A SURVAY ON LOG-OPTIMUM PORTFOLIO SELECTION

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Résumé. Cet article passe des investissements log-optimales en revue. Considerous un commerçant qui vent distribuer ses moyens financiers entre des titres variés. Le comportement de son capital engagé pendant long temps est analysé en detail. Quand la distribution de probabilités dans la bourse est connue, on peut compter un "portefeuille" log-optimale. Le propriétés de cet investissement sont discutées. Des stratégies d'apprentissage, qui re dépendentes que d'observation de la bourse, sont ansi révisées.

Abstract. The present paper provides a survey on log-optimum investment. Suppose an investor who is facing the problem how to distribute his funds among various shares. The long-run behaviour of his capital is in the focus of the present paper. If the probability law (probability distribution) governing the stock market is known then the so called *log-optimal* portfolio can be calculated. The properties of *log-optimal* investment are analysed. Learning strategies which depend solely on observation of the stock market are also surveyed.

1. INTRODUCTION

The present paper deals with the portfolio selection problem as follows. Suppose an investor who wants to distribute his funds among various shares. He makes transactions, that is, buys and sells shares at discrete times. (Let us say at every noon. In this case the investment period is one day, but any other time interval is also good.) So, he goes to the stock market every noon, and buys and sells shares. The obvious question is how to choose the shares to sell and the shares to buy. Consider the assumptions as follows:

1. Transactions can be made at discrete times.
2. The investor cannot lose more than the invested capital.
3. There is no cost on transactions.
4. The investor can buy arbitrary small fraction of shares.

Assumption 1 is not really a restriction since the time interval can be chosen arbitrary small. Assumption 2 is natural. Assumption 3 and 4 are quite unrealistic, but they make the analysis easier. However, if a good investment strategy is given which performs well under these assumptions, it can be adapted to real world situation.

Suppose the probability law governing the stock market is known. That is, the investor knows the statistical behaviour of the stock market. In this case, the so called *log-optimal* strategy can be calculated (the definition will be given later).

Let S_n^*, S_n denote capitals achieved by the *log-optimal* and by an arbitrary strategy after n investment periods, respectively. In this paper the focus is on the long run behaviour of the capitals. For short-run properties of log-investment we refer to Bell and Cover (1980, 1988). Then without any condition on the probability law (the probability distribution which governs the stock market) the following properties hold (cf. Algoet and Cover (1988)).

(1) $$E\left(\frac{S_n}{S_n^*}\right) \leq 1 \text{ for all } n.$$

(2) $$E\left(\lim_{n\to\infty} \frac{S_n}{S_n^*}\right) \leq 1.$$

(3) $$\limsup_{n\to\infty} \frac{1}{n} \ln S_n \leq \lim_{n\to\infty} \frac{1}{n} \ln S_n^* \quad \text{with probability one,}$$

provided that $\lim_{n\to\infty} \frac{1}{n} \ln S_n^*$ exists with probability one.

(4) $$\limsup_{n\to\infty} \frac{1}{n} \ln \frac{S_n}{S_n^*} = \limsup_{n\to\infty} \left(\frac{1}{n} \ln S_n - \frac{1}{n} \ln S_n^*\right) \leq 0 \quad \text{with probability one.}$$

For arbitrary $\epsilon > 0$,

(5) $$S_n < S_n^* e^{n\epsilon} \text{ eventually with probability one.}$$

Property 1 says that the expected value of the ratio of the capitals achieved by an arbitrary strategy and the *log-optimal* one is at most one.

Property 2 states that the limit of the ratio of the capitals is a random variable with the expected value less or equal one.

Property 3 is very significant. It says that if the asymptotic growth rate of capital $\lim_{n\to\infty} \frac{1}{n} \ln S_n^*$ achieved by the *log-optimal* strategy exists with probability one, then it is the highest one (note that expression $\frac{1}{n} \ln S_n^*$ may diverge).

To see how common the notion of capital growth rate $\frac{1}{n} \ln S_n$ is consider a bank account with interest rate 10 %. Let S_n denote capital on this account after n years and suppose that the initial capital is one unit. Clearly, $S_n = (1+0.1)^n = e^{n \ln(1+0.1)}$, hen ce capital S_n grows exponentially fast with exponent $\ln(1+0.1)$. Thus, the asymptotic growth rate (in strict sense the asymptotic exponential growth rate) is

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n = \ln(1+0.1).$$

This quantity is closely related to the interest rate: the higher is the interest rate the higher the growth rate. So, it is the interest rate (hence the growth rate) that really matters.

Property 4 eliminates the previous assumption on the existence of $\lim_{n\to\infty} \frac{1}{n} \ln S_n^*$. It says that the asymptotic difference between the growth rate of capital achieved by an arbitrary and the *log-optimal* strategy can not be positive.

Property 5 is of great significance as well. Roughly, it says that the *log-optimal* strategy can not be beaten by any other strategy very often by very much. It states that the *log-optimal* strategy can not be beaten by an arbitrarily small exponential factor infinitelly often. Choose arbitrarily small $\epsilon > 0$. Then $S_n \geq S_n^* e^{n\epsilon}$ will happen only for finitely many n. That is, for n large enough, $S_n < S_n^* e^{n\epsilon}$ will hold. So, it is the capital achieved by the *log-optimal* strategy that grows with the highest exponent.

The significance of properties 3, 4, and 5 is that they compare the performances of the *log-optimal* strategy and an arbitrary strategy on stock market sequences, instead of merely comparing their expected values.

Now suppose that the price relatives of shares constitute a stationary and ergodic process. (We will make the meaning of this precise later.) Then

(6) $$\lim_{n\to\infty} \frac{1}{n} \ln S_n^* = constant \quad \text{with probability one.}$$

Property 6 says that in this case the asymptotic growth rate of capital $\lim_{n\to\infty} \frac{1}{n} \ln S_n^*$ exists, and is constant. So, the capital S_n^* grows with the highest asymptotic exponential rate.

In real world situation, the probability distribution which governs the stock market is not known in advance. In this case the *log-optimal* strategy can not be calculated. What then can be done? A natural goal is to find a strategy which solely depends on the observetions (outcomes) of the stock market (that is, it does not assume foreknowledge about the probability distribution), and performs asymptotically as well as the *log-optimal* one, at least to first order in the exponent. That is, the goal is to find a strategy which learns the behaviour of the stock market and achieves the same asymptotic growth rate of capital as the *log-optimal* strategy does. We will refer to such strategies as optimal learning strategies.

To make it more precise, let S_n, S_n^*, and $\hat{S}_n$ denote capitals achieved by an arbitrary, the *log-optimal*, and an optimal learning strategy after n investment periods, respectively. Then

(7) $$\lim_{n\to\infty} \frac{1}{n} \ln \hat{S}_n = \lim_{n\to\infty} \frac{1}{n} \ln S_n^* \quad \text{with probability one.}$$

For arbitrary $\epsilon > 0$,

(8) $$S_n^* e^{-n\epsilon} < \hat{S}_n < S_n^* e^{n\epsilon} \quad \text{eventually with probability one.}$$

(9) $$\lim_{n\to\infty} \frac{1}{n} \ln \hat{S}_n \geq \limsup_{n\to\infty} \frac{1}{n} \ln S_n \quad \text{with probability one.}$$

For arbitrary $\epsilon > 0$,

(10) $$S_n < \hat{S}_n e^{n\epsilon} \quad \text{eventually with probability one.}$$

Property 7 says that an optimal learning strategy achieves the same asymptotic growth rate of capital as the *log-optimal* strategy does provided $\lim_{n\to\infty} \frac{1}{n} \ln S_n^*$ exists almost surely. $\hat{S}_n$ grows with the same exponent as S_n^* does.

Property 8 says that the *log-optimal*, and an optimal learning strategy will not beat each other by any small exponential factor infinitely often. That is, for n large enough, $S_n^* e^{-n\epsilon} < \hat{S}_n < S_n^* e^{n\epsilon}$ will hold, with probability one.

Property 9 says that no other strategy can achieve higher asymptotic growth rate of capital than an optimal learning strategy.

Property 10 says that no other strategy will beat an optimal learning strategy infinitelly often by an exponential factor.

2. THE MATHEMATICAL MODEL OF THE STOCK MARKET

A stock market is represented as a random vector of stocks $\underline{X} = (X_1, X_2, \ldots, X_m)$, $X_i \geq 0$, $i = 1, 2, \ldots, m$, where m is the number of stocks and the price relative X_i represents the ratio of the price at the end of the investment period to the price at the beginning of the period. For example if $X_i = 1.03$ it means that the ith stock went up 3 % during that period.

A portfolio $\underline{b} = (b_1, b_2, \ldots, b_m)$, $b_i \geq 0$, $\sum_{i=1}^{m} b_i = 1$ is an allocation of wealth across the stocks. Here b_i is the fraction of one's wealth invested in stock i. Let B denote the set of portfoios, that is, $B = \{\underline{b} : b_i \geq 0, \sum_{i=1}^{m} b_i = 1\}$. If one uses a portfolio $\underline{b}$ and the stock vector is $\underline{X}$, the ratio of wealth at the end of the period to the beginning of the period is $S = \underline{b}\underline{X}$.

(We assume that $P(\underline{X} = \underline{0}) = 0$, that is, all the stocks can not yield zero capital at the same time. This is the case e.g. when one may always keep some money in the pocket. This could be included in the model by a virtual share $X_1 \equiv 1$, that is, one keep b_1 fraction of the capital in the pocket.)

Our objective is to maximize S in some sense. But S is a random variable. Which distribution of S is the best? The standard theory of stock market investment is based on the consideration of the moments of S. The objective is to maximize the expected value of S, subject to a constraint on the variance. Since it is easy to calculate these moments, the theory is simpler than the theory that deals with the entire distribution of S. Looking at the mean of a random variable gives information of the sum of samples of a random variable. But in the stock market, one normally reinvests every period, so that the wealth after n periods is the product of factors, one for each period of the market. The behaviour of the product is determined not by the expected value but by the expected logarithm.

More precisely, suppose an investor who wants to distribute his funds among various stocks. The stocks are described at period i by their price relatives, that is, by vector $\underline{X}_i$. Let $\{\underline{X}_i\}_{i=1}^{\infty}$ denote the stock market process. The investor distributes his funds according to portfolio $\underline{b}_i$ at the beginning of period i. So, if he has initial capital one unit, then his capital after period 1 is $S_1 = \underline{b}_1\underline{X}_1$. He reinvest his entire capital according to portfolio $\underline{b}_2$ at the beginning of period 2, and so his compounded capital at the end of period 2 is $S_2 = (\underline{b}_1\underline{X}_1)(\underline{b}_2\underline{X}_2)$. After n periods of investment, he has capital

$$S_n = \Pi_{i=1}^{n} \underline{b}_i \underline{X}_i.$$

We allow that the portfolio $\underline{b}_i$ used in period i may depend on the past outcomes $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1}$. Where we want to emphasize this fact, we will use the notation $\underline{b}_i(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1})$. In fact, $\underline{b}_i(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1})$ is a function which maps from the past outcomes of the stock market to the set of portfolios B. A strategy is described by a series of such functions $\{\underline{b}_i(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1})\}_{i=1}^{\infty}$.

Suppose one has a good strategy $\{\underline{b}_i(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1})\}_{i=1}^{\infty}$, and he would like to use it, but there is cost on each transaction, and an arbitrary fraction of share can not be bought. First of all, notice that if one already has some shares, and he wants to restructure his possessions according to portfolio $\underline{b}$ he usually has to sell only some of his shares, and then buy some other ones. He need not sell all of them. In this way, transaction costs will not be high. In practice, one need not follow the theoretically optimal strategy all the time. If there is only a small difference between the theoretically optimal structure of possession and the actual one, an investor need not change it. One may consider the optimal strategy as a guideline. But if there is already big difference between the theoretically optimal structure and the actual one, one should follow the guideline, and restructure his possession.

3. THE LOG-OPTIMAL STRATEGY

Now we can introduce the notion of *log-optimal* strategy (cf. Cover and Thomas (1991)).

Definition. A strategy $\{\underline{b}_i^*(\underline{X}_1, \underline{X}_2, \dots, \underline{X}_{i-1})\}_{i=1}^{\infty}$ is said to be *log-optimal* if and only if, for all i,

$$E\left(\ln \frac{\underline{b}\underline{X}_i}{\underline{b}_i^*(\underline{X}_1, \underline{X}_2, \dots, \underline{X}_{i-1})\underline{X}_i} \middle| \underline{X}_1, \underline{X}_2, \dots, \underline{X}_{i-1}\right) \leq 0 \text{ for all } \underline{b} \in B. \tag{11}$$

Nota bene, the *log-optimal* strategy always exists (cf. Algoet and Cover (1988)).

Essentially, definition 11 says that if the past sequence of the stock market variables is $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{i-1}$ then portfolio $\underline{b}_i^*(\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{i-1})$ is to be chosen so that

$$\begin{aligned} &E\left(\ln \underline{b}_i^*(\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{i-1})\underline{X}_i \middle| \underline{X}_1 = \underline{x}_1, \underline{X}_2 = \underline{x}_2, \dots, \underline{X}_{i-1} = \underline{x}_{i-1}\right) = \\ &\max_{\underline{b}\in B} E\left(\ln \underline{b}\underline{X}_i \middle| \underline{X}_1 = \underline{x}_1, \underline{X}_2 = \underline{x}_2, \dots, \underline{X}_{i-1} = \underline{x}_{i-1}\right). \end{aligned} \tag{12}$$

This form shows that the *log-optimal* strategy maximizes the conditional expected log-return.

Consider an arbitrary stock market process $\{\underline{X}_i\}_{i=1}^{\infty}$. Then it follows from Algoet and Cover (1988) that properties 1, 2, 3, 4, and 5 hold.

If the stock market process $\{\underline{X}_i\}_{i=1}^{\infty}$ is assumed to be stationary and ergodic then property 6 holds as well (cf. ibid.).

In a special case, when the stock market process $\{\underline{X}_i\}_{i=1}^{\infty}$ is independent, and identically distributed, the *log-optimal* strategy is a constant $\underline{b}^*$, that is, $\underline{b}_i^*(\underline{X}_1, \underline{X}_2, \dots, \underline{X}_{i-1}) = \underline{b}^*$ for all i, and definition 11 has the simple form,

$$E\left(\ln \frac{\underline{b}\underline{X}}{\underline{b}^*\underline{X}}\right) \leq 0 \quad \text{for all } \underline{b} \in B. \tag{13}$$

Essentially, it says that portfolio $\underline{b}^*$ maximizes expected log-return, that is,

$$E \ln \underline{b}^*\underline{X} = \max_{\underline{b}\in B} E \ln \underline{b}\underline{X}. \tag{14}$$

Note that Vajda and Österreicher (1993) used exactly this property as a definition of log-optimal portfolio $\underline{b}^*$ and found necessary and sufficient conditions for the existence and uniqueness.

As we have seen, the *log-optimal* strategy is optimal in some sense as we have seen but in order to calculate it we have to know the probability distribution which governs the stock market process. In fact, there exits an algorithm which calculates the *log-optimal* strategy, provided the probability distribution is known (cf. Cover (1991)). So, if we know the statistical behaviour of the stock market process, the best we can do is to employ the *log-optimal* strategy.

4. OPTIMAL LEARNING STRATEGIES FOR INDEPENDENT AND IDENTICALLY DISTRIBUTED PROCESSES

The only problem is that the probability distribution is usually unknown, hence we can not calculate the *log-optimal* strategy. (We can not evaluate expressions 11,12, or 13 without knowing the probability distribution.) The goal is to find a strategy which performs as well as the *log-optimal* strategy (at least to first order in the exponent), and depends solely on the past outcomes of the stock market. In other words, this strategy learns the stock market from observations, in some sense.

Let the stock market process $\{\underline{X}_i\}_{i=1}^{\infty}$ be independent and identically distributed.
Let further

$$\underline{b}_1 = (1/m, 1/m, \ldots, 1/m)$$

$$\underline{b}_i\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1}\right) = \arg\max_{\underline{b}\in B} \Pi_{k=1}^{i-1} \underline{b}\underline{X}_i \quad \text{for } i \geq 2.$$

(That is, $\underline{b}_i\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1}\right)$ denotes the portfolio which would have gained the biggest capital if we had known the past sequence in advance.)

The proposed strategy is

$$\tilde{\underline{b}}_i\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1}\right) = \lambda_i \underline{b}_i\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1}\right) + (1-\lambda_i)\,\underline{b}_1$$

where $0 < \lambda_i < 1$, and $\lim_{i\to\infty} \lambda_i = 1$.

This strategy is due to Morvai (1992). Let $\tilde{S}_n$ denote the capital achieved by this strategy. Properties 7-10 hold $\tilde{S}_n$ in place of $\hat{S}_n$. It also holds

$$(15) \qquad \lim_{i\to\infty} \tilde{\underline{b}}_{n_i} = \underline{b} \Longrightarrow \underline{b} \text{ is } log-optimal.$$

Property 15 says that the accumulation points of $\{\tilde{\underline{b}}_i\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{i-1}\right)\}_{i=1}^{\infty}$ are *log-optimal*. That is, each convergent subsequence $\{\tilde{\underline{b}}_{n_i}\}_{i=1}^{\infty}$ tends to a *log-optimal* portfolio.

However, a very simple strategy achieves the same goal (cf. Cover (1991)). Let $B_0 = \{\underline{b}^{(0)}, \underline{b}^{(1)}, \ldots\}$ be a dense subset of B. Let $\mu_i > 0$, $\sum_{i=1}^{\infty}\mu_i = 1$. Use initial capital μ_i to play with portfolio $\underline{b}^{(i)}$. It will yield $\mu_i S_n^{(i)}$ amount of money at time n, where $S_n^{(i)} = \prod_{j=1}^{n} \underline{b}^{(i)}\underline{X}_j$. Thus $\limsup_{n\to\infty} \frac{1}{n}\ln\left(\sum_{i=1}^{\infty}\mu_i S_n^{(i)}\right) \geq \limsup_{n\to\infty}\frac{1}{n}\ln \mu_i S_n^{(i)} = \lim_{n\to\infty}\frac{1}{n}\ln\mu_i + \lim_{n\to\infty}\frac{1}{n}\ln S_n^{(i)} = \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\ln \underline{b}^{(i)}\underline{X}_j = E\ln \underline{b}^{(i)}\underline{X}$ a.s. by the strong law of large numbers. Thus $\limsup_{n\to\infty}\frac{1}{n}\ln\left(\sum_{i=1}^{\infty}\mu_i S_n^{(i)}\right) \geq \sup_{\underline{b}\in B_0} E\ln\underline{b}\underline{X} = \sup_{\underline{b}\in B} E\ln \underline{b}\underline{X} = \lim_{\to\infty}\frac{1}{n}\ln S_n^{*} a.s.$ Hence this strategy achieves the highest exponential growth rate and properties 7-10 hold. It suffices to put $\grave{S}_n = \prod_{j=1}^{n}\grave{\underline{b}}_j\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{j-1}\right)\underline{X}_j$ in place of $\hat{S}_n$.

In a closed form this strategy can be expressed as follows

$$\grave{\underline{b}}_n\left(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}\right) = \frac{\sum_{i=1}^{\infty}\mu_i S_{n-1}^{(i)}\underline{b}^{(i)}.}{\sum_{i=1}^{\infty}\mu_i S_{n-1}^{(i)}}$$

Let us mention that the Cover's (1991) universal strategy has a very similar form:

$$\underline{\check{b}}_i\left(\underline{X}_1,\underline{X}_2,\ldots,\underline{X}_{i-1}\right)=\frac{\int_B \underline{b}S_{i-1}\left(\underline{b}\right)d\underline{b}}{\int_B S_{i-1}\left(\underline{b}\right)d\underline{b}},$$

where $S_i\left(\underline{b}\right)=\prod_{j=1}^{i}\underline{b}\underline{X}_j$.

T. M. Cover examined the performance of this strategy on individual sequences. He did not make any statistical assumption. Here we will not go into details. However, if $\underline{b}^*$ is unique and lies in the interior of B then properties 7-10 hold – just put $\dot{S}_n$ in place of $\hat{S}_n$.

For the rest of this chapter suppose one can always keep some money in the pocket. Formulating this, let $X_1\equiv 1$ (this is a 'virtual' share). Assume that the *log-optimal* portfolio $\underline{b}^*$ is unique and $b_i^*>0$ for $i=2,\ldots,m$. T. F. Móri (1982) proposed the strategy

$$\underline{\check{b}}_1=(1,0,\ldots,0)$$

$$\underline{\check{b}}_i\left(\underline{X}_1,\underline{X}_2,\ldots,\underline{X}_{i-1}\right)=\arg\max_{\underline{b}\in B_{\epsilon_n}}\Pi_{k=1}^{i-1}\underline{b}\underline{X}_i\quad\text{for i}\geq 2,$$

where $B_{\epsilon_n}=\{\underline{b}\in B: b_1\geq\epsilon_n\}$, $1\geq\epsilon_n>0$, a nd $\lim_{n\to\infty}\epsilon_n=0$. (Note that now $X_1\equiv 1$). Let $\check{S}_n$ denote the capital achieved by Móri's strategy. Then properties 7- 10 hold just replace $\hat{S}_n$ by $\check{S}_n$.

(16) $$\lim_{n\to\infty}\underline{\check{b}}_n=\underline{b}^*.$$

The following properties hold as well:

For arbitrary $\epsilon>0$, $\delta>0$,

(17) $$\lim_{n\to\infty}P\left(n^{\frac{1}{2}-\epsilon}\|\underline{\check{b}}_n-\underline{b}^*\|>\delta\right)=0.$$

For arbitrary $\epsilon>0$,

(18) $$S_n^*n^{-\frac{m}{2}-\epsilon}<\check{S}_n<S_n^*n^{-\frac{m}{2}+\epsilon}\text{ eventually with probability one.}$$

Property 17 determines how fast the empirical strategy converges to the *log-optimal* one. Roughly, it says that the rate of convergence is of order $n^{-1/2}$.

Property 18 says that $\check{S}_n$ is worse than S_n^* by a polynomial factor. This is a much stronger result than property 8. It is interesting (but not surprising) that there appears the dimension m, that is, the number of shares in the stock market. Property 18 implies that $\frac{\check{S}_n}{S_n^*}$ tends to zero with probability one. That is the price we have to pay for not knowing the probability distribution in advance. Note however that usually both S_n^*, $\check{S}_n$ tend to infinity.

5. OPTIMAL LEARNING STRATEGIES FOR STATIONARY AND ERGODIC PROCESSES

Under the assumption that the stock market process $\{\underline{X}_i\}_{i=1}^{\infty}$ is stationary and ergodic, there also exits a strategy $\{\check{\underline{b}}_n(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1})\}_{n=1}^{\infty}$ (this strategy is due to Algoet (1992) such that it does not assume the foreknowledge of the probability distribution, and achieves the best (optimal) asymptotic growth rate of capital. That is, properties 7-10 hold.

Let $\{G_k\}_{k=1}^{\infty}$ be a series of finite quantizers such that they asymptotically generate the Borel σ-algebra. Let $G_k(\underline{X})$ be a quantized version of $\underline{X}$ using the finite quantizer G_k. Consider the estimate of the conditional distribution as follows:

$$\hat{P}_s^k = \frac{\delta_{\underline{x}_0}(dx) + \sum_{\tau \in I_s^k} \delta_{\underline{X}_\tau}(dx)}{1 + \|I_s^k\|},$$

where $\underline{x}_0 \neq \underline{0}$ otherwise it is an arbitrary stock vector, and $I_s^k = \{\tau : 1 \leq \tau \leq s; (G_k(\underline{X}_{s+k}) \ldots \ldots G_k(\underline{X}_{s+1})) = (G_k(\underline{X}_{\tau+k-1}) \ldots G_k(\underline{X}_\tau))\}$.

Now let

$$\underline{b}_n^{(k)}(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}) = (1/m, \ldots, 1/m) \quad \text{if} \quad n-1 \leq k,$$
$$\underline{b}_n^{(k)}(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}) = \arg\max_{\underline{b} \in B} \int \ln \underline{b}\underline{x} \hat{P}_{n-k-1}^k(d\underline{x}) \quad \text{if} \quad n-1 > k.$$

Let

(19) $$\check{\underline{b}}_n^{(k)}(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}) = \lambda_n \underline{b}_n^{(k)}(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}) + (1 - \lambda_n)\underline{\beta},$$

where $\underline{\beta} = (1/m, \ldots, 1/m), 0 < \lambda_n < 1$, and $\lim_{n\to\infty} \lambda_n = 1$.

Now the same procedure is applied as before. Let $\mu_k > 0$, $\sum_{k=1}^{\infty} \mu_k = 1$. Algoet's strategy is as follows:

(20) $$\check{\underline{b}}_n(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}) = \frac{\sum_{k=1}^{\infty} \mu_k S_{n-1}^{(k)} \check{\underline{b}}_n^{(k)}(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1})}{\sum_{k=1}^{\infty} \mu_k S_{n-1}^{(k)}},$$

where $S_n^{(k)} = \prod_{j=1}^{n} \check{\underline{b}}_j^{(k)}(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{j-1})\underline{X}_j$.

For capital $\check{S}_n$ properties 7-10 hold. To this end it suffices to write

$$\check{S}_n = \prod_{j=1}^{n} \check{\underline{b}}_j(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{j-1})\underline{X}_j$$

in place of $\hat{S}_n$.

REFERENCES

A. H. Algoet (1992): Universal schemes for prediction, gambling and portfolio selection. Ann. Probab. *20*, 901–941.

A. H. Algoet and T. M. Cover (1988): Asymptotic optimality and asymptotic equipartition properties of log-optimum investment. Ann. Probab. *16*, 876–898.A. R. Barron and T. M. Cover (1988): A bound on the financial value of information. IEEE Trans. Inform. Theory, IT-*34*, 1097–1100.

R. Bell and T. M. Cover (1988): Game Theoretic Optimal Portfolios. Management Sci. *34*, 724–733.

R. Bell and T. M. Cover (1980): Competitive optimality of logarithmic investment. Math. Oper. Res. *5*, 161–166.

L. Breiman (1960): Investment policies for expanding businesses optimal in long-run sense. Naval Res. Logist. Quart. 7, 647–651.

L. Breiman (1961): Optimal gambling systems for favorable games. In: Fourth Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley, pp. 65-78.

T. M. Cover (1984): An algorithm for maximizing expected log-investment return. IEEE Trans. Inform. Theory IT-*30*, 369–373.

T. M. Cover (1991): Universal portfolios. Math.Finance *12*, 1–29.

T. M. Cover and J. A. Thomas (1991): Elements of Information Theory. Wiley, New York 1991.

T. M. Cover and D. H. Gluss (1986): Empirical Bayes stock market portfolios. Adv. Appl. Math. *7*, 170–181.

M. Finkelstein and R. Whitley (1981): Optimal strategies for repeated games. Adv. Appl. Probab. *13*, 415–428.

J. Kelly (1956): A new interpretation of information rates. Bell. Sys. Tech. J. *35*, 917–926.

M. Feder (1991): Gambling using a finite state machine. IEEE Trans. Inform. Theory *37*, 1459–1465.

T. F. Móri (1982): Asymptotic properties of the empirical strategy in favourable stochastic games. In: Limit Theorems in Probability and Statistics *36*, pp. 777–790.

T. F. Móri (1982): How to win if you can? In: Limit Theorems in Probability and Statistics *36*, pp. 791-806.

T. F. Móri (1986): Is the empirical strategy optimal? Statistics and Decisions *4*, 45–60.

G. Morvai (1991): Empirical log-optimal portfolio selection. Problems Control Inform. Theory *20*, 453–463.

G. Morvai (1992): Portfolio choice based on the empirical distribution. Kybernetika *28*, 484–493.

I. Vajda and F. Österreicher (1993): Existence uniquess and evaluation of log-optimal investment portfolio. Kybernetika *29*, 105–120.