth resulting from a unit investment allocated to the m stocks according to portfolio $\mathbf{b}$. If the current capital is reallocated according to portfolio $\mathbf{b}_i$ at time i in repeated investments against stock vectors $\mathbf{X}_1$, $\mathbf{X}_2, \ldots$ then the wealth S_n at time n is given by

$$S_n = \prod_{i=1}^{n} \mathbf{b}_i \mathbf{X}_i .$$

Suppose the stock market process $\mathbf{X}_1$, $\mathbf{X}_2, \ldots$ is independent and identically distributed. A portfolio $\mathbf{b}^*$ is called log-optimal if $E \ln \mathbf{b}^* \mathbf{X} = \sup\limits_{\mathbf{b} \in B} E \ln \mathbf{bX}$. Let B^* denote the set of log-optimal portfolios. It can be shown that $\limsup\limits_{n\to\infty} \frac{1}{n} \ln S_n \leq$

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$\lim_{n\to\infty} \frac{1}{n} \ln S_n^* = E \ln \mathbf{b}^*\mathbf{X}$ a. s., where S_n, S_n^* denote capitals achieved by an arbitrary and the log-optimal portfolio in n repeated games, respectively. For more about the log-optimal portfolio see [1]–[8].

If the probability distribution of the stocks is not known in advance, consider as a goal to find a portfolio selector $\hat{\mathbf{b}}(\cdot)$ which achieves the same asymptotic capital growth rate as the log-optimal portfolio does, that is,

$$\lim_{n\to\infty} \frac{1}{n} \ln \hat{S}_n = E \ln \mathbf{b}^*\mathbf{X} \qquad \text{a. s.,}$$

where $\hat{S}_n = \prod_{i=1}^{n} \hat{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i$.

2. The empirical log-optimal portfolio

We suppose that the sequence of random stock market variables $\mathbf{X}_1$, $\mathbf{X}_2, \ldots$ is stationary and ergodic. We examine the performance of the following portfolio selector:

$$\hat{\mathbf{b}}(\cdot) = (1/m, 1/m, \ldots, 1/m) \qquad \text{for } n = 0$$

$$\hat{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) = \arg\max_{\mathbf{b}\in B} \frac{1}{n} \sum_{i=1}^{n} \ln \mathbf{b}\mathbf{X}_i = \arg\max_{\mathbf{b}\in B} \int \ln \mathbf{b}\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) \qquad \text{for } n \geq 1$$

where

$$\hat{\mu}_n(A) = \frac{1}{n} \sum_{i=1}^{n} I_{\{\mathbf{X}_i \in A\}}$$

and

$$I_{\{\mathbf{X}_i\in A\}} = \begin{Bmatrix} 1 & \text{if} & \mathbf{X}_i \in A \\ 0 & \text{if} & \mathbf{X}_i \notin A \end{Bmatrix}.$$

In other words, we choose the log-optimal portfolio according to the empirical distribution of the past.

The following theorem implies that the asymptotically optimal growth rate is achieved by the proposed portfolio selector if the sequence of random stock vectors is independent and identically distributed rather than merely ergodic. The portfolio selector proposed in Cover [9] achieves this goal but our selector is much simpler.

THEOREM 1. Suppose the sequence of random stock market variables $\mathbf{X}_1$, $\mathbf{X}_2, \ldots$, is stationary, ergodic, and $E|\ln X_j| < \infty$ for $j = 1, 2, \ldots, m$. Then

$$\lim_{n\to\infty} \frac{1}{n} \ln \hat{S}_n = E \ln \mathbf{b}^*\mathbf{X} \qquad \text{a. s.,}$$

where $\hat{S}_n = \prod_{i=1}^{n} \hat{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i$ and $E \ln \mathbf{b}^*\mathbf{X} = \sup_{\mathbf{b}\in B} E \ln \mathbf{b}\mathbf{X}$.

Let μ denote the distribution of the random stock vector $\mathbf{X}$.

Lemma 1. Suppose $-\infty < \sup_{\mathbf{b}\in B} E \ln \mathbf{b}\mathbf{X} < \infty$. Let $\{\mathbf{b}_n\}$ be a fixed sequence of portfolios. If $\lim_{n\to\infty} \int \ln \mathbf{b}_n\mathbf{x}\, \mu(d\mathbf{x}) = E \ln \mathbf{b}^*\mathbf{X}$ then the accumulation points of $\{\mathbf{b}_n\}$ are log-optimal according to the true distribution μ.

Proof. Suppose $\mathbf{b}'$ is an accumulation point which is not log-optimal.

Let $\{\mathbf{b}_{n_i}\}$ be a subsequence of $\{\mathbf{b}_n\}$ converging to $\mathbf{b}'$. Since the function $E \ln \mathbf{b}\mathbf{X}$ is continuous in $\mathbf{b}$,

$$E \ln \mathbf{b}'\mathbf{X} = \int \ln \lim_{i\to\infty} \mathbf{b}_{n_i}\mathbf{x}\, \mu(d\mathbf{x}) = \lim_{i\to\infty} \int \ln \mathbf{b}_{n_i}\mathbf{x}\, \mu(d\mathbf{x}).$$

Since $\mathbf{b}'$ is not log-optimal, $E \ln \mathbf{b}'\mathbf{X} < E \ln \mathbf{b}^*\mathbf{X}$. Thus

$$\lim_{i\to\infty} \int \ln \mathbf{b}_{n_i}\mathbf{x}\, \mu(d\mathbf{x}) < E \ln \mathbf{b}^*\mathbf{X}.$$

But this contradicts the assumption $\lim_{n\to\infty} \int \ln \mathbf{b}_n\mathbf{x}\, \mu(d\mathbf{x}) = E \ln \mathbf{b}^*\mathbf{X}$.

Lemma 2 (Cover [8]). Suppose $-\infty < \sup_{\mathbf{b}\in B} E \ln \mathbf{b}\mathbf{X} < \infty$. Let L be the subspace of $\mathbb{R}^m$ of least dimension satisfying $P(\mathbf{X} \in L) = 1$. Each log-optimal portfolio $\mathbf{b}^* \in B^*$ has the same orthogonal projection $\mathbf{b}_L$ onto L.

Lemma 3. Suppose $-\infty < \sup_{\mathbf{b}\in B} E \ln \mathbf{b}\mathbf{X} < \infty$. Let the process $\mathbf{X}_1, \mathbf{X}_2, \ldots,$ be stationary and ergodic. Consider any function $\mathbf{b}^*(\cdot)$ such that $\mathbf{b}^*(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_i) \in B^*$ for all i. Let $\mathbf{b}^*$ be a log-optimal portfolio. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \mathbf{b}^*(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i = E \ln \mathbf{b}^*\mathbf{X} \qquad \text{a. s.}$$

Proof.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \mathbf{b}^*(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i =$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln(\mathbf{b}_L\mathbf{X}_i + (\mathbf{b}^*(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1}) - \mathbf{b}_L)\mathbf{X}_i) =$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \mathbf{b}_L\mathbf{X}_i =$$

$$E \ln \mathbf{b}_L\mathbf{X} =$$

$$E \ln \mathbf{b}^*\mathbf{X} \qquad \text{a. s.,}$$

since $(\mathbf{b}^*(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1}) - \mathbf{b}_L)\mathbf{x} = 0$ for $\mathbf{x} \in L$ and $P(\mathbf{X} \in L) = 1$ by Lemma 2, where $\mathbf{b}_L$ is the unique projection of the log-optimal portfolios.

Lemma 4. Suppose $-\infty < \sup_{\mathbf{b} \in B} E \ln \mathbf{bX} < \infty$. Then the set of log-optimal portfolios B^* is closed.

Proof. Suppose $\mathbf{b}'$ is from the boundary of B^* but $\mathbf{b}' \notin B^*$. Let $\mathbf{b}_n^*$ be a sequence converging to $\mathbf{b}'$. By the continuity of the function $E \ln \mathbf{bX}$, $E \ln \mathbf{b}'\mathbf{X} = E \ln \lim_{n\to\infty} \mathbf{b}_n^* \mathbf{X} = \lim_{n\to\infty} E \ln \mathbf{b}_n^* \mathbf{X} = E \ln \mathbf{b}^* \mathbf{X}$. Thus $\mathbf{b}'$ is log-optimal. But this contradicts the assumption $\mathbf{b}' \notin B^*$.

Lemma 5.

$$I_{\{\mathbf{X} \in A_\epsilon\}} \ln \mathbf{bX} - I_{\{\mathbf{X} \in A_\epsilon\}} \ln \mathbf{b}'\mathbf{X} \le \frac{m}{\epsilon^2} \|\mathbf{b} - \mathbf{b}'\|,$$

where $0 < \epsilon < 1$ and

$$I_{\{\mathbf{X} \in A_\epsilon\}} = \left\{ \begin{matrix} 1 & \text{if} & \mathbf{X} \in [\epsilon, 1/\epsilon]^m \\ 0 & \text{if} & \mathbf{X} \notin [\epsilon, 1/\epsilon]^m \end{matrix} \right\}.$$

Proof.

$$I_{\{\mathbf{X} \in A_\epsilon\}} \ln \mathbf{bX} - I_{\{\mathbf{X} \in A_\epsilon\}} \ln \mathbf{b}'\mathbf{X} =$$

$$I_{\{\mathbf{X} \in A_\epsilon\}} \ln \frac{\mathbf{bX}}{\mathbf{b}'\mathbf{X}} = I_{\{\mathbf{X} \in A_\epsilon\}} \ln \left(1 + \frac{(\mathbf{b} - \mathbf{b}')\mathbf{X}}{\mathbf{b}'\mathbf{X}}\right) \le$$

$$I_{\{\mathbf{X} \in A_\epsilon\}} \ln \left(1 + \frac{\sum_{i=1}^m |b_i - b_i'| \frac{1}{\epsilon}}{\sum_{i=1}^m b_i' \epsilon}\right) \le I_{\{\mathbf{X} \in A_\epsilon\}} \ln \left(1 + \frac{\sum_{i=1}^m |b_i - b_i'|}{\epsilon^2}\right) \le$$

$$I_{\{\mathbf{X} \in A_\epsilon\}} \frac{\sum_{i=1}^m |\, b_i - b_i' \,|}{\epsilon^2} \le \frac{m}{\epsilon^2} \|\mathbf{b} - \mathbf{b}'\|.$$

Lemma 6. For $0 < \epsilon < 1$

$$\limsup_{n\to\infty} \sup_{\mathbf{b} \in B} \int I_{\{\mathbf{x} \in A_\epsilon\}} \ln \mathbf{bx} \, \hat{\mu}_n(d\mathbf{x}) - \int I_{\{\mathbf{x} \in A_\epsilon\}} \ln \mathbf{bx} \, \mu(d\mathbf{x}) \le 0 \qquad \text{a. s.}$$

Proof. We cover the simplex $B = \left\{ \mathbf{b} : \sum_{i=1}^m b_i = 1, \; b_i \ge 0 \text{ for } i = 1, 2, \ldots, m \right\}$ by regions D_j with diameter Δ, where $j = 1, 2, \ldots, r(\Delta)$. Let $\mathbf{b}_j$ denote a portfolio from the region D_j.

$$\sup_{\mathbf{b} \in B} \int I_{\{\mathbf{x} \in A_\epsilon\}} \ln \mathbf{bx} \, \hat{\mu}_n(d\mathbf{x}) - \int I_{\{\mathbf{x} \in A_\epsilon\}} \ln \mathbf{bx} \, \mu(d\mathbf{x}) =$$

$$\max_j \sup_{\mathbf{b}\in D_j} \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) - \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\, \mu(d\mathbf{x}) \le$$

$$\max_j \sup_{\mathbf{b}\in D_j} \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) - \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}_j\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) +$$

$$\max_j \sup_{\mathbf{b}\in D_j} \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}_j\mathbf{x}\, \mu(d\mathbf{x}) - \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\, \mu(d\mathbf{x}) +$$

$$\max_j \sup_{\mathbf{b}\in D_j} \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}_j\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) - \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}_j\mathbf{x}\, \mu(d\mathbf{x}).$$

From Lemma 5,

$$\sup_{\mathbf{b}\in D_j} \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) - \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}_j\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) \le \frac{m}{\epsilon^2}\Delta.$$

Similarly,

$$\sup_{\mathbf{b}\in D_j} \int \ln \mathbf{b}_j\mathbf{x} I_{\{\mathbf{x}\in A_\epsilon\}}\mu(d\mathbf{x}) - \int \ln \mathbf{b}\mathbf{x} I_{\{\mathbf{x}\in A_\epsilon\}}\mu(d\mathbf{x}) \le \frac{m}{\epsilon^2}\Delta.$$

Thus

$$\sup_{\mathbf{b}\in B} \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) - \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\, \mu(d\mathbf{x}) \le$$

$$\frac{2m\Delta}{\epsilon^2} + \max_j \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}_j\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) - \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}_j\mathbf{x}\, \mu(d\mathbf{x}).$$

By the strong law of large numbers for ergodic sequence,

$$\lim_{n\to\infty} \max_j \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}_j\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) - \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}_j\mathbf{x}\, \mu(d\mathbf{x}) = 0 \qquad \text{a. s.},$$

hence

$$\limsup_{n\to\infty} \sup_{\mathbf{b}\in B} \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) - \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\, \mu(d\mathbf{x}) \le \frac{2m\Delta}{\epsilon^2} \qquad \text{a. s.}$$

Since Δ was arbitrary,

$$\limsup_{n\to\infty} \sup_{\mathbf{b}\in B} \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) - \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\, \mu(d\mathbf{x}) \le 0 \qquad \text{a. s.}$$

Lemma 7. Under the conditions of Theorem 1,

$$\limsup_{n\to\infty} \sup_{\mathbf{b}\in B} \int \ln \mathbf{b}\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) - \int \ln \mathbf{b}\mathbf{x}\, \mu(d\mathbf{x}) \le 0 \qquad \text{a. s.}$$

Proof.

$$\sup_{\mathbf{b}\in B}\int \ln \mathbf{b}\mathbf{x}\,\hat{\mu}_n(d\mathbf{x}) - \int \ln \mathbf{b}\mathbf{x}\,\mu(d\mathbf{x}) \le$$

$$\sup_{\mathbf{b}\in B}\int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\,\hat{\mu}_n(d\mathbf{x}) - \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\,\mu(d\mathbf{x}) +$$

$$\sup_{\mathbf{b}\in B}\int I_{\{\mathbf{x}\notin A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\,\hat{\mu}_n(d\mathbf{x}) - \int I_{\{\mathbf{x}\notin A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\,\mu(d\mathbf{x}).$$

From Lemma 6, for arbitrary $\epsilon > 0$,

$$\limsup_{n\to\infty}\sup_{\mathbf{b}\in B}\int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\,\hat{\mu}_n(d\mathbf{x}) - \int I_{\{\mathbf{x}\in A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\,\mu(d\mathbf{x}) \le 0 \qquad \text{a. s.}$$

Furthermore,

$$\sup_{\mathbf{b}\in \mathbf{B}}\int I_{\{\mathbf{x}\notin A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\,\hat{\mu}_n(d\mathbf{x}) - \int I_{\{\mathbf{x}\notin A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\,\mu(d\mathbf{x}) \le$$

$$\sup_{\mathbf{b}\in \mathbf{B}}\left|\int I_{\{\mathbf{x}\notin A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\,\hat{\mu}_n(d\mathbf{x})\right| + \sup_{\mathbf{b}\in \mathbf{B}}\left|\int I_{\{\mathbf{x}\notin A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\,\mu(d\mathbf{x})\right| \le$$

$$\sup_{\mathbf{b}\in \mathbf{B}}\int I_{\{\mathbf{x}\notin A_\epsilon\}} \left|\ln \sum_{i=1}^m b_i x_i\right| \hat{\mu}_n(d\mathbf{x}) + \sup_{\mathbf{b}\in B}\int I_{\{\mathbf{x}\notin A_\epsilon\}} \left|\ln \sum_{i=1}^m b_i x_i\right| \mu(d\mathbf{x}) \le$$

$$\sup_{\mathbf{b}\in \mathbf{B}}\int I_{\{\mathbf{x}\notin A_\epsilon\}} \max\left\{\max_{i=1,2,\dots,m} \ln x_i,\ -\min_{i=1,2,\dots,m} \ln x_i\right\}\hat{\mu}_n(d\mathbf{x}) +$$

$$\sup_{\mathbf{b}\in \mathbf{B}}\int I_{\{\mathbf{x}\notin A_\epsilon\}} \max\left\{\max_{i=1,2,\dots,m} \ln x_i,\ -\min_{i=1,2,\dots,m} \ln x_i\right\}\mu(d\mathbf{x}) \le$$

$$\int I_{\{\mathbf{x}\notin A_\epsilon\}} \sum_{i=1}^m |\ln x_i|\hat{\mu}_n(d\mathbf{x}) + \int I_{\{\mathbf{x}\notin A_\epsilon\}} \sum_{i=1}^m |\ln x_i|\mu(d\mathbf{x}).$$

It follows that

$$\limsup_{n\to\infty}\sup_{\mathbf{b}\in B}\int I_{\{\mathbf{x}\notin A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\,\hat{\mu}_n(d\mathbf{x}) - \int I_{\{\mathbf{x}\notin A_\epsilon\}} \ln \mathbf{b}\mathbf{x}\,\mu(d\mathbf{x}) \le$$

$$2\int I_{\{\mathbf{x}\notin A_\epsilon\}} \sum_{i=1}^m |\ln x_i|\mu(d\mathbf{x}) \qquad \text{a. s.},$$

since

$$\lim_{n\to\infty}\int I_{\{\mathbf{x}\notin A_\epsilon\}} \sum_{i=1}^m |\ln x_i|\hat{\mu}_n(d\mathbf{x}) = \int I_{\{\mathbf{x}\notin A_\epsilon\}} \sum_{i=1}^m |\ln x_i|\mu(d\mathbf{x}) \qquad \text{a. s.}$$

Thus for arbitrary $\epsilon > 0$,

$$\limsup_{n\to\infty} \sup_{\mathbf{b}\in B} \int \ln \mathbf{bx}\, \hat{\mu}_n(d\mathbf{x}) - \int \ln \mathbf{bx}\, \mu(d\mathbf{x}) \le 2 \int I_{\{\mathbf{x}\notin A_\epsilon\}} \sum_{i=1}^{m} |\ln x_i| \mu(d\mathbf{x}) \qquad \text{a. s.}$$

Since $\epsilon > 0$ was arbitrary and by assumption $E|\ln X_i| < \infty$ for $i = 1, 2, \dots, m$, $\lim_{\epsilon\to 0} 2 \int I_{\{\mathbf{x}\notin A_\epsilon\}} \sum_{i=1}^{m} |\ln x_i| \mu(d\mathbf{x}) = 0$ by the Lebesgue dominated convergence theorem. Thus

$$\limsup_{n\to\infty} \sup_{\mathbf{b}\in B} \int \ln \mathbf{bx}\, \hat{\mu}_n(d\mathbf{x}) - \int \ln \mathbf{bx}\, \mu(d\mathbf{x}) \le 0 \qquad \text{a. s.}$$

Lemma 8. Under the conditions of Theorem 1, the accumulation points of $\hat{\mathbf{b}}(\cdot)$ are log-optimal with probability one.

Proof. By the definition of log-optimality,

$$\begin{aligned} 0 \le \int \ln \mathbf{b}^*\mathbf{x}\, \mu(d\mathbf{x}) - \int \ln \hat{\mathbf{b}}(\mathbf{X}_1\mathbf{X}_2 \dots, \mathbf{X}_n)\mathbf{x}\, \mu(d\mathbf{x}) = \\ \int \ln \mathbf{b}^*\mathbf{x}\, \mu(d\mathbf{x}) - \int \ln \mathbf{b}^*\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) + \int \ln \mathbf{b}^*\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) - \\ \int \ln \hat{\mathbf{b}}(\mathbf{X}_1\mathbf{X}_2 \dots, \mathbf{X}_n)\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) + \int \ln \hat{\mathbf{b}}(\mathbf{X}_1\mathbf{X}_2 \dots, \mathbf{X}_n)\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) - \\ \int \ln \hat{\mathbf{b}}(\mathbf{X}_1\mathbf{X}_2 \dots, \mathbf{X}_n)\mathbf{x}\, \mu(d\mathbf{x}). \end{aligned}$$

Since

$$\int \ln \mathbf{b}^*\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) - \int \ln \hat{\mathbf{b}}(\mathbf{X}_1\mathbf{X}_2 \dots, \mathbf{X}_n)\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) \le 0$$

by the definition of $\hat{\mathbf{b}}(\cdot)$, and

$$\lim_{n\to\infty} \int \ln \mathbf{b}^*\mathbf{x}\, \mu(d\mathbf{x}) - \int \ln \mathbf{b}^*\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) = 0 \qquad \text{a. s.}$$

by ergodicity, we have,

$$\begin{aligned} 0 \le \limsup_{n\to\infty} \int \ln \mathbf{b}^*\mathbf{x}\, \mu(d\mathbf{x}) - \int \ln \hat{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n)\mathbf{x}\, \mu(d\mathbf{x}) \le \\ \limsup_{n\to\infty} \int \ln \hat{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n)\mathbf{x}\, \hat{\mu}_n(d\mathbf{x}) - \int \ln \hat{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n)\mathbf{x}\, \mu(d\mathbf{x}) \le \\ \limsup_{n\to\infty} \sup_{\mathbf{b}\in B} \int \ln \mathbf{bx}\, \hat{\mu}_n(d\mathbf{x}) - \int \ln \mathbf{bx}\, \mu(d\mathbf{x}) \le 0 \qquad \text{a. s.} \end{aligned}$$

where the last step follows from Lemma 7.

Thus we have,

$$\liminf_{n\to\infty} \int \ln \hat{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)\mathbf{x}\, \mu(d\mathbf{x}) \geq \int \ln \mathbf{b}^*\mathbf{x}\, \mu(d\mathbf{x}) \qquad \text{a. s.,}$$

and

$$\int \ln \hat{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)\mathbf{x}\, \mu(d\mathbf{x}) \leq \int \ln \mathbf{b}^*\mathbf{x}\, \mu(d\mathbf{x})$$

hence

$$\lim_{n\to\infty} \int \ln \hat{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)\mathbf{x}\, \mu(d\mathbf{x}) = \int \ln \mathbf{b}^*\mathbf{x}\, \mu(d\mathbf{x}) \qquad \text{a. s.}$$

Now the statement follows from Lemma 1.

Lemma 9. Let the process $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be stationary and ergodic. Suppose that

$$-\infty < \sup_{\mathbf{b}} E \ln \mathbf{bx} < \infty.$$

Consider a portfolio selector $\tilde{\mathbf{b}}(\cdot)$ such that $P(\tilde{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i = 0) = 0$ for all i and the accumulation points of $\tilde{\mathbf{b}}(\cdot)$ are log-optimum with probability one. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \tilde{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i = E \ln \mathbf{b}^*\mathbf{X} \qquad \text{a. s.}$$

Proof. Let $\mathbf{b}^{*\prime}$ be a log-optimum portfolio such that $b_j^{*\prime} = 0 \Rightarrow b_j^* = 0$ for $j = 1, 2, \ldots, m$ and for all $\mathbf{b}^* \in B^*$, where B^* denotes the set of log-optimal portfolios. Such a portfolio exists, since suppose $b_{1,j}^* = 0$ and $b_{2,j}^* \neq 0$ for some j. Then for any $\lambda \in (0,1), \lambda \mathbf{b}_1^* + (1-\lambda)\mathbf{b}_2^* \in B^*$ and contains less number of zeros than $\mathbf{b}_1^*$ does. (Note $E(\lambda \mathbf{b}_1^*\mathbf{X} + (1-\lambda)\mathbf{b}_2^*\mathbf{X}) = E \ln \mathbf{b}_1^*\mathbf{X} = E \ln \mathbf{b}_2^*\mathbf{X}$ by Lemma 2.) If this new portfolio does not satisfy the condition we can repeat this procedure. After at most m steps we get a proper portfolio.

Since $\mathbf{b}^*\mathbf{X} = \mathbf{b}^{*\prime}\mathbf{X}$ a. s. (see Lemma 2),

$$\frac{1}{n} \sum_{i=1}^{n} \ln \frac{\tilde{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i}{\mathbf{b}^*\mathbf{X}_i} = \frac{1}{n} \sum_{i=1}^{n} \ln \frac{\tilde{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i}{\mathbf{b}^{*\prime}\mathbf{X}_i} =$$
$$\frac{1}{n} \sum_{i=1}^{n} \ln \left(\frac{\tilde{\mathbf{b}}^*(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i}{\mathbf{b}^{*\prime}\mathbf{X}_i} + \right.$$
$$\left. \frac{\tilde{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i - \tilde{\mathbf{b}}^*(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i}{\mathbf{b}^{*\prime}\mathbf{X}_i} \right)$$

where $\tilde{\mathbf{b}}^*(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})$ denotes the closest log-optimal portfolio to $\tilde{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})$ in Euclidean distance. (Such a portfolio exists since the set of log-optimal portfolios is closed by Lemma 4.) Thus

$$\frac{1}{n}\sum_{i=1}^{n} \ln \frac{\tilde{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i}{\mathbf{b}^*\mathbf{X}_i} =$$

$$\frac{1}{n}\sum_{i=1}^{n} \ln \left(1 + \frac{\left(\tilde{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1}) - \tilde{\mathbf{b}}^*(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\right)\mathbf{X}_i}{\mathbf{b}^{*\prime}\mathbf{X}_i}\right) \geq$$

$$\frac{1}{n}\sum_{i=1}^{k(\omega)} \ln \frac{\tilde{\mathbf{b}}(\mathbf{X}_1(\omega), \mathbf{X}_2(\omega), \ldots, \mathbf{X}_{i-1}(\omega))\mathbf{X}_i(\omega)}{\mathbf{b}^{*\prime}\mathbf{X}_i(\omega)} + \frac{1}{n}\sum_{i=k(\omega)+1}^{n} \ln\left(1 - \frac{\epsilon \mathbf{a}\mathbf{X}_i(\omega)}{\mathbf{b}^{*\prime}\mathbf{X}_i(\omega)}\right),$$

where $a_j = 1$ if $b_j^{*\prime} \neq 0$, $a_j = 0$ if $b_j^{*\prime} = 0$ and $k(\omega)$ is an integer such that $\|\tilde{\mathbf{b}}(\mathbf{X}_1(\omega), \mathbf{X}_2(\omega), \ldots, \mathbf{X}_i(\omega)) - \tilde{\mathbf{b}}^*(\mathbf{X}_1(\omega), \mathbf{X}_2(\omega), \ldots, \mathbf{X}_i(\omega))\| < \epsilon$ for $i > k(\omega)$, where $0 < \epsilon < 0.5 \min_{j \in I} b_j^{*\prime}$, $I = \{j : b_j^{*\prime} \neq 0\}$.

Thus

$$\frac{1}{n}\sum_{i=1}^{n} \ln \frac{\tilde{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i}{\mathbf{b}^*\mathbf{X}_i} \geq$$

$$\frac{1}{n}\sum_{i=1}^{k(\omega)} \ln \frac{\tilde{\mathbf{b}}(\mathbf{X}_1(\omega), \mathbf{X}_2(\omega), \ldots, \mathbf{X}_{i-1}(\omega))\mathbf{X}_i(\omega)}{\mathbf{b}^{*\prime}\mathbf{X}_i(\omega)}$$

$$-\frac{1}{n}\sum_{i=1}^{k(\omega)} \ln\left(1 - \frac{\epsilon \mathbf{a}\mathbf{X}_i(\omega)}{\mathbf{b}^{*\prime}\mathbf{X}_i(\omega)}\right) + \frac{1}{n}\sum_{i=1}^{n} \ln\left(1 - \frac{\epsilon \mathbf{a}\mathbf{X}_i(\omega)}{\mathbf{b}^{*\prime}\mathbf{X}_i(\omega)}\right).$$

Thus

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \ln \frac{\tilde{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i}{\mathbf{b}^*\mathbf{X}_i} \geq E \ln\left(1 - \frac{\epsilon \mathbf{a}\mathbf{X}}{\mathbf{b}^{*\prime}\mathbf{X}}\right) \qquad \text{a. s.}$$

Expanding the function $\ln\left(1 - \frac{y\mathbf{a}\mathbf{X}}{\mathbf{b}^{*\prime}\mathbf{X}}\right)$ into Taylor series around 0 in the interval $[0, \epsilon]$, we have, $\left|\ln\left(1 - \frac{y\mathbf{a}\mathbf{X}}{\mathbf{b}^{*\prime}\mathbf{X}}\right)\right| = \left|\ln(1) + \frac{-y\mathbf{a}\mathbf{X}}{\mathbf{b}^{*\prime}\mathbf{X} - t\mathbf{a}\mathbf{X}}\right|$ for some $t \in [0, \epsilon]$.

Thus

$$\left|\ln\left(1 - \frac{\epsilon \mathbf{a}\mathbf{X}}{\mathbf{b}^{*\prime}\mathbf{X}}\right)\right| \leq \frac{\epsilon \mathbf{a}\mathbf{X}}{\mathbf{b}^{*\prime}\mathbf{X} - t\mathbf{a}\mathbf{X}} \leq \frac{\epsilon \mathbf{a}\mathbf{X}}{\mathbf{b}^{*\prime}\mathbf{X} - \epsilon\mathbf{a}\mathbf{X}} \leq \frac{\epsilon \mathbf{a}\mathbf{X}}{0.5\mathbf{b}^{*\prime}\mathbf{X}} = \frac{2\epsilon \mathbf{a}\mathbf{X}}{\mathbf{b}^{*\prime}\mathbf{X}}.$$

Since $E\frac{X_j}{\mathbf{b}^*\mathbf{X}} \leq 1$ for $j = 1, 2, \ldots, m$ by log-optimality (see Bell and Cover [7]), hence

$$E\frac{2\epsilon \mathbf{a}\mathbf{X}}{\mathbf{b}^{*\prime}\mathbf{X}} \leq 2\epsilon m < \infty.$$

Since ϵ was arbitrary,

$$\lim_{\epsilon \to 0} E \ln \left(1 - \frac{\epsilon \mathbf{aX}}{\mathbf{b}^{*\prime}\mathbf{X}} \right) = E \ln 1 = 0$$

by the Lebesgue dominated convergence theorem. The upper bound follows similarly,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \frac{\tilde{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i}{\mathbf{b}^*\mathbf{X}_i} =$$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \left(1 + \frac{\left(\tilde{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1}) - \tilde{\mathbf{b}}^*(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1}) \right) \mathbf{X}_i}{\mathbf{b}^{*\prime}\mathbf{X}_i} \right) \leq$$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \left(1 + \frac{\epsilon \mathbf{eX}_i}{\mathbf{b}^{*\prime}\mathbf{X}_i} \right) = E \ln \left(1 + \frac{\epsilon \mathbf{eX}}{\mathbf{b}^{*\prime}\mathbf{X}} \right),$$

where $\mathbf{e} = (1, 1, \ldots, 1)$. Since ϵ was arbitrary,

$$\lim_{\epsilon \to 0} E \ln \left(1 + \frac{\epsilon \mathbf{eX}}{\mathbf{b}^{*\prime}\mathbf{X}} \right) = 0 \qquad \text{a. s.,}$$

by the Lebesgue dominated convergence theorem. Hence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \tilde{\mathbf{b}}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i = E \ln \mathbf{b}^*\mathbf{X} \qquad \text{a. s.}$$

Proof of Theorem 1. The accumulation points of $\hat{\mathbf{b}}(\cdot)$ are log-optimal by Lemma 8. Then the theorem follows from Lemma 9.

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Эмпирическая лог-оптимальная селекция портфеля

Г. МОРВАИ

(Будапешт)

Показано, что эмпирический лог-оптимальный портфель ведет себя асимптотически как оптимально при некоторых условиях.

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