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Intermittent estimation of stationary time series.

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Abstract

Let $\{X_n\}_{n=0}^\infty$ be a stationary real-valued time series with unknown distribution. Our goal is to estimate the conditional expectation of X_{n+1} based on the observations X_i , $0 \leq i \leq n$ in a strongly consistent way. Bailey and Ryabko proved that this is not possible even for ergodic binary time series if one estimates at all values of n . We propose a very simple algorithm which will make prediction infinitely often at carefully selected stopping times chosen by our rule. We show that under certain conditions our procedure is strongly (pointwise) consistent, and L_2 consistent without any condition. An upper bound on the growth of the stopping times is also presented in this paper.

1 Introduction

Let $\{X_n\}_{n=0}^{\infty}$ be a real-valued time series. We are interested in estimating the random variable X_{n+1} given the past observations X_0, \dots, X_n . If the random variable X_{n+1} has finite expectation and we are to minimize the conditional mean squared error then the solution is to choose the conditional expectation $E(X_{n+1}|X_0, \dots, X_n)$. Usually, the distribution is not known a priori. In this case we may try to estimate the above quantity from observations.

Assume the distribution of the real-valued time series $\{X_n\}_{n=0}^{\infty}$ is stationary. Now the goal is to estimate the conditional expectation $E(X_{n+1}|X_0, \dots, X_n)$ from the data segment X_0, \dots, X_n such that the difference between the estimate and the conditional expectation should tend to zero almost surely as the number of observations n tends to infinity. However [14] proved that there is no such estimator if one estimates for all values of n , even for all stationary and ergodic first order Markov chains taking values from the unit interval $[0, 1]$. This problem was posed originally in [6]. [5] (applying the method of cutting and stacking developed in [22] and [27]) constructed a family of stationary and ergodic binary processes such that for any estimation scheme there was a process in his family for which the difference between the estimate and the true conditional expectation did not tend to zero. (Cf. [25] also.)

However, for the class of all stationary and ergodic binary Markov chains of some finite order one can solve this problem. Indeed, if the time series is a Markov chain of some finite (but unknown) order, we can estimate the order (cf. [9], and [8]) and count frequencies of blocks with length equal to the order.

In another special case, for certain Gaussian processes, [26] constructed an estimator such that for that family of processes the error between his estimator and the true conditional expectation tends to zero almost surely as the number of observations increases.

Here we note that a totally different problem is when the goal is to estimate the conditional expectation in such a way that the *time average* of the squared error is required to vanish as the number of observations tends to infinity. This problem can be easily solved, cf. [5], [21], [1, 2, 3], [19], [20], [13] and [12]. (See also [28] and [11].)

In this paper the setting is different. We do not weaken the error criterion, that is, we will further consider the difference between our estimate and the true conditional expectation (rather than time averages) but we do not

require to estimate for every time instance n , but rather, merely along a stopping time sequence. That is, looking at the data segment X_0, \dots, X_n our rule will decide if we dare to estimate for this n or not, but anyhow we will definitely estimate for infinitely many n .

Such algorithm was proposed for binary time series in [17] but there the growth of the stopping times is like an exponential tower, and so that scheme is not feasible at all. A more practical algorithm was proposed in [18] for certain binary time series. In this paper we provide an algorithm for real-valued processes.

2 Definition of the Estimator and Main Results

For some technical reason, we will consider two-sided stationary real-valued processes $\{X_n\}_{n=-\infty}^{\infty}$. Note that a one-sided stationary time series $\{X_n\}_{n=0}^{\infty}$ can be extended to be a two-sided stationary time series $\{X_n\}_{n=-\infty}^{\infty}$.

For notational convenience, let $X_m^n = (X_m, \dots, X_n)$, where $m \leq n$. Let $\{\mathcal{P}_k\}_{k=0}^{\infty}$ denote a nested sequence of finite or countably infinite partitions of the real line by intervals. Let $x \rightarrow [x]^k$ denote a quantizer that assigns to any point $x \in \mathbb{R}$ the unique interval in \mathcal{P}_k that contains x . For a set $C \subseteq \mathbb{R}$ let $\text{diam}(C) = \sup_{y,z \in C} |z - y|$. We assume that

$$\lim_{k \rightarrow \infty} \text{diam}([x]^k) = 0 \quad \text{for all } x \in \mathbb{R}. \quad (1)$$

Let $[X_m^n]^k = ([X_m]^k, \dots, [X_n]^k)$. Let $1 \leq l_k \leq k$ be a nondecreasing sequence of positive integers such that $\lim_{k \rightarrow \infty} l_k = \infty$. Put

$$J(n) = \min\{j \geq 1 : l_{j+1} > n\}.$$

Define the stopping times as follows. Set $\zeta_0 = 0$. For $k = 1, 2, \dots$, define the sequences η_k and ζ_k recursively. Define

$$\eta_1 = \min\{t > 0 : [X_{\zeta_0 - (l_1 - 1) + t}^{\zeta_0 + t}]^1 = [X_{\zeta_0 - (l_1 - 1)}^{\zeta_0}]^1\} \quad \text{and} \quad \zeta_1 = \eta_1.$$

Next we refine the quantization and look for the next occurrence of the block of length l_2 , namely

$$\eta_2 = \min\{t > 0 : [X_{\zeta_1 - (l_2 - 1) + t}^{\zeta_1 + t}]^2 = [X_{\zeta_1 - (l_2 - 1)}^{\zeta_1}]^2\} \quad \text{and} \quad \zeta_2 = \zeta_1 + \eta_2.$$

In general, we refine the quantization, and slowly increase the block length of the next repetition, as follows:

$$\eta_k = \min\{t > 0 : [X_{\zeta_{k-1}-(l_{k-1})+t}^{\zeta_{k-1}+t}]^k = [X_{\zeta_{k-1}-(l_{k-1})}^{\zeta_{k-1}}]^k\} \text{ and } \zeta_k = \zeta_{k-1} + \eta_k. \quad (2)$$

One denotes the k th estimate of $E(X_{\zeta_k+1}|X_0^{\zeta_k})$ by g_k , and defines it to be

$$g_k = \frac{1}{k} \sum_{j=0}^{k-1} X_{\zeta_j+1}. \quad (3)$$

Let \mathbb{R} be the set of all real numbers and put \mathbb{R}^{*-} the set of all one-sided sequences of real numbers, that is,

$$\mathbb{R}^{*-} = \{(\dots, x_{-1}, x_0) : x_i \in \mathbb{R} \text{ for all } -\infty < i \leq 0\}.$$

Define a metric on sequences (\dots, x_{-1}, x_0) and (\dots, y_{-1}, y_0) as follows. Let

$$d^*((\dots, x_{-1}, x_0), (\dots, y_{-1}, y_0)) = \sum_{i=0}^{\infty} 2^{-i-1} \frac{|x_{-i} - y_{-i}|}{1 + |x_{-i} - y_{-i}|}. \quad (4)$$

(For details see [10] p. 51.)

Definition 1 (Almost surely continuous conditional expectation.) *The conditional expectation $E(X_1|\dots, X_{-1}, X_0)$ is almost surely continuous if for some set $C \subseteq \mathbb{R}^{*-}$ which has probability one the conditional expectation $E(X_1|\dots, X_{-1}, X_0)$ restricted to this set C is continuous with respect to metric $d^*(\cdot, \cdot)$ in (4).*

Example 1 *A stationary and ergodic time series with almost surely continuous conditional expectation which is not continuous on the whole space.*

We will define a transformation S on the unit interval. Consider the binary expansion r_1^∞ of each real-number $r \in [0, 1)$, that is, $r = \sum_{i=1}^{\infty} r_i 2^{-i}$. When there are two expansions, use the representation which contains finitely many 1's. Now let

$$\tau(r) = \min\{i > 0 : r_i = 1\}.$$

Notice that, aside from the exceptional set $\{0\}$, which has Lebesgue measure zero τ is finite and well-defined on the closed unit interval. The transformation is defined by

$$(Sr)_i = \begin{cases} 1 & \text{if } 0 < i < \tau(r) \\ 0 & \text{if } i = \tau(r) \\ r_i & \text{if } i > \tau(r). \end{cases} \quad (5)$$

Notice that in fact, $Sr = r - 2^{-\tau(r)} + \sum_{l=1}^{\tau(r)-1} 2^{-l}$. All iterations S^k of S for $-\infty < k < \infty$ are well defined and invertible with the exception of the set of dyadic rationals which has Lebesgue measure zero. This transformation S could be defined recursively as

$$Sr = \begin{cases} r - 0.5 & \text{if } 0.5 \leq r < 1 \\ \frac{1+S(2r)}{2} & \text{if } 0 \leq r < 0.5. \end{cases} \quad (6)$$

Now choose r uniformly on the unit interval. Set $X_0(r) = r$ and put $X_n(r) = S^n r$. Notice that the resulting time series $\{X_n\}$ is a stationary and ergodic Markov chain with order one, cf. [14]. What more, one observation determines the whole orbit of the process. Observe that $E(X_{n+1}|X_0^n) = E(X_{n+1}|X_n)$ and $E(X_{n+1}|X_n = x) = Sx$. Since S is a continuous mapping disregarding the set of dyadic rationals, the resulting conditional expectation is almost surely continuous. However, the conditional expectation is not continuous on the whole unit interval, since it can not be made continuous, for example, at 0.5.

The next theorem establishes the strong (pointwise) consistency of the proposed estimator.

Theorem 1 *Let $\{X_n\}$ be a real-valued stationary time series with $E(|X_0|^2) < \infty$. For the estimator g_k defined in (3) and for the stopping time ζ_k defined in (2),*

$$\lim_{k \rightarrow \infty} \left| g_k - E(X_{\zeta_k+1} | X_0^{\zeta_k}) \right| = 0 \quad \text{almost surely}$$

provided that the conditional expectation $E(X_1 | X_{-\infty}^0)$ is almost surely continuous.

The proof of Theorem 1 involves both the martingale convergence theorem and classical convergence results for an auxiliary sequence of orthogonal

random variables. The assumption on the almost sure continuity of the conditional expectation is crucial in going from the auxiliary variables to the actual random variables that take part in the estimator.

The consistency holds independently of how the sequence l_k and the partitions are chosen as long as l_k goes to infinity and the partitions become finer. However, the choice of these sequences has a great influence on the growth of the stopping times.

From the proof of [5], [25] and [14] it is clear that even for the class of all stationary and ergodic binary time series with almost surely continuous conditional expectation $E(X_1 | \dots, X_{-1}, X_0)$ one can not estimate $E(X_{n+1} | X_0^n)$ for all n strongly (pointwise) consistently.

Note that the processes constructed by the method of cutting and stacking (cf. [22] and [27]) are stationary processes with almost surely continuous conditional expectations.

The stationary processes with almost surely continuous conditional expectation generalize the processes for which the conditional expectation is actually continuous. (Cf. [15] or [16].)

If one's goal is to estimate the conditional mean merely in L_2 then the problem becomes very easy and even for all time instances one can estimate it, cf. [19]. We will prove that our proposed estimator $\{g_n\}$ along the stopping time sequence $\{\zeta_n\}$ is not just strongly consistent under the above mentioned continuity condition but also consistent in L_2 without any continuity condition. The point here is that our scheme achieves two goals simultaneously. In this way, if one runs our algorithm he can be sure that if the above mentioned continuity condition holds then the algorithm achieves strong consistency and if unfortunately that condition fails to hold then even in that case it achieves L_2 consistency. Precisely:

Theorem 2 *Let $\{X_n\}$ be a real-valued stationary time series with $E(|X_0|^2) < \infty$. For the estimator defined in (3) and for the stopping time ζ_k defined in (2),*

$$\lim_{k \rightarrow \infty} E \left(\left| g_k - E(X_{\zeta_k+1} | X_0^{\zeta_k}) \right|^2 \right) = 0. \quad (7)$$

The next theorem gives an upper bound on the growth of the stopping times $\{\zeta_k\}$ in case when finite partitions are used.

Theorem 3 *Let $\{X_n\}$ be a stationary real-valued time series. Assume \mathcal{P}_k is a nested sequence of finite partitions of the real line by intervals. If for some $\epsilon > 0$, $\sum_{k=1}^{\infty} (k+1)2^{-l_k\epsilon} < \infty$ then for the stopping time ζ_k defined in (2),*

$$\zeta_k < |\mathcal{P}_k|^{l_k} 2^{l_k\epsilon}$$

eventually almost surely.

Example 2 *One may set $\epsilon = 1$, $l_k = \lfloor 3 \log_2 k \rfloor$, and $|\mathcal{P}_k| = \lfloor 2^{f_k} \rfloor$ where f_k is an increasing sequence of positive real numbers tending to infinity arbitrary slowly. By Theorem 3, $\zeta_k < k^{3(1+f_k)}$, which is almost a polynomial growth.*

In case of finite alphabet processes you can achieve a slightly better upper bound than in Theorem 3. Indeed, let H denote the entropy rate associated with the stationary and ergodic finite alphabet time series $\{X_n\}$, cf. [7]. Note that in this case no quantization is needed. Then it is easy to see, that $\zeta_k < 2^{l_k(H+\epsilon)}$ eventually almost surely provided that $(k+1)2^{-l_k\epsilon}$ is summable. (Cf. [18], [23], [19].)

If one desires to estimate X_{ζ_j+1} in L_2 sense based on data X_0, \dots, X_{ζ_j} then the best he can do is to choose the conditional expectation

$$g_j^* = E(X_{\zeta_j+1} | X_0^{\zeta_j}).$$

Now we show that the conditional mean squared error $E((X_{\zeta_j+1} - g_j)^2 | X_0^{\zeta_j})$ with regard to g_j is close to that of the best possible $E((X_{\zeta_j+1} - g_j^*)^2 | X_0^{\zeta_j})$ for large j . Indeed, this is an immediate consequence of Theorem 1, Theorem 2, and the fact that $E((X_{\zeta_j+1} - g_j)^2 | X_0^{\zeta_j}) - E((X_{\zeta_j+1} - g_j^*)^2 | X_0^{\zeta_j}) = (g_j - g_j^*)^2$.

Corollary 1 *Let $\{X_n\}$ be a stationary real-valued time series. Assume $E(|X_0|^2) < \infty$. Then*

$$\left| E \left((X_{\zeta_j+1} - g_j)^2 | X_0^{\zeta_j} \right) - E \left((X_{\zeta_j+1} - g_j^*)^2 | X_0^{\zeta_j} \right) \right| \rightarrow 0 \quad (8)$$

in L_1 . Moreover, if in addition, the conditional expectation $E(X_1 | X_{-\infty}^0)$ is almost surely continuous, then (8) holds almost surely.

Note that X_n can not be estimated for all n in such a way that the conditional mean squared error tend to zero in the pointwise sense even in case of almost surely continuous conditional expectation. (Cf. [5], [25], [14].) The main point here is that along a sequence of stopping times one can achieve that property.

3 Auxiliary Results

It will be useful to define other processes for $k \geq 0$ $\{\hat{X}_n^{(k)}\}_{n=-\infty}^{\infty}$ as follows. Let

$$\hat{X}_{-n}^{(k)} = X_{\zeta_k - n} \quad \text{for } -\infty < n < \infty. \quad (9)$$

For an arbitrary real-valued stationary time series $\{Y_n\}$, for $k \geq 0$ let $\hat{\zeta}_0^k(Y_{-\infty}^0) = 0$ and for all $k \geq 1$ and $1 \leq i \leq k$ define

$$\hat{\eta}_i^k(Y_{-\infty}^0) = \min\{t > 0 : [Y_{\hat{\zeta}_{i-1}^k - (l_{k-i+1} - 1) - t}^{\hat{\zeta}_{i-1}^k - t}]^{(k-i+1)} = [Y_{\hat{\zeta}_{i-1}^k - (l_{k-i+1} - 1)}]^{\hat{\zeta}_{i-1}^k - (l_{k-i+1} - 1)}\}$$

and

$$\hat{\zeta}_i^k(Y_{-\infty}^0) = \hat{\zeta}_{i-1}^k(Y_{-\infty}^0) - \hat{\eta}_i^k(Y_{-\infty}^0).$$

When it is obvious on which time series $\hat{\eta}_i^k(Y_{-\infty}^0)$ and $\hat{\zeta}_i^k(Y_{-\infty}^0)$ are evaluated, we will use the notation $\hat{\eta}_i^k$ and $\hat{\zeta}_i^k$. Let T denote the left shift operator, that is, $(Tx_{-\infty}^0)_i = x_{i+1}$. It is easy to see that if $\zeta_k(x_{-\infty}^0) = l$ then $\hat{\zeta}_k^k(T^l x_{-\infty}^0) = -l$. We will need the next lemmas for later use.

Lemma 1 *Let $\{X_n\}_{n=-\infty}^{\infty}$ be a real-valued stationary process. Then the time series $\{\hat{X}_n^{(k)}\}_{n=-\infty}^{\infty}$, $\{X_n\}_{n=-\infty}^{\infty}$ have identical distribution, that is, for all $k \geq 0$, $n \geq 0$, $m \geq 0$, and Borel set $F \subseteq \mathbb{R}^{n+1}$,*

$$P((\hat{X}_{m-n}^{(k)}, \dots, \hat{X}_m^{(k)}) \in F) = P(X_{m-n}^m \in F).$$

Thus all the time series $\{\hat{X}_n^{(k)}\}_{n=-\infty}^{\infty}$ for $k = 0, 1, \dots$ are stationary.

Proof. Since the time series $\{X_n\}$ is stationary and for all $k \geq 0$, $n \geq 0$, $l \geq 0$, $F \subseteq \mathbb{R}^{n+1}$,

$$T^l\{X_{\zeta_k + m - n}^{\zeta_k + m} \in F, \zeta_k = l\} = \{X_{m-n}^m \in F, \hat{\zeta}_k^k(X_{-\infty}^0) = -l\}, \quad (10)$$

and by the construction in (9), we have

$$\begin{aligned} & P((\hat{X}_{m-n}^{(k)}, \dots, \hat{X}_m^{(k)}) \in F) \\ &= P(X_{\zeta_k + m - n}^{\zeta_k + m} \in F) = \sum_{l=0}^{\infty} P(X_{\zeta_k + m - n}^{\zeta_k + m} \in F, \zeta_k = l) \\ &= \sum_{l=0}^{\infty} P(X_{m-n}^m \in F, \hat{\zeta}_k^k(X_{-\infty}^0) = -l) = P(X_{m-n}^m \in F). \end{aligned}$$

The proof of the Lemma 1 is complete.

For a given n , the partition cell $[X_{\zeta_j-n}]^j$ is a random set and is varying as $j \rightarrow \infty$. However, we will prove that eventually it shrinks.

Lemma 2 *Let $\{X_n\}_{n=-\infty}^{\infty}$ be a real-valued stationary process. Then for all $n \geq 0$, $\lim_{j \rightarrow \infty} \text{diam}([X_{\zeta_j-n}]^j) = 0$ almost surely.*

Proof. Observe, that by the definition of stopping times in (2), for a given n , $\{[X_{\zeta_j-n}]^j\}_{j=J(n)}^{\infty}$ is a decreasing sequence of intervals. Now, if for some $j \geq J(n)$, $\text{diam}([X_{\zeta_j-n}]^j) < \infty$ then $\lim_{i \rightarrow \infty} \text{diam}([X_{\zeta_i-n}]^i) = 0$. To see this notice that if $\lim_{i \rightarrow \infty} \text{diam}([X_{\zeta_i-n}]^i) > 0$ then $\bigcap_{i=J(n)}^{\infty} [X_{\zeta_i-n}]^i \neq \emptyset$ and let z denote a real number from this set. For this z , $\lim_{i \rightarrow \infty} \text{diam}([z]^i) > 0$ contradicting our assumption in (1). What remains is to prove that

$$P(\text{diam}[X_{\zeta_j-n}]^j = \infty \text{ for all } j \geq J(n)) = 0.$$

Indeed by Lemma 1 and assumption (1),

$$\begin{aligned} & P(\text{diam}([X_{\zeta_j-n}]^j) = \infty \text{ for all } j \geq J(n)) \\ & \leq \lim_{j \rightarrow \infty} P(\text{diam}([X_{\zeta_j-n}]^j) = \infty) = \lim_{j \rightarrow \infty} P(\text{diam}([\hat{X}_{-n}^{(j)}]^j) = \infty) \\ & = \lim_{j \rightarrow \infty} P(\text{diam}([X_{-n}]^j) = \infty) = \lim_{j \rightarrow \infty} P(\text{diam}([X_1]^j) = \infty) = 0. \end{aligned}$$

The proof of Lemma 2 is complete.

Define the time series $\{\tilde{X}_n\}_{n=-\infty}^0$

$$\tilde{X}_{-n} = \lim_{j \rightarrow \infty} X_{\zeta_j-n} \text{ for } n \geq 0, \quad (11)$$

where the limit exists since $\{[X_{\zeta_j-n}]^j\}_{j=J(n)}^{\infty}$ is a random sequence of nested intervals and by Lemma 2 their lengths tend to zero.

Lemma 3 *Let $\{X_n\}_{n=-\infty}^{\infty}$ be a real-valued stationary process. Then the distribution of $\{\tilde{X}_n\}_{n=-\infty}^0$ equals the distribution of $\{X_n\}_{n=-\infty}^0$.*

Proof. By Lemma 1 it is enough to prove that for any $i \geq 0$, for all $j \geq J(i)$, $[\tilde{X}_{-i}]^j = [\hat{X}_{-i}^{(j)}]^j$. Let R_k be the set of right end-points of the right open intervals in the k -th partition, that is,

$$R_k = \{b \in \mathbb{R} : \exists -\infty < a < b [a, b) \in \mathcal{P}_k \text{ or } \exists -\infty \leq a < b (a, b) \in \mathcal{P}_k\}.$$

Similarly, let L_k be the set of left end-points of the left open intervals in the k -th partition, that is,

$$L_k = \{b \in \mathbb{R} : \exists b < a < \infty (b, a] \in \mathcal{P}_k \text{ or } \exists b < a \leq \infty (b, a) \in \mathcal{P}_k\}.$$

If $[\tilde{X}_{-i}]^j = [\hat{X}_{-i}^{(j)}]^j$ fails for some $j \geq J(i)$ then this must happen at some end point, that is, $\tilde{X}_{-i} \in \bigcup_{k=0}^{\infty} R_k$ or $\tilde{X}_{-i} \in \bigcup_{k=0}^{\infty} L_k$. (Since the partition sequence is a nested sequence and $\tilde{X}_{-i} = \lim_{j \rightarrow \infty} \hat{X}_{-i}^{(j)}$.) Therefore we can estimate: By (11), Lemma 2, and Lemma 1, we have

$$\begin{aligned} & 1 - P(\tilde{X}_{-i} \in [\hat{X}_{-i}^{(j)}]^j \text{ for all } j \geq J(i)) \\ & \leq \sum_{k=J(i)}^{\infty} \sum_{s \in R_k} P(\tilde{X}_{-i} = s, \hat{X}_{-i}^{(j)} < \tilde{X}_{-i} \text{ for all } j \geq k) \\ & + \sum_{k=J(i)}^{\infty} \sum_{s \in L_k} P(\tilde{X}_{-i} = s, \hat{X}_{-i}^{(j)} > \tilde{X}_{-i} \text{ for all } j \geq k) \\ & \leq \sum_{k=J(i)}^{\infty} \sum_{s \in R_k} \lim_{j \rightarrow \infty} P(s - \text{diam}([\hat{X}_{-i}^{(j)}]^j) \leq \hat{X}_{-i}^{(j)} < s) \\ & + \sum_{k=J(i)}^{\infty} \sum_{s \in L_k} \lim_{j \rightarrow \infty} P(s < \hat{X}_{-i}^{(j)} \leq s + \text{diam}([\hat{X}_{-i}^{(j)}]^j)) \\ & = \sum_{k=J(i)}^{\infty} \sum_{s \in R_k} \lim_{j \rightarrow \infty} P(s - \text{diam}([X_{-i}]^j) \leq X_{-i} < s) \\ & + \sum_{k=J(i)}^{\infty} \sum_{s \in L_k} \lim_{j \rightarrow \infty} P(s < X_{-i} \leq s + \text{diam}([X_{-i}]^j)) \\ & = \sum_{k=J(i)}^{\infty} \sum_{s \in R_k} \lim_{j \rightarrow \infty} P(s - \text{diam}([X_1]^j) \leq X_1 < s) \\ & + \sum_{k=J(i)}^{\infty} \sum_{s \in L_k} \lim_{j \rightarrow \infty} P(s < X_1 \leq s + \text{diam}([X_1]^j)) = 0. \end{aligned}$$

The proof of Lemma 3 is complete.

Now it is immediate that the time series $\{\tilde{X}_n\}_{n=-\infty}^0$ is stationary, since $\{X_n\}_{n=-\infty}^0$ is stationary, and it can be extended to be a two-sided time series $\{\tilde{X}_n\}_{n=-\infty}^\infty$. We will use this fact only for the purpose of defining the conditional expectation $E(\tilde{X}_1|\tilde{X}_{-\infty}^0)$.

4 Proof of Theorem 1

Proof. Define the function $e : \mathbb{R}^{*-} \rightarrow (-\infty, \infty)$ as

$$e(x_{-\infty}^0) = E(X_1|X_{-\infty}^0 = x_{-\infty}^0).$$

Recall (3) and consider

$$\begin{aligned} g_k &= \frac{1}{k} \sum_{j=0}^{k-1} \left(X_{\zeta_{j+1}} - E(X_{\zeta_{j+1}}|X_{-\infty}^{\zeta_j}) \right) \\ &+ \frac{1}{k} \sum_{j=0}^{k-1} E(X_{\zeta_{j+1}}|X_{-\infty}^{\zeta_j}) \\ &= \frac{1}{k} \sum_{j=0}^{k-1} \Gamma_j + \frac{1}{k} \sum_{j=0}^{k-1} E(X_{\zeta_{j+1}}|X_{-\infty}^{\zeta_j}). \end{aligned} \quad (12)$$

Consider the first term and observe that $\{\Gamma_j\}$ is a sequence of orthogonal random variables with $E\Gamma_j = 0$ and $E(\Gamma_j^2) \leq E((X_1)^2) < \infty$ since $E(\Gamma_j^2) \leq E((X_{\zeta_{j+1}})^2)$ and, by Lemma 1, $X_{\zeta_{j+1}}$ has the same distribution as X_1 . Now by Theorem 3.2.2 in [24],

$$\frac{1}{k} \sum_{j=0}^{k-1} \Gamma_j \rightarrow 0 \text{ almost surely.}$$

(Alternatively, you can apply Theorem A6 in [11])

Now we deal with the second term. For arbitrary $j \geq 0$, by the constructions in (9),(11)

$$\lim_{j \rightarrow \infty} d^*(\tilde{X}_{-\infty}^0, (\dots, \hat{X}_{-1}^{(j)}, \hat{X}_0^{(j)})) = 0 \text{ almost surely.} \quad (13)$$

By assumption, the function $e(\cdot)$ is continuous on a set $C \subseteq \mathbb{R}^{*-}$ with $P(X_{-\infty}^0 \in C) = 1$. By Lemma 1 and Lemma 3,

$$P(\tilde{X}_{-\infty}^0 \in C, (\dots, \hat{X}_{-1}^{(j)}, \hat{X}_0^{(j)}) \in C \text{ for all } j \geq 0) = 1. \quad (14)$$

Now by the continuity of $e(\cdot)$ on the set C , and by (13) and (14),

$$E(X_{\zeta_j+1} | X_{-\infty}^{\zeta_j}) = e(\dots, \hat{X}_{-1}^{(j)}, \hat{X}_0^{(j)}) \rightarrow e(\tilde{X}_{-\infty}^0) = E(\tilde{X}_1 | \tilde{X}_{-\infty}^0). \quad (15)$$

Thus $g_k \rightarrow E(\tilde{X}_1 | \tilde{X}_{-\infty}^0)$ almost surely.

What remains to be proven is that almost surely, $E(X_{\zeta_j+1} | X_0^{\zeta_j}) \rightarrow E(\tilde{X}_1 | \tilde{X}_{-\infty}^0)$. For any set $A \subseteq \mathbb{R}$ let $\text{closure}(A)$ denote the smallest closed subset of the real line containing A . Put

$$S_j(X_0^{\zeta_j}) = \{z_{-\infty}^0 \in \mathbb{R}^{*-} : z_{-l_{j+1}+1} \in \text{closure}([X_{\zeta_j-l_{j+1}+1}]^j), \dots, z_0 \in \text{closure}([X_{\zeta_j}]^j)\}.$$

By (2), (11) and (14), almost surely, for all j ,

$$X_{-\infty}^{\zeta_j} \in S_j(X_0^{\zeta_j}) \cap C \text{ and } \tilde{X}_{-\infty}^0 \in S_j(X_0^{\zeta_j}) \cap C. \quad (16)$$

Put

$$\Delta_j(X_0^{\zeta_j}) = \sup_{y_{-\infty}^0, z_{-\infty}^0 \in S_j(X_0^{\zeta_j}) \cap C} |e(y_{-\infty}^0) - e(z_{-\infty}^0)|.$$

Now since $e(\cdot)$ is continuous at $\tilde{X}_{-\infty}^0$ on set C and by (16) and Lemma 2,

$$\lim_{j \rightarrow \infty} \Delta_j(X_0^{\zeta_j}) = 0 \text{ almost surely.} \quad (17)$$

By (17) almost surely,

$$\begin{aligned} & \limsup_{j \rightarrow \infty} \left| E \left(e(\tilde{X}_{-\infty}^0) | X_0^{\zeta_j} \right) - E \left(e(X_{-\infty}^{\zeta_j}) | X_0^{\zeta_j} \right) \right| \\ & \leq \limsup_{j \rightarrow \infty} E \left(\left| e(\tilde{X}_{-\infty}^0) - e(X_{-\infty}^{\zeta_j}) \right| | X_0^{\zeta_j} \right) \\ & \leq \limsup_{j \rightarrow \infty} E \left(\Delta_j(X_0^{\zeta_j}) | X_0^{\zeta_j} \right) \\ & = \limsup_{j \rightarrow \infty} \Delta_j(X_0^{\zeta_j}) = 0. \end{aligned} \quad (18)$$

Now consider

$$E\left(X_{\zeta_j}|X_0^{\zeta_j}\right) = E\left(e(\tilde{X}_{-\infty}^0)|X_0^{\zeta_j}\right) - \left\{E\left(e(\tilde{X}_{-\infty}^0)|X_0^{\zeta_j}\right) - E\left(e(X_{-\infty}^{\zeta_j})|X_0^{\zeta_j}\right)\right\}.$$

The first term it is a martingale and tends to $e(\tilde{X}_{-\infty}^0)$ by Theorem 7.6.2 in [4] since by Lemma 3, $E\left|e(\tilde{X}_{-\infty}^0)\right| = E\left|e(X_{-\infty}^0)\right| \leq E|X_1| < \infty$, and $\tilde{X}_{-\infty}^0$ is measurable with respect to $\sigma(X_0^\infty)$. The second term tends to zero by (18). The proof of Theorem 1 is complete.

5 Proof of Theorem 2

Proof. By Jensen's inequality, (9) and Lemma 1,

$$\begin{aligned} & \frac{1}{4}E\left(\left|g_k - E(X_{\zeta_{k+1}}|X_0^{\zeta_k})\right|^2\right) \\ & \leq E\left(\left|\frac{1}{k}\sum_{j=0}^{k-1}\left(X_{\zeta_{j+1}} - E(X_{\zeta_{j+1}}|X_{-\infty}^{\zeta_j})\right)\right|^2\right) \\ & + \frac{1}{k}\sum_{j=0}^{k-1}E\left(\left|E(\hat{X}_1^{(j)}|\dots, \hat{X}_{-1}^{(j)}, \hat{X}_0^{(j)}) - E(\hat{X}_1^{(j)}|[\hat{X}_{-(l_{j+1}-1)}^{(j)}, \dots, \hat{X}_0^{(j)}]^{j+1})\right|^2\right) \\ & + \frac{1}{k}\sum_{j=0}^{k-1}E\left(\left|E(\hat{X}_1^{(k)}|[\hat{X}_{-(l_{j+1}-1)}^{(k)}, \dots, \hat{X}_0^{(k)}]^{j+1}) - E(\hat{X}_1^{(k)}|\dots, \hat{X}_{-1}^{(k)}, \hat{X}_0^{(k)})\right|^2\right) \\ & + E\left(\left|E(\hat{X}_1^{(k)}|\dots, \hat{X}_{-1}^{(k)}, \hat{X}_0^{(k)}) - E(\hat{X}_1^{(k)}|\hat{X}_{\zeta_k}^{(k)}, \dots, \hat{X}_0^{(k)})\right|^2\right), \end{aligned}$$

where $\hat{\zeta}_k^k$ is evaluated on $\{\hat{X}_n^{(k)}\}_{n=-\infty}^0$. The first term converges to zero since $\Phi_j = X_{\zeta_{j+1}} - E(X_{\zeta_{j+1}}|X_{-\infty}^{\zeta_j})$ is a sequence of orthogonal random variables with $E(|X_{\zeta_{j+1}}|^2) = E(|X_0|^2) < \infty$, and

$$E\left(\left|\frac{1}{k}\sum_{j=0}^{k-1}\Phi_j\right|^2\right) = \frac{1}{k^2}\sum_{j=0}^{k-1}E(|\Phi_j|^2) \leq \frac{1}{k^2}\sum_{j=0}^{k-1}E(|X_{\zeta_{j+1}}|^2) = \frac{1}{k}E(|X_1|^2) \rightarrow 0. \quad (19)$$

Applying (9) and Lemma 1, one can estimate the sum of the last three terms by the sum

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{j=0}^{k-1} E \left(\left| E(X_1 | X_{-\infty}^0) - E(X_1 | [X_{-(l_{j+1}-1)}^0]^{j+1}) \right|^2 \right) \\ & + \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{j=0}^{k-1} E \left(\left| E(X_1 | [X_{-(l_{j+1}-1)}^0]^{j+1}) - E(X_1 | X_{-\infty}^0) \right|^2 \right) \\ & + \limsup_{k \rightarrow \infty} E \left(\left| E(X_1 | X_{-\infty}^0) - E(X_1 | X_{\hat{\zeta}_k^k}^0) \right|^2 \right), \end{aligned}$$

where $\hat{\zeta}_k^k$ is now evaluated on $\{X_n\}_{n=-\infty}^0$. All of these terms converge to zero since $\lim_{j \rightarrow \infty} E(X_1 | X_{-j}^0) = E(X_1 | X_{-\infty}^0)$ and $\lim_{j \rightarrow \infty} E(X_1 | [X_{-(l_{j+1}-1)}^0]^{j+1}) = E(X_1 | X_{-\infty}^0)$ in L_2 by the martingale convergence theorem, cf. Theorem 7.6.10 and Theorem 7.6.2 in [4], and thus the limit in fact exists and equals zero. The proof of Theorem 2 is complete.

6 Proof of Theorem 3

Proof. Let $\mathbb{R}^{\mathbb{Z}}$ be the set of all two-sided sequences of real numbers, that is,

$$\mathbb{R}^{\mathbb{Z}} = \{(\dots, x_{-1}, x_0, x_1, \dots) : x_i \in \mathbb{R} \text{ for all } -\infty < i < \infty\}.$$

Let $y_{-l_k+1}^0 \in \mathcal{P}_k^{l_k}$. Define the set $Q_k(y_{-l_k+1}^0)$ as follows:

$$Q_k(y_{-l_k+1}^0) = \{z_{-\infty}^0 \in \mathbb{R}^{\mathbb{Z}} : -\hat{\zeta}_k^k(z_{-\infty}^0) \geq |\mathcal{P}_k|^{l_k} 2^{l_k \epsilon}, [z_{-l_k+1}^0]^k = y_{-l_k+1}^0\}.$$

We will estimate the probability of $Q_k(y_{-l_k+1}^0)$ by means of the ergodic theorem. To do this apply the ergodic decomposition theorem, cf. [10], and denote the distribution according to the ergodic mode ω by P_ω . Let $x_{-\infty}^\infty \in \mathbb{R}^{\mathbb{Z}}$ be a typical sequence according to P_ω . Define $\alpha_0(y_{-l_k+1}^0) = 0$ and for $i \geq 1$ let

$$\alpha_i(y_{-l_k+1}^0) = \min\{l > \alpha_{i-1}(y_{-l_k+1}^0) : T^{-l} x_{-\infty}^\infty \in Q_k(y_{-l_k+1}^0)\}.$$

Define also $\beta_0(y_{-l_k+1}^0) = 0$ and for $i \geq 1$ let

$$\beta_i(y_{-l_k+1}^0) = \min\{l > \beta_{i-1}(y_{-l_k+1}^0) + |\mathcal{P}_k|^{l_k} 2^{l_k \epsilon} : T^{-l} x_{-\infty}^\infty \in Q_k(y_{-l_k+1}^0)\}.$$

Observe that for arbitrary $i > 0$,

$$\sum_{j=1}^{\infty} 1_{\{\beta_{i-1}(y_{-l_k+1}^0) < \alpha_j(y_{-l_k+1}^0) \leq \beta_i(y_{-l_k+1}^0)\}} \leq k + 1.$$

By Lemma 1 and ergodicity,

$$\begin{aligned} P_{\omega}((\dots, \hat{X}_{-1}^{(k)}, \hat{X}_0^{(k)}, \hat{X}_1^{(k)}, \dots) \in Q_k(y_{-l_k+1}^0)) &= P_{\omega}(X_{-\infty}^{\infty} \in Q_k(y_{-l_k+1}^0)) \\ &= \lim_{t \rightarrow \infty} \frac{1}{\beta_t(y_{-l_k+1}^0)} \sum_{j=1}^{\infty} 1_{\{\alpha_j(y_{-l_k+1}^0) \leq \beta_t(y_{-l_k+1}^0)\}} \\ &= \lim_{t \rightarrow \infty} \frac{1}{\beta_t(y_{-l_k+1}^0)} \sum_{i=1}^t \sum_{j=1}^{\infty} 1_{\{\beta_{i-1}(y_{-l_k+1}^0) < \alpha_j(y_{-l_k+1}^0) \leq \beta_i(y_{-l_k+1}^0)\}} \\ &\leq \lim_{t \rightarrow \infty} \frac{t(k+1)}{t|\mathcal{P}_k|^{l_k} 2^{l_k \epsilon}} = \frac{(k+1)}{|\mathcal{P}_k|^{l_k} 2^{l_k \epsilon}}. \end{aligned}$$

Since the right hand side does not depend on ω , the same upper bound applies for the original stationary time series $\{X_n\}$, that is,

$$P((\dots, \hat{X}_{-1}^{(k)}, \hat{X}_0^{(k)}, \hat{X}_1^{(k)}, \dots) \in Q_k(y_{-l_k+1}^0)) \leq \frac{(k+1)}{|\mathcal{P}_k|^{l_k} 2^{l_k \epsilon}}.$$

By the construction in (9) $-\hat{\zeta}_k^k(\dots, \hat{X}_{-1}^{(k)}, \hat{X}_0^{(k)}) = \zeta_k(X_0^{\infty})$ we get

$$\begin{aligned} P(\zeta_k(X_0^{\infty}) \geq |\mathcal{P}_k|^{l_k} 2^{l_k \epsilon}) &= P(-\hat{\zeta}_k^k(\dots, \hat{X}_{-1}^{(k)}, \hat{X}_0^{(k)}) \geq |\mathcal{P}_k|^{l_k} 2^{l_k \epsilon}) \\ &= \sum_{y_{-l_k+1}^0 \in \mathcal{P}_k^{l_k}} P((\dots, \hat{X}_{-1}^{(k)}, \hat{X}_0^{(k)}, \hat{X}_1^{(k)}, \dots) \in Q_k(y_{-l_k+1}^0)) \leq (k+1)2^{-l_k \epsilon}. \end{aligned}$$

By assumption, the right hand side sums, the Borel-Cantelli Lemma yields that $\zeta_k < |\mathcal{P}_k|^{l_k} 2^{l_k \epsilon}$ eventually almost surely and Theorem 3 is proved.

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