— Corrected answer sheets are available for review to all participants in office H666 from 13:00 till 15:00 on 4 May. All interested are cordially invited to the ceremonial announcement of results and to the following reception at the cafeteria on the 4th floor of building Q from 18:15 on 15 May.
— Each exercise is worth 10 points. Partial solutions are also considered. In some instances even more than 10 points are awarded on a problem (e.g. by providing interesting generalizations or multiple solutions).
— **Each problem should be answered on a separate piece of paper** carrying the exercise number as well as the name and NEPTUN code of the participant.

---

1. The continuous function $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ admits a constant $K > 0$ such that $\forall x \in \mathbb{R}_0^+$: $f(x) \leq K \int_0^x f(t)dt$. Prove that $f(x) = 0$ for every $x \in \mathbb{R}_0^+$.

2. A letter "L" is two orthogonal line-segments (of nonzero lengths) with a common endpoint and a length ratio of 2 : 1. An "X" is two orthogonal line-segments of equal (positive) lengths with a common midpoint. One can easily exhibit an uncountable set of disjoint "L" on the plane. (How?) But what about placing so many disjoint "X" there? (The letters in the collection can have different sizes and orientations.)

3. A graph with n edges is drawn on the plane in a manner so that only a finite number of points lie in more than one edges; each in at most 2 edges and each being a tangent, rather than an intersection point. Depending on n, what is the maximum number of edge-pairs with at least one common tangent point?

4. 66 people dance hand-to-hand in a circle when the names of 13 of them are drawn in a raffle. By leaving to take the prizes, they incidentally separate the remaining dancers into N connected groups. Find the expected value of N.

5. There are several foxes and rabbits in a field. Because of the tall grass, the foxes sense other animals only within 100 meters. If within this circle, a fox finds at least as many rabbits as foxes (including itself), then it remains unworried about food. At the moment, no fox is worrying. At most how many times more foxes are in the field than rabbits?

6. Prove that the edges of the complete graph with $2n$ vertices can be colored with n colors so that between any two vertices there will be a unicolor path of each color and moreover, any two of these n paths will have only their start and endpoints in common.

7. The square matrix X has rational entries and satisfies the equation $X^5 = I$. Does it follow that all entries of X are integers? And that its *trace*, $\mathrm{Tr}(X)$ is an integer?
For bonus points, consider the problem with arbitrary power rather than 5 and / or the issue of finding a basis in which X has integer entries only.

8. Let φ be Euler's totient function; i.e. $\varphi(n)$ is the number of elements in $\{1, 2, \ldots, n\}$ that are relative primes to n. Prove that for all $k \in \mathbb{N}$ and $\varepsilon > 0$ there exist k consecutive natural numbers each satisfying the inequality $\varphi(n)/n < \varepsilon$.

9. Let f be a twice continuously differentiable real function with $f(0) = f'(0) = 0$. Show that $\int_0^1 (f(x))^2\, dx \leq \frac{1}{12} \int_0^1 (f''(x))^2\, dx$.

10. Let m be a positive number, $M = m^{-1}$ and X_1, X_2 two identically distributed independent random variables with values in the interval $[m, M]$ only. At most how much can be the expected value of the ratio X_1/X_2? Generalize to cover the case $M \neq m^{-1}$.