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PORTFOLIO CHOICE BASED ON THE EMPIRICAL DISTRIBUTION

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It is shown that a slightly modified version of the empirical log-optimal portfolio selector achieves the asymptotically optimal growth rate of capital on independent and identically distributed random stock market return vectors.

1. INTRODUCTION

Let $\underline{X} \in \mathsf{R}^m$ denote a random stock market return vector, where X_j is the value of a one unit investment in stock j at the end of the trading day. We require that $X_j \geq 0$ for $j = 1, 2, \ldots, m$, that is, an investor cannot lose more than the invested capital. Let $\underline{b}$, $b_j \geq 0, \sum_{j=1}^m b_j = 1$, denote a portfolio, that is, an allocation of investor's capital across the investment alternatives. Let B denote the set of such portfolios. Thus b_j is the proportion of current capital invested in stock j. The resulting wealth is $S = \sum_{j=1}^m b_j X_j = \underline{bX}$. This is the wealth resulting from a unit investment allocated to the m stocks according to portfolio $\underline{b}$. If the current capital is reallocated according to portfolio $\underline{b}_i$ at time i in repeated investments against stock vectors $\underline{X}_1, \underline{X}_2, \ldots$ then the wealth S_n at time n is given by

$$S_n = \prod_{i=1}^n \underline{b}_i \underline{X}_i.$$

Suppose the stock market return vectors $\underline{X}_1, \underline{X}_2, \ldots$ are independent and identically distributed. A portfolio $\underline{b}^*$ is called log-optimal if $\mathsf{E} \ln \underline{b}^* \underline{X} = \sup_{\underline{b} \in B} \mathsf{E} \ln \underline{bX}$. Let B^* denote the set of log-optimal portfolios. Since the portfolio selection may depend on the past outcomes, an investment scheme can be described as a portfolio selector

$$\left\{\underline{b}_n\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}\right)\right\}_{n=1}^\infty,$$

that is, a series of measurable functions $\underline{b}_n\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}\right)$ mapping from the past outcomes of the stock market return vectors to the set of portfolios. It can be shown that

$$\limsup_{n\to\infty} \frac{1}{n} \ln S_n \leq \lim_{n\to\infty} \frac{1}{n} \ln S_n^* = \mathsf{E} \ln \underline{b}^* \underline{X} \qquad \text{a.s.},$$

where $S_n = \prod_{i=1}^n \underline{b}_i(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1})\underline{X}_i$ and $S_n^* = \prod_{i=1}^n \underline{b}^* \underline{X}_i$ denote capitals achieved by an arbitrary portfolio selector $\left\{\underline{b}_n\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}\right)\right\}_{n=1}^\infty$ and the log-optimal portfolio $\underline{b}^*$ in n repeated games, respectively. (That is, $\sup_{\underline{b} \in B} \mathsf{E} \ln \underline{bX}$ is the highest asymptotic growth rate of capital a portfolio selector may achieve. See Algoet and Cover [1].) For more about the log-optimal portfolio see [1], [3]–[11], and [13].

If the probability distribution of the stocks is not known in advance, consider as a goal to find a portfolio selector $\{\underline{\hat{b}}_n(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1})\}_{n=1}^{\infty}$ which achieves the asymptotically optimal growth rate of capital, that is,

$$\lim_{n\to\infty} \frac{1}{n} \ln \hat{S}_n = \mathsf{E} \ln \underline{b}^* \underline{X} \qquad \text{a.s.},$$

where $\hat{S}_n = \prod_{i=1}^{n} \underline{\hat{b}}_i(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1})\underline{X}_i$.

2. THE PROPOSED PORTFOLIO SELECTOR

Let $\{\underline{b}_n(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1})\}_{n=1}^{\infty}$ be a measurable selector of the empirical log-optimal portfolios, that is,

$$\underline{b}_1 = (1/m, 1/m, \ldots, 1/m)$$
$$\underline{b}_n(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}) \in \arg\max_{\underline{b}\in B} \tfrac{1}{n-1} \sum_{i=1}^{n-1} \ln \underline{b}\underline{X}_i \qquad \text{if } n \geq 2.$$

The proposed portfolio selector $\{\underline{\hat{b}}_n(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1})\}_{n=1}^{\infty}$ is defined by

$$\underline{\hat{b}}_n(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}) = (1-\lambda_n)\underline{b}_n(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}) + \lambda_n \underline{e},$$

where $\lim_{n\to\infty} \lambda_n = 0$, $\lambda_n \in (0,1)$ for all n, and $\underline{e} = (1/m, 1/m, \ldots, 1/m)$. That is, the empirical log-optimal portfolio is combined with the uniform one.

The following theorem says that the asymptotically optimal growth rate of capital is achieved by the proposed portfolio selector if the random stock market return vectors are independent and identically distributed. The portfolio selector proposed in Cover [8] achieves this goal also but our selector is simpler. It has been shown in Morvai [13] that even the pure empirical log-optimal portfolio selector $\{\underline{b}_n(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1})\}_{n=1}^{\infty}$ achieves the asymptotically optimal growth rate of capital if the random stock market return vectors are independent, identically distributed, and none of the stocks X_j, $j = 1, 2, \ldots, m$, may take on the value of zero.

Theorem 1. Suppose the random stock market return vectors $\underline{X}_1, \underline{X}_2, \ldots$ are independent, identically distributed, and $-\infty < \sup_{\underline{b}\in B} \mathsf{E} \ln \underline{b}\underline{X} < \infty$. (Note $\mathsf{E} \ln \underline{b}^* \underline{X} = \sup_{\underline{b}\in B} \mathsf{E} \ln \underline{b}\underline{X}$.) Then

$$\lim_{n\to\infty} \frac{1}{n} \ln \hat{S}_n = \mathsf{E} \ln \underline{b}^* \underline{X} \qquad \text{a.s.},$$

where $\hat{S}_n = \prod_{i=1}^{n} \underline{\hat{b}}_i(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1})\underline{X}_i$.

3. PERFORMANCE ANALYSIS OF THE PROPOSED PORTFOLIO SELECTOR

Here we prove several lemmas in order to be able to prove Theorem 1.

Definition. Consider a function $h(\cdot) : \mathsf{R}^n \to \mathsf{R} \cup \{\infty\}$. epi $h(\cdot)$ denotes the set $\{(\underline{y}, \alpha) \in \mathsf{R}^n \times \mathsf{R} : h(\underline{y}) \leq \alpha\}$.

Lemma 1. Let $(\Xi,\mathcal{U},\mathcal{P})$ be a probability space, where $\Xi = \mathsf{R}^m$, and $\mathcal{U}$ denotes the Borel σ-algebra completed with respect to $\mathcal{P}$. Let Θ denote the set $\{\underline{x}\in \mathsf{R}^m : x_i \geq 0 \quad \text{for} \quad i = 1, 2, \dots, m\}$. Consider the function

$$f\left(\underline{b},\underline{x}\right)=\begin{cases}-\ln \underline{b}\underline{x} & : \underline{b}\in B, \text{ and } \underline{x}\in\Theta\\ \infty & : \text{otherwise}\end{cases},$$

that is, $f\left(\,\cdot\,,\,\cdot\,\right) : \mathsf{R}^m\times\Xi\to\mathsf{R}\cup\{\infty\}$. Then
a) for each $\underline{x}\in\Xi$ the set $\operatorname{epi} f\left(\,\cdot\,,\underline{x}\right)=\{(\underline{b},\alpha)\in\mathsf{R}^m\times\mathsf{R} : f\left(\underline{b},\underline{x}\right)\leq\alpha\}$ is convex,
b) the set $\operatorname{epi} f\left(\,\cdot\,,\underline{x}\right)$ is closed, and
c) $\{\underline{x}\in\Xi : \operatorname{epi} f\left(\,\cdot\,,\underline{x}\right)\cap F\neq\emptyset\}\in\mathcal{U}$ for all closed subsets $F\subseteq\mathsf{R}^{m+1}$.
d) Consider the product probability space $\left(\Xi^{(n)},\mathcal{U}^{(n)},\mathcal{P}^{(n)}\right)$.
Then for each $(\underline{x}_1,\underline{x}_2,\dots,\underline{x}_n)\in\Xi^{(n)}$ the set $\operatorname{epi}\frac{1}{n}\sum_{i=1}^n f\left(\,\cdot\,,\underline{x}_i\right)$ is closed, and
e) $\{(\underline{x}_1,\underline{x}_2,\dots,\underline{x}_n)\in\Xi^{(n)} : \operatorname{epi}\frac{1}{n}\sum_{i=1}^n f\left(\,\cdot\,,\underline{x}_i\right)\cap F\neq\emptyset\}\in\mathcal{U}^{(n)}$ for all closed subsets $F\subseteq\mathsf{R}^{m+1}$.

Proof. a) If $\underline{x}\notin\Theta$ then $\operatorname{epi} f\left(\,\cdot\,,\underline{x}\right)$ is empty, hence it is trivially convex. Suppose $\underline{x}\in\Theta$. If $(\underline{b}_1,\alpha_1)$, $(\underline{b}_2,\alpha_2)\in\operatorname{epi} f\left(\,\cdot\,,\underline{x}\right)$ then $\underline{b}_1,\underline{b}_2\in B$, and $\alpha_1\geq-\ln\underline{b}_1\underline{x}$, $\alpha_2\geq-\ln\underline{b}_2\underline{x}$. Furthermore,

$$\lambda\alpha_1+(1-\lambda)\,\alpha_2\geq-\lambda\ln\underline{b}_1\underline{x}-(1-\lambda)\ln\underline{b}_2\underline{x}=$$

$$-\left(\lambda\ln\underline{b}_1\underline{x}+(1-\lambda)\ln\underline{b}_2\underline{x}\right)\geq-\ln\left(\left(\lambda\underline{b}_1+(1-\lambda)\,\underline{b}_2\right)\underline{x}\right)$$

by Jensen's inequality. Thus the set $\operatorname{epi} f\left(\,\cdot\,,\underline{x}\right)$ is convex.
b) If $\underline{x}\notin\Theta$ then $\operatorname{epi} f\left(\,\cdot\,,\underline{x}\right)$ is empty.
Suppose $\underline{x}\in\Theta$. If $\left(\underline{b}',\alpha'\right)$ is from the boundary of the set $\operatorname{epi} f\left(\,\cdot\,,\underline{x}\right)$ then let $(\underline{b}_i,\alpha_i)$ converge to $\left(\underline{b}',\alpha'\right)$ such that $(\underline{b}_i,\alpha_i)\in\operatorname{epi} f\left(\,\cdot\,,\underline{x}\right)$. (Thus $\underline{b}_i,\underline{b}'\in B$, $\alpha_i,\alpha'\in\mathsf{R}$.) Let $\varepsilon>0$ be arbitrary. Since α' is finite, $\alpha_i\leq\alpha'+\varepsilon$ for large i. Thus
$-\ln\underline{b}_i\underline{x}\leq\alpha_i\leq\alpha'+\varepsilon$ for large i.
The function $-\ln\underline{b}\underline{x}$ is continuous in $\underline{b}\in B$, hence

$$-\ln\underline{b}'\underline{x}=\lim_{i\to\infty}-\ln\underline{b}_i\underline{x}\leq\alpha'+\varepsilon.$$

Since ε was arbitrary, $-\ln\underline{b}'\underline{x}\leq\alpha'$, that is, $\left(\underline{b}',\alpha'\right)\in\operatorname{epi} f\left(\,\cdot\,,\underline{x}\right)$.

c) Suppose G is a bounded and closed set from R^{m+1}.
Let $g_G\left(\underline{x}\right)=\max\limits_{(\underline{b},\alpha)\in B\times\mathsf{R}\cap G}-f\left(\underline{b},\underline{x}\right)+\alpha=\sup\limits_{(\underline{b},\alpha)\in D}-f\left(\underline{b},\underline{x}\right)+\alpha$, where D is a countable dense subset of $B\times\mathsf{R}\cap G$. (We used the continuity of the function $\ln\underline{b}\underline{x}$, and the compactness of the set $B\times\mathsf{R}\cap G$.) Since for each $\underline{b}\in B$, $\alpha\in\mathsf{R}$, the function $f\left(\underline{b},\underline{x}\right)+\alpha$ is measurable thus the function $\sup\limits_{(\underline{b},\alpha)\in D} f\left(\underline{b},\underline{x}\right)+\alpha$ is measurable as well. (The supremum is taken over a countable set.)
Let F denote a closed set from R^{m+1}. The following statements are equivalent:

$$\{\underline{x}\in\Xi\,:\,\operatorname{epi} f\left(\,\cdot\,,\underline{x}\right)\cap F\neq\emptyset\}\in\mathcal{U}.$$

$$\{\underline{x}\in\Theta\,:\,\{(\underline{b},\alpha)\in B\times\mathsf{R}\,:\,-\ln\underline{b}\underline{x}\leq\alpha\}\cap F\neq\emptyset\}\in\mathcal{U}.$$

$$\bigcup_i\{\underline{x}\in\Theta\,:\,\{(\underline{b},\alpha)\in B\times\mathsf{R}\,:\,-\ln\underline{b}\underline{x}\leq\alpha\}\cap F_i\neq\emptyset\}\in\mathcal{U},$$

where F_i are bounded and closed sets (countable many) such that $\bigcup\limits_i F_i=F$.

$$\bigcup_i\{\underline{x}\in\Theta\,:\,\max_{(\underline{b},\alpha)\in B\times\mathsf{R}\cap F_i}\ln\underline{b}\underline{x}+\alpha\geq 0\}\in\mathcal{U}.$$

(We used the continuity of the function $\ln \underline{bx} + \alpha$, and the compactness of the set $B \times \mathsf{R} \cap F_i$.)

$$\bigcup_i \{\underline{x} \in \Xi : \max_{(\underline{b},\alpha)\in B\times\mathsf{R}\cap F_i} -f(\underline{b},\underline{x}) + \alpha \geq 0\} \in \mathcal{U}.$$

$$\bigcup_i \{\underline{x} \in \Xi : g_{F_i}(\underline{x}) \geq 0\} \in \mathcal{U}.$$

We have already proved that $\{\underline{x} \in \Xi : g_{F_i}(\underline{x}) \geq 0\} \in \mathcal{U}$. (The function $g_{F_i}(\underline{x})$ is measurable.) Thus

$$\bigcup_i \{\underline{x} \in \Xi : g_{F_i}(\underline{x}) \geq 0\} \in \mathcal{U}.$$

d) Similar argument works as in b. e) Similar argument works as in c. □

Lemma 2. Suppose $X_i \geq 0$ a.s. for $i = 1, 2, \ldots, m$ and $-\infty < \sup_{\underline{b}\in B} \mathsf{E} \ln \underline{bX} < \infty$. Let $\underline{\tilde{b}} = (1/m, 1/m, \ldots, 1/m)$. Then $\mathsf{E}f\left(\underline{\tilde{b}}, \underline{X}\right) < \infty$, and there exists a measurable function $\underline{u}(\cdot) : \Xi \to \mathsf{R}^m$ such that

$$f(\underline{b},\underline{x}) \geq f\left(\underline{\tilde{b}},\underline{x}\right) + \underline{u}(\underline{x})\left(\underline{b} - \underline{\tilde{b}}\right) \qquad \text{for all } \underline{b} \in \mathsf{R}^m, \underline{x} \in \Xi, \text{and}$$
$$\mathsf{E} \parallel \underline{u}(\underline{X}) \parallel < \infty.$$

P r o o f. Since

$$-\infty < \sup_{\underline{b}\in B} \mathsf{E} \ln \underline{bX} \leq \mathsf{E} \ln \sum_{i=1}^{m} X_i = \mathsf{E} \ln \left(m \sum_{i=1}^{m} \frac{1}{m} X_i \right) = \mathsf{E} \ln \underline{\tilde{b}X} + \ln(m),$$

hence $\mathsf{E}\left(-\ln \underline{\tilde{b}X}\right) < \infty$. Thus $\mathsf{E}f\left(\underline{\tilde{b}}, \underline{X}\right) < \infty$.

Let $\underline{u}(\underline{x}) = \begin{cases} -\underline{x}/\underline{\tilde{b}x} & : \text{ if } \underline{x} \neq \underline{0} \text{ and } \underline{x} \in \Theta \\ \underline{0} & : \text{ otherwise} \end{cases}$. Obviously, $\underline{u}(\underline{X})$ is measurable.

If $\underline{x} = \underline{0}$ or $\underline{x} \notin \Theta$ then $\infty \geq \infty + 0$.

If $\underline{b} \notin B, \underline{x} \neq \underline{0}$ and $\underline{x} \in \Theta$ then $\infty \geq \ln \underline{\tilde{b}x} + -\underline{x}/\underline{\tilde{b}x}\left(\underline{b} - \underline{\tilde{b}}\right)$, since $\ln \underline{\tilde{b}x}$ is finite.

If $\underline{b} \in B$, $\underline{x} \neq \underline{0}$ and $\underline{x} \in \Theta$ then

$$-\ln \underline{bx} + \ln \underline{\tilde{b}x} = -\ln \frac{\underline{bx}}{\underline{\tilde{b}x}} = -\ln \left(\frac{\left(\underline{b} - \underline{\tilde{b}}\right)\underline{x} + \underline{\tilde{b}x}}{\underline{\tilde{b}x}} \right) =$$

$$-\ln \left(\frac{\left(\underline{b} - \underline{\tilde{b}}\right)\underline{x}}{\underline{\tilde{b}x}} + 1 \right) \geq -\frac{\left(\underline{b} - \underline{\tilde{b}}\right)\underline{x}}{\underline{\tilde{b}x}} = \frac{-\underline{x}}{\underline{\tilde{b}x}}\left(\underline{b} - \underline{\tilde{b}}\right).$$

Furthermore, $\mathsf{E} \parallel -\underline{X}/\underline{\tilde{b}X} \parallel \leq m^3 < \infty$. □

Lemma 3. (A.J. King and R.J.-B. Wets [12]) Let $\underline{Y}$ be a directly given random variable on the probability space $(\Gamma, \mathcal{A}, \mathcal{Q})$, where $\mathcal{A}$ denotes a σ-algebra completed with respect to $\mathcal{Q}$. Consider the following assumptions:

Assumption A. $g(\cdot,\cdot) : \mathsf{R}^k \times \Gamma \to \mathsf{R} \cup \{\infty\}$ is a convex normal integrand, that is,

(i) the set $\operatorname{epi} g\left(\cdot, \underline{y}\right)$ is closed for each $\underline{y} \in \Gamma$,

(ii) the set valued mapping $\underline{y} \to \operatorname{epi} g\left(\cdot, \underline{y}\right)$ is measurable, that is, for all closed subsets $F \subseteq \mathsf{R}^{k+1}$, $\{\underline{y} \in \Gamma : \operatorname{epi} g\left(\cdot, \underline{y}\right) \cap F \neq \emptyset\} \in \mathcal{A}$,

(iii) the set $\operatorname{epi} g\left(\cdot, \underline{y}\right)$ is convex for each $\underline{y} \in \Gamma$, and it is not empty a.s.

Assumption B. There exist a $\underline{\tilde{c}} \in \mathsf{R}^k$ such that $\int g\left(\underline{\tilde{c}}, \underline{y}\right) \mathcal{Q}\left(d\underline{y}\right)$ is finite, and a measurable function $\underline{u}(\cdot) : \Gamma \to \mathsf{R}^k$ such that

(i) $g\left(\underline{c}, \underline{y}\right) \geq g\left(\underline{\tilde{c}}, \underline{y}\right) + \underline{u}\left(\underline{y}\right)\left(\underline{c} - \underline{\tilde{c}}\right)$ for all $\underline{c} \in \mathsf{R}^k$, $\underline{y} \in \Gamma$, and
(ii) $\int \| \underline{u}\left(\underline{y}\right) \| \mathcal{Q}\left(\mathrm{d}\underline{y}\right)$ is finite.

Assumption C. The random variables $\underline{Y}_i$ are independent and identically distributed.

Under assumptions A, B, and C if $\underline{\hat{c}}$ is a cluster point of any sequence $\{\underline{c}_n\}$ with $\underline{c}_n \in \arg\min_{\underline{c} \in \mathsf{R}^k} \frac{1}{n} \sum_{i=1}^{n} g\left(\underline{c}, \underline{Y}_i\right)$ then $\underline{\hat{c}} \in \arg\min_{\underline{c} \in \mathsf{R}^k} \mathsf{E} g\left(\underline{c}, \underline{Y}\right)$ a.s.

P r o o f. See King and Wets [12] Theorem 2.3 and Proposition 2.1, or Wets [15] Theorem 3.4. $\square$

Lemma 4. Suppose that the random stock market return vectors $\underline{X}_i$ are independent, identically distributed and $-\infty < \sup_{\underline{b} \in B} \mathsf{E} \ln \underline{b}\underline{X} < \infty$. Then there exists a measurable selector $\left\{\underline{b}_n\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}\right)\right\}_{n=1}^{\infty}$ such that

$$\underline{b}_n\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}\right) \in \arg\max_{\underline{b} \in B} \frac{1}{n-1} \sum_{i=1}^{n-1} \ln \underline{b}\underline{X}_i$$

and the cluster points of $\left\{\underline{b}_n\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}\right)\right\}_{n=1}^{\infty}$ are log-optimal a.s.

P r o o f. The existence of the measurable selector $\left\{\underline{b}_n\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}\right)\right\}_{n=1}^{\infty}$ follows from Lemma 1, Rockafellar [14] Theorems 1C and 2K, and the fact that

$$\arg\max_{\underline{b} \in B} \frac{1}{n} \sum_{i=1}^{n} \ln \underline{b}\underline{X}_i = \arg\min_{\underline{b} \in B} \frac{1}{n} \sum_{i=1}^{n} - \ln \underline{b}\underline{X}_i = \arg\min_{\underline{b} \in \mathsf{R}^m} \frac{1}{n} \sum_{i=1}^{n} f\left(\underline{b}, \underline{X}_i\right) \text{ a.s.}$$

Since $\frac{1}{n} \sum_{i=1}^{n} - \ln \underline{b}\underline{X}_i = \frac{1}{n} \sum_{i=1}^{n} f\left(\underline{b}, \underline{X}_i\right)$ for all $\underline{b} \in B$ a.s., the log-optimality of the cluster points follows immediately from Lemmas 1,2 and 3. $\square$

Lemma 5. Suppose that the random stock market return vectors $\underline{X}_1, \underline{X}_2, \ldots$ are independent and identically distributed. Then the cluster points of the empirical log-optimal portfolio selector $\left\{\underline{b}_n\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}\right)\right\}_{n=1}^{\infty}$ and the proposed portfolio selector $\left\{\underline{\hat{b}}_n\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}\right)\right\}_{n=1}^{\infty}$ coincide and hence the cluster points of the sequence
$\left\{\underline{\hat{b}}_n\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}\right)\right\}_{n=1}^{\infty}$ are log-optimal with probability one.

P r o o f. Suppose $\lim_{n \to \infty} \underline{b}_{i_n}\left(\underline{X}_1(\omega), \underline{X}_2(\omega), \ldots, \underline{X}_{i_n-1}(\omega)\right) = \underline{b}'$. Since

$$\underline{\hat{b}}_{i_n}\left(\underline{X}_1(\omega), \underline{X}_2(\omega), \ldots, \underline{X}_{i_n-1}(\omega)\right) =$$
$$\left(1 - \lambda_{i_n}\right) \underline{b}_{i_n}\left(\underline{X}_1(\omega), \underline{X}_2(\omega), \ldots, \underline{X}_{i_n-1}(\omega)\right) + \lambda_{i_n} \underline{e}$$

and $\lambda_{i_n} \to 0$, hence

$$\lim_{n \to \infty} \underline{\hat{b}}_{i_n}\left(\underline{X}_1(\omega), \underline{X}_2(\omega), \ldots, \underline{X}_{i_n-1}(\omega)\right) =$$
$$\lim_{n \to \infty} \underline{b}_{i_n}\left(\underline{X}_1(\omega), \underline{X}_2(\omega), \ldots, \underline{X}_{i_n-1}(\omega)\right) = \underline{b}'.$$

The other direction follows similarly.
Suppose $\lim_{n \to \infty} \underline{\hat{b}}_{i_n}\left(\underline{X}_1(\omega), \underline{X}_2(\omega), \ldots, \underline{X}_{i_n-1}(\omega)\right) = \underline{b}'$.

$$\lim_{n\to\infty} \underline{b}_{i_n}\left(\underline{X}_1(\omega), \underline{X}_2(\omega), \ldots, \underline{X}_{i_n-1}(\omega)\right) =$$

$$\lim_{n\to\infty} \frac{1}{1-\lambda_{i_n}} \underline{\hat{b}}_{i_n}\left(\underline{X}_1(\omega), \underline{X}_2(\omega), \ldots, \underline{X}_{i_n-1}(\omega)\right) - \frac{\lambda_{i_n}}{1-\lambda_{i_n}} \underline{e} = \underline{b}'.$$

Now the log-optimality follows from Lemma 4. □

Lemma 6. Let the random stock market return vectors $\underline{X}_1, \underline{X}_2, \ldots$ be independent, identically distributed, and $-\infty < \sup_{\underline{b}\in B} \mathsf{E} \ln \underline{b}\underline{X} < \infty$. Suppose $\{\underline{\tilde{b}}_n\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}\right)\}_{n=1}^{\infty}$ is a measurable selector of portfolios such that $\underline{\tilde{b}}_n(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}) \in \mathrm{int}B$ for all n, and the cluster points of $\{\underline{\tilde{b}}_n\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}\right)\}_{n=1}^{\infty}$ are log-optimal a.s. (Here $\mathrm{int}B$ denotes the interior of the set B.) Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \underline{\tilde{b}}_i\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1}\right) \underline{X}_i = \mathsf{E} \ln \underline{b}^* \underline{X} \qquad \text{a.s.}$$

P r o o f. Let $\underline{b}^{*\prime}$ be a log-optimal portfolio such that $b_j^{*\prime} = 0$ implies $b_j^* = 0$ for all $\underline{b}^* \in B^*$, $j = 1, 2, \ldots, m$, where B^* denotes the set of log-optimal portfolios. Such a portfolio exists, since suppose $b_{1,j}^* = 0$ and $b_{2,j}^* \neq 0$ for some j. Then for any $\lambda \in (0,1)$, $\lambda\underline{b}_1^* + (1-\lambda)\underline{b}_2^* \in B^*$ and contains less number of zeros than $\underline{b}_1^*$ does. (Note that the set of log-optimum portfolios form a convex set. See Cover [7].) If this new portfolio does not satisfy the condition we can repeat this prucedure. After at most m steps we get a proper portfolio.

Since there exists a set L such that $P(\underline{X} \in L) = 1$ and $\underline{b}_1^* \underline{x} = \underline{b}_2^* \underline{x}$ if $\underline{x} \in L$, $\underline{b}_1^*$, $\underline{b}_2^* \in B^*$ (see Cover [7]),

$$\frac{1}{n} \sum_{i=1}^{n} \ln \frac{\underline{\tilde{b}}_i\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1}\right) \underline{X}_i}{\underline{b}^* \underline{X}_i} = \frac{1}{n} \sum_{i=1}^{n} \ln \frac{\underline{\tilde{b}}_i\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1}\right) \underline{X}_i}{\underline{b}^{*\prime} \underline{X}_i} =$$

$$\frac{1}{n} \sum_{i=1}^{n} \ln \left(\frac{\underline{\tilde{b}}_i^*\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1}\right) \underline{X}_i}{\underline{b}^{*\prime} \underline{X}_i} + \right.$$

$$\left. \frac{\underline{\tilde{b}}_i\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1}\right) \underline{X}_i - \underline{\tilde{b}}_i^*(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1})\underline{X}_i}{\underline{b}^{*\prime} \underline{X}_i} \right)$$

where $\underline{\tilde{b}}_i^*(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1})$ denotes the closest log-optimal portfolio to $\underline{\tilde{b}}_i\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1}\right)$ in Euclidean distance. (Such a portfolio exists since the set of log-optimal portfolios $B^* \subseteq B$ is closed by the continuity of the function $\mathsf{E} \ln \underline{b}\underline{X}$.) Thus

$$\frac{1}{n} \sum_{i=1}^{n} \ln \frac{\underline{\tilde{b}}_i\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1}\right) \underline{X}_i}{\underline{b}^* \underline{X}_i} =$$

$$\frac{1}{n} \sum_{i=1}^{n} \ln \left(1 + \frac{\left(\underline{\tilde{b}}_i\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1}\right) - \underline{\tilde{b}}_i^*\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1}\right)\right) \underline{X}_i}{\underline{b}^{*\prime} \underline{X}_i} \right) \geq$$

$$\frac{1}{n} \sum_{i=1}^{k(\omega)} \ln \frac{\underline{\tilde{b}}_i\left(\underline{X}_1(\omega), \underline{X}_2(\omega), \ldots, \underline{X}_{i-1}(\omega)\right) \underline{X}_i(\omega)}{\underline{b}^{*\prime} \underline{X}_i(\omega)} + \frac{1}{n} \sum_{i=k(\omega)+1}^{n} \ln \left(1 - \frac{\varepsilon \underline{a} \underline{X}_i(\omega)}{\underline{b}^{*\prime} \underline{X}_i(\omega)} \right),$$

where $a_j = 1$ if $b_j^{*\prime} \neq 0$, $a_j = 0$ if $b_j^{*\prime} = 0$ and $k(\omega)$ is an integer such that $\| \tilde{\underline{b}}_i \left(\underline{X}_1(\omega), \underline{X}_2(\omega), \dots, \underline{X}_{i-1}(\omega)\right) - \tilde{\underline{b}}_i^* \left(\underline{X}_1(\omega), \underline{X}_2(\omega), \dots, \underline{X}_{i-1}(\omega)\right) \| < \varepsilon$ for $i > k(\omega)$, where $0 < \varepsilon < \frac{1}{2} \min_{j \in I} b_j^{*\prime}$, and $I = \{j : b_j^{*\prime} \neq 0\}$.

Thus

$$\frac{1}{n} \sum_{i=1}^{n} \ln \frac{\tilde{\underline{b}}_i \left(\underline{X}_1, \underline{X}_2, \dots, \underline{X}_{i-1}\right) \underline{X}_i}{\underline{b}^* \underline{X}_i} \geq$$

$$\frac{1}{n} \sum_{i=1}^{k(\omega)} \ln \frac{\tilde{\underline{b}}_i \left(\underline{X}_1(\omega), \underline{X}_2(\omega), \dots, \underline{X}_{i-1}(\omega)\right) \underline{X}_i(\omega)}{\underline{b}^{*\prime} \underline{X}_i(\omega)}$$

$$-\frac{1}{n} \sum_{i=1}^{k(\omega)} \ln \left(1 - \frac{\varepsilon \underline{a} \underline{X}_i(\omega)}{\underline{b}^{*\prime} \underline{X}_i(\omega)}\right) + \frac{1}{n} \sum_{i=1}^{n} \ln \left(1 - \frac{\varepsilon \underline{a} \underline{X}_i(\omega)}{\underline{b}^{*\prime} \underline{X}_i(\omega)}\right).$$

Thus

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln \frac{\tilde{\underline{b}}_i \left(\underline{X}_1, \underline{X}_2, \dots, \underline{X}_{i-1}\right) \underline{X}_i}{\underline{b}^* \underline{X}_i} \geq \mathsf{E} \ln \left(1 - \frac{\varepsilon \underline{a} \underline{X}}{\underline{b}^{*\prime} \underline{X}}\right) \qquad \text{a.s.}$$

Expanding the function $\ln\left(1 - y\underline{a}\underline{X}/\underline{b}^{*\prime}\underline{X}\right)$ into Taylor series around 0 in the interval $[0, \varepsilon]$, we have,

$$\left|\ln \left(1 - \frac{y\underline{a}\underline{X}}{\underline{b}^{*\prime}\underline{X}}\right)\right| = \left|\ln(1) + \frac{-y\underline{a}\underline{X}}{\underline{b}^{*\prime}\underline{X} - t\underline{a}\underline{X}}\right|$$

for some $t \in [0, \varepsilon]$. Thus

$$\left|\ln \left(1 - \frac{\varepsilon\underline{a}\underline{X}}{\underline{b}^{*\prime}\underline{X}}\right)\right| \leq \frac{\varepsilon\underline{a}\underline{X}}{\underline{b}^{*\prime}\underline{X} - t\underline{a}\underline{X}} \leq \frac{\varepsilon\underline{a}\underline{X}}{\underline{b}^{*\prime}\underline{X} - \varepsilon\underline{a}\underline{X}} \leq \frac{\varepsilon\underline{a}\underline{X}}{0.5\underline{b}^{*\prime}\underline{X}} = \frac{2\varepsilon\underline{a}\underline{X}}{\underline{b}^{*\prime}\underline{X}}.$$

Since $\mathsf{E}X_j/\underline{b}^*\underline{X} \leq 1$ for $j = 1, 2, \dots, m$ by log-optimality (see Bell and Cover [4]), hence

$$\mathsf{E}\frac{2\varepsilon\underline{a}\underline{X}}{\underline{b}^{*\prime}\underline{X}} \leq 2\varepsilon m < \infty.$$

Since ε was arbitrary,

$$\lim_{\varepsilon \to 0} \mathsf{E} \ln \left(1 - \frac{\varepsilon\underline{a}\underline{X}}{\underline{b}^{*\prime}\underline{X}}\right) = \mathsf{E} \ln 1 = 0$$

by Lebesgue's dominated convergence theorem. The upper bound follows similarly,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln \frac{\tilde{\underline{b}}_i \left(\underline{X}_1, \underline{X}_2, \dots, \underline{X}_{i-1}\right) \underline{X}_i}{\underline{b}^* \underline{X}_i} =$$

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln \left(1 + \frac{\left(\tilde{\underline{b}}_i \left(\underline{X}_1, \underline{X}_2, \dots, \underline{X}_{i-1}\right) - \tilde{\underline{b}}_i^* \left(\underline{X}_1, \underline{X}_2, \dots, \underline{X}_{i-1}\right)\right) \underline{X}_i}{\underline{b}^{*\prime} \underline{X}_i}\right) \leq$$

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln \left(1 + \frac{\varepsilon\underline{e}\underline{X}_i}{\underline{b}^{*\prime}\underline{X}_i}\right) = \mathsf{E} \ln \left(1 + \frac{\varepsilon\underline{e}\underline{X}}{\underline{b}^{*\prime}\underline{X}}\right),$$

where $\underline{e} = (1, 1, \ldots, 1)$. Since ε was arbitrary,

$$\lim_{\varepsilon \to 0} \mathsf{E} \ln \left(1 + \frac{\varepsilon \underline{e} \underline{X}}{\underline{b}^{*\prime} \underline{X}} \right) = 0 \qquad \text{a.s.,}$$

by Lebesgue's dominated convergence theorem. Hence

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln \underline{\tilde{b}}_i \left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1} \right) \underline{X}_i = \mathsf{E} \ln \underline{b}^* \underline{X} \qquad \text{a.s.} \qquad \square$$

Proof of Theorem 1. The cluster points of $\{\underline{\hat{b}}_n \left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1} \right)\}_{n=1}^{\infty}$ are log-optimal by Lemma 5. Since $\underline{\hat{b}}_n \left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1} \right) \in \text{int B}$ for all n, hence the theorem follows from Lemma 6. (Here int B denotes the interior of the set B.) Note that we could not have used the pure empirical log-optimal portfolio selector since it might lead us to ultimate ruin. That is why it was necessary to combine the pure empirical log-optimal portfolio with the uniform one. In order to force the cluster points of the proposed portfolio selector into the set of log-optimal portfolios B^* we made the uniform portfolio vanish asymptotically. $\square$

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