Gyakorlás

$$\ln z = \ln r + i\varphi + 2\pi i \cdot k \quad (z = re^{i\varphi}),\ k \text{ egész}$$

① $e^{2z} + ie^{z} + 2 = 0$

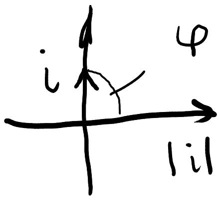
$w = e^{z}$

$w^2 + iw + 2 = 0$

$$w_{1,2} = \frac{-i \pm \sqrt{-1-4\cdot 1\cdot 2}}{2} = \frac{-i \pm \sqrt{-9}}{2} = \frac{-i \pm i\cdot 3}{2} = \left\langle \begin{matrix} i \\ -2i \end{matrix} \right.$$

$e^{z} = i$ | $e^{z} = -2i$

$z = \ln i = \delta$ | $z = \ln(-2i)$

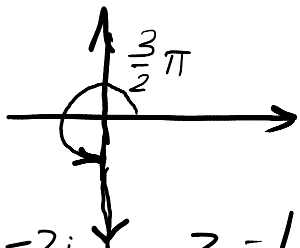


$\varphi = \frac{\pi}{2}$

$|i| = 1 = r$

$\delta = \underbrace{\ln 1}_{0} + i\frac{\pi}{2} + 2\pi i k$

$z_1 = i\frac{\pi}{2} + 2\pi i k$



$\varphi = \frac{3}{2}\pi$

$r = |-2i| = 2$

$z_2 = \ln 2 + i\frac{3}{2}\pi + 2\pi i k$

k egész

② $\sin z = 2$; $\frac{e^{iz} - e^{-iz}}{2i} = 2$; $e^{iz} - e^{-iz} = 4i$

$w = e^{iz}$ $\quad \left(e^{-iz} = \frac{1}{e^{iz}} = \frac{1}{w}\right)$ $\quad \left(\sqrt{12} = \sqrt{3\cdot 4} = 2\sqrt{3}\right)$

$w - \frac{1}{w} = 4i \quad / \cdot w$

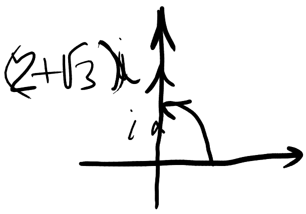
$w^2 - 1 - 4iw = 0$

$w^2 - 4iw - 1 = 0$

$$w_{1,2} = \frac{4i \pm \sqrt{16(-1) - 4\cdot 1\cdot(-1)}}{2} = \frac{4i \pm \sqrt{-12}}{2} = \frac{\cancel{4}^{2}i \pm i\cdot \cancel{2}^{1}\sqrt{3}}{\cancel{2}_{1}} = i(2 \pm \sqrt{3})$$

$e^{iz} = \underbrace{i\cdot(2+\sqrt{3})}_{|\cdot| = r = 2+\sqrt{3}}$

$$\boxed{\frac{1}{i} = \frac{1}{i}\cdot\frac{i}{i} = \frac{i}{-1} = -i}$$



$\varphi = \frac{\pi}{2}$

$iz = \ln(2+\sqrt{3}) + i\frac{\pi}{2} + 2\pi i k \quad / : i$

$z = (-i)\ln(2+\sqrt{3}) + \pi/2 + 2\pi k$

$e^{iz} = i(2-\sqrt{3})$

$\varphi = \pi/2$

$r = 2-\sqrt{3}$

$z = (-i)\ln(2-\sqrt{3}) + \pi/2 + 2\pi k$

Komplex differenciálhatóság

$\partial\partial\partial\partial\partial\partial$

$$\left[\ (z^2)' = 2z\ ;\quad (\sin z)' = \cos z\ \right]$$

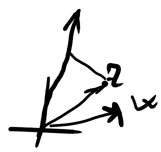
Ism.: $f: \mathbb{R}^2 \to \mathbb{R}^2$ fv deriváltja: Jacobi-

1\. megj. mátrix; $f(x,y) \equiv \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ $\qquad (\partial_x u = u'_x)$

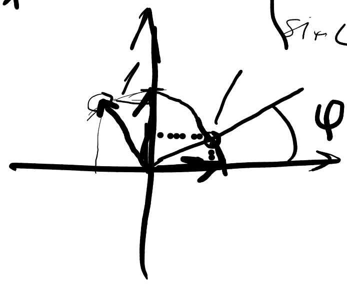
$$J^f(x,y) = \begin{bmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{bmatrix}$$

$\cancel{z} = r\cdot e^{i\varphi}$

$z \cdot w$

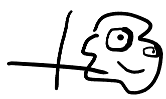


$\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$



2\. megj. $\mathbb{C} = \{ M \in \mathbb{R}^{2\times 2} \mid M \text{ forg. nyú.} \}$

$$\left[L\begin{pmatrix} b_1 \end{pmatrix};\ L\begin{pmatrix} b_2 \end{pmatrix} \right] = \begin{bmatrix} r\cos\varphi & -r\sin\varphi \\ r\sin\varphi & r\cos\varphi \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cong a + bi$$



Tétel: $f: \mathbb{C} \rightarrowtail \mathbb{C}$; $f \equiv u + iv$; $z_0 = x_0 + iy_0 \in \operatorname{int} \operatorname{Dom}(f)$;

$$f \in \mathrm{Diff}_{\mathbb{C}}(x_0 + iy_0) \iff \begin{cases} u; v \in \mathrm{Diff}_{\mathbb{R}^2}(x_0, y_0) \\ \text{és} \\ J^f_{(x_0, y_0)} \in \mathbb{C} \end{cases}$$

Cauchy–Riemann-egyenletrendszer

$$\boxed{\begin{array}{l} \text{I.)}\ \ \partial_x u = \partial_y v \\ \text{II.)}\ \ \partial_y u = -\partial_x v \end{array}}$$

③ $z \mapsto z^2$ komplex módon deri. válható és $(z^2)' = 2z$

$$z^2 = (x+iy)^2 = x^2 + 2ixy + \underset{-1}{i^2} y^2 = \underbrace{x^2 - y^2}_{u} + i\underbrace{(2xy)}_{v}$$

u és v $\mathbb{R}^2$-diffható

$$\begin{cases} 2x = 2x \\ -2y = -2y \end{cases}$$ minden $x+iy$-ra teljesül

$$f'(z) \cong J^f(x+iy) = \begin{bmatrix} \boxed{\partial_x u} & \partial_y u \\ \boxed{\partial_x v} & \partial_y v \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \hat{=}\, a+ib$$

$$\Rightarrow \boxed{f'(z) = \partial_x u + i\partial_x v}$$

$$(z^2)' = 2x + i2y = 2(x+iy) = 2z$$

④ $z \xmapsto{f} \bar{z}$ deriválható-e, és ha igen mi a deriváltja?

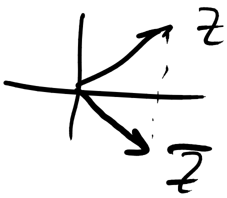
$$\overline{x+iy} = x - iy = \underbrace{x}_{u} + i\underbrace{(-y)}_{v}$$

u, v valósan deriválható

$$\begin{aligned} \partial_x u &= \partial_y v &&\Rightarrow& 1 &= -1 \\ \partial_y u &= -\partial_x v &&\Rightarrow& 0 &= 0 \end{aligned}$$

$\Rightarrow$ sehol sem teljesülnek a C-R egyenletek

$\Rightarrow f$ sehol sem diff-ható



Borzasztó függvények:

$\bar{z}$; $\mathrm{Re}(z)$; $\mathrm{Im}(z)$; $|z|$

HF.: diff.

Def: $f: \mathbb{C} \to \mathbb{C}$; $z_0 \in \operatorname{int} \operatorname{Dom}(f)$; $w \in \mathbb{C}$

Azt mondjuk, hogy az f deriválható a z_0 pontban és deriváltja a w szám, ha

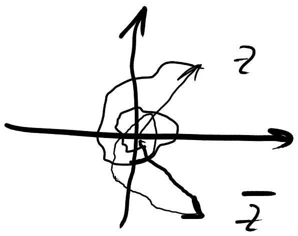
$$\exists \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = w. \quad (f'(z_0) = w)$$

---

(5) Igazoljuk, hogy az $f(z) = z \cdot \overline{z}$ deriválható a 0-ban (def. szerint)

$z_0 = 0$

$$\lim_{z \to 0} \frac{z \cdot \overline{z} - 0 \cdot \overline{0}}{z - 0} = \lim_{z \to 0} \frac{\cancel{z} \cdot \overline{z}}{\cancel{z}} = \lim_{z \to 0} \overline{z} = 0$$



$f'(0) = 0$