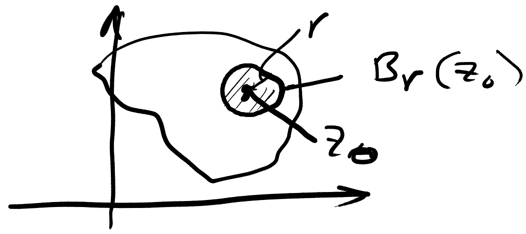
Harmonikus társ keresés

Def.: $u(x,y)$ harmonikus egy $T \subseteq \mathbb{R}^2$ tartományon, ha

$$\partial^2_{xx} u + \partial^2_{yy} u \equiv 0$$

T-n.



Def.: $f : \mathbb{C} \to \mathbb{C}$ reguláris az $z_0 \in \operatorname{int} \operatorname{Dom}(f)$ pontban, ha van $B_r(z_0)$ teljes környezete z_0-nak melynek minden egyes pontjában f deriválható.

Tétel $f : T \to \mathbb{C}$ reguláris függvény, T egyszeresen öf. tartomány. Ekkor $f = u + iv$ esetén u, v harmonikus fv.-ek

Tétel 2. $u : T \to \mathbb{R}$ harmonikus fv és T egysz. öf. Ekkor van olyan $v : T \to \mathbb{R}$ harmonikus fv, hogy $u + iv$ reguláris.
(itt v harmonikus társa u-nak)

Feladatok

① $u(x,y) = e^{2x} \cdot \cos(2y) + y$

Van-e u-nak h. társa és ha igen mi az?

Ell.: $\partial_x u = 2e^{2x} \cdot \cos(2y)$ $\quad \partial_y u = e^{2x} \cdot (-2\sin 2y) + 1$

$\partial^2_{xx} u = 4e^{2x} \cdot \cos(2y)$ $\quad \partial^2_{yy} u = -4e^{2x} \cos 2y$

$\oplus$

$$\partial^2_{xx} u + \partial^2_{yy} u \equiv 0$$

$\Rightarrow u$ harmonikus $\Rightarrow \exists$ harm. társa

C-R $\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases}$ $v = ?$ $\begin{cases} \partial_x v = -\partial_y u \\ \partial_y v = \partial_x u \end{cases}$

$$\int f(ax+b) = \frac{F(ax+b)}{a}$$

$$\int e^{2x}dx = \frac{e^{2x}}{2}$$

$$\begin{cases} \partial_x v = +2e^{2x}\sin 2y - 1 \xrightarrow{\int \dots dx} v(x,y) = e^{2x}\sin 2y - x + C(y) \\ \partial_y v = 2e^{2x}\cos 2y \end{cases}$$

$$\partial_y\left(e^{2x}\sin 2y - x + C(y)\right) = 2e^{2x}\cos 2y$$

$$2\cdot e^{2x}\cos 2y + C'(y) = 2e^{2x}\cos 2y$$

$$C'(y) = 0$$

$$-iz = -i(x+iy) = -ix + y$$

$$C(y) = c \in \mathbb{R}$$

$$f(x,y) = \underbrace{e^{2x}\cos 2y + y}_{u} + i\underbrace{(e^{2x}\sin 2x - x + c)}_{v}$$

megj.: $e^{2z} = e^{2x+2yi} = e^{2x}\cdot \underbrace{e^{2yi}}_{\cos 2y + i\sin 2y} =$

$$= e^{2x}\cos 2y + ie^{2x}\sin 2y$$

$$\boxed{f(z) = e^{2z} - iz}$$

(2) $u(x,y) = 2xy - x$

$v(x,y) = ?$ $f = u + iv$ reguláris és $f(0) = i$

C-R

$$\begin{cases} \partial_x v = -\partial_y u = -2x \xrightarrow{\int \dots dx} v = -x^2 + C(y) \\ \partial_y v = \partial_x u = 2y - 1 \end{cases}$$

$$\partial_y(-x^2 + C(y)) = 2y - 1$$

$$C'(y) = 2y - 1 \quad / \int \dots dy$$

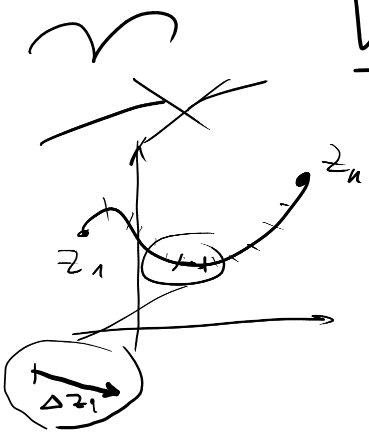
$f(0)$: $z = 0 = \underset{x}{0} + i\underset{y}{0}$

$$C(y) = y^2 - y + c$$

$$f = u + iv = 2xy - x + i(-x^2 + y^2 - y + c)$$

$f(0) = i$ | $f(0,0) = i\cdot c = i \Rightarrow c = 1$

Komplex integrál



$G : [t_1, t_2] \to \mathbb{C}$

$t \mapsto z(t)$

- folytonos
- véges sok kivétellel folyt. diff.

$$\sum_{i=1}^{n} f(z_i)\cdot \Delta z_i \xrightarrow[\substack{n\to\infty \\ |\Delta z_i|\to 0 \\ z_i\in G}]{} I$$

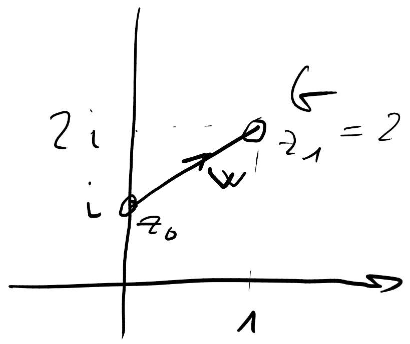
$$I = \int_G f$$

$$\boxed{\int_G f = \int_{t_1}^{t_2} f(z(t))\cdot \dot{z}(t)\, dt}$$

megj.:

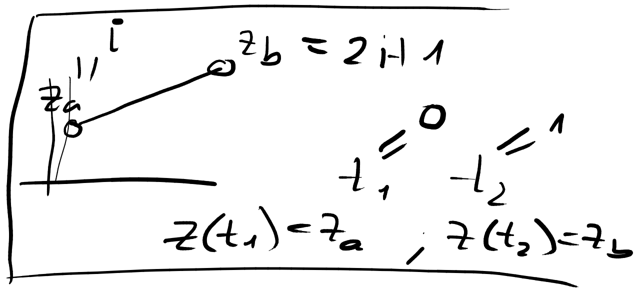
- Bonyolult függvény integrálása paraméterezéssel
  $\mathrm{Re},\ \mathrm{Im},\ \overline{(.)},\ |(.)|$
- Primitív függvénnyel rendelkező reguláris fv-ek:
  Newton—Leibniz-formula
  $$F(b)-F(a) = \int_a^b f$$
- néhány szinguláris függvény
  Cauchy-féle integrálformula
  $\frac{1}{z}$; $\frac{1}{\sin z}$ . . .

① Mi az $f(z)=Re(z)^2+i\,Im(z)$ integrálja az $[i;2i+1]$ szakaszon?



$$f(z) = Re(z)^2 + i\cdot Im\, z$$

$$\int_G f = ?$$



szakasz $[z_0;z_1]$: $\boxed{z = z_0 + w\cdot t}$ $t\in[0;1]$

param. $w = z_1 - z_0$

$$\boxed{\int_G f = \int_{t_1}^{t_2} f(z(t))\cdot \dot z(t)\,dt} = \ldots$$

S₂FE

$$\underbrace{t + i(1+t)}_{i + it + t}$$

$z(t) = i + (i+1)\cdot t$

$\dot z(t) = i+1$

$t_1 = 0$

$t_2 = 1$

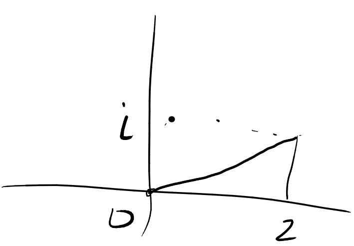
$$f(z(t)) = [Re(i+(i+1)\cdot t)]^2 + i\cdot Im(i+(i+1)t) = t^2 + i\cdot(1+t)$$

S₂FE

$$= \int_0^1 (t^2 + i(1+t))\cdot(i+1)\,dt = (i+1)\cdot\int_0^1 t^2 + i + it\,dt =$$

$$= (i+1)\cdot\left[\frac{t^3}{3} + it + \frac{it^2}{2}\right]_0^1 = (i+1)\cdot\left(\frac{1}{3} + i + \frac{i}{2}\right) - 0 = \; = -\frac{7}{6} + \frac{11}{6}$$

„Szép" fv-ek integrálja



$$\frac{1}{2}\int_0^{2+i} 2\cdot z\cdot e^{z^2}\,d(z) = \frac{1}{2}\int_0^{2+i} 2z\cdot e^{z^2}\,dz = \frac{1}{2}\left[e^{z^2}\right]_0^{2+i} = \; = 1/2\cdot\left(e^{(2+i)^2} - 1\right)$$

$$\int f(g(x))\cdot g'(x)\,dx = F(g(x)) + C$$

helyettesítéses integrálás