BME Mathematics Contest **3 May 2022** 14^{15}–18^{00}, **CHFMAX**

— Corrected answer sheets are available for review to all participants in office H666 from 14:15 till 17:00 on 12 May. All interested are cordially invited to the ceremonial announcement of results and to the following reception with times and places to be determined at a later date.
— The first exercise is only for first year students. The remaining 9 exercises are open for every participant (both first year and higher year students).
— Each exercise is worth 10 points. Partial solutions are also considered. In some instances even more than 10 points are awarded on a problem (e.g. by providing interesting generalizations or multiple solutions).
— **Each problem should be answered on a separate sheet of paper** carrying the exercise number as well as the name and NEPTUN code of the participant.

THE FIRST EXERCISE IS FOR FIRST YEAR STUDENTS ONLY!

1. Given 12 numbers in the open interval (1,12), show that one can surely choose 3 of them for which there exists an *acute* triangle whose side lengths are the chosen values!

---

2. Let $1 < q \leq 2$ and consider the set S of numbers that can be written as sums of finitely many different (integer, non-negative) powers of q. (For example, the single term q^7 in itself or $q^0 + q^4 + q^5 = 1 + q^4 + q^5$ are such numbers.) Prove that any real greater than or equal to 1/2 can be approximated by such numbers with error at most 1/2; i.e. that for any $x \geq 1/2$ there exists an $s \in S$ such that $|x - s| \leq 1/2$ holds!

3. Does there exist a constant $c \in \mathbb{R}$ such that for any monotonously decreasing function $f : [0, 1] \to \mathbb{R}$, the inequality

$$\int_0^1 e^{-x}\, x\, f(x)\, dx \ \leq c \int_0^1 f(x)\, dx$$

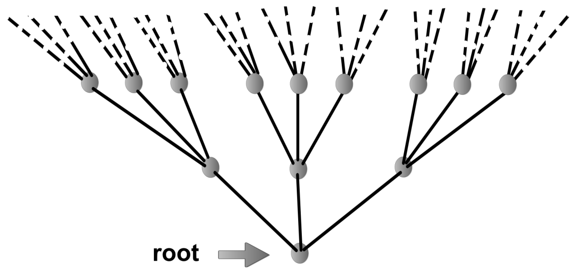
holds? (*Attention, all we require is that* $x \leq y \ \Rightarrow f(x) \geq f(y)$, *but there is no restriction on the sign of* f!)

4. You are swimming in a circular lake. You are almost at the coast, ready to get out, when just in front of you the terrifying human-eating monster appears on the shore. Luckily, it does not like to enter the water and on dry land you are the faster runner – so it only poses a threat if it catches you while you are getting out of the water. You are not yet tired; so keeping your eye on the monster, you decide to swim on and find a better place / moment to get out of the water. Is there a strategy that guarantees your survival if the monster can run a) 4 times, b) 8 times as fast as you can swim? (These are two separate questions.)

5. Let $c_1, \ldots c_n$ and $\alpha_1, \ldots \alpha_n$ be real parameters and $f : \mathbb{R} \to \mathbb{R}$ the function defined by the formula $f(x) = \sum_{k=1}^{n} c_k\, e^{\alpha_k x}$. Prove that if f has at least n different zeros, then f is the constant zero function!

6. Can 16 points of the plane determine precisely 17 lines? (*Here is a question that may help to consider. Suppose some points of the plane are painted black and certain two lines are such that there are x_1 and x_2 black points on them, respectively. Can you give, in terms of x_1 and x_2, an estimate of the number of lines determined by the black points?*)

7. With an independent and uniform (fixed) probability p, edges of a complete, infinite *ternary tree* (see the figure) are painted red. (Unpainted edges remain black.) Determine the probability q (as a function of p) that the resulting graph will contain (as a subgraph) a complete infinite *binary* tree with red edges only, sprouting from the root of the original tree!



8. Can it happen that $p(1) = 0$, $p(3) = 2$ and $p(7) = 18$, if p is a polynomial with integer coefficients, only?

9. Alice and Bob play. There is a game board between them with 6 consecutive squares and a single figure on one of them. If the figure arrives to square nr 1 (i.e. to the position closest to Alice), Alice wins and the game ends. If it arrives to square nr 6 (i.e. to the position closest to Bob), the game ends with Bob's win. Movement of the figure happens once every round according to the following rules. First Alice can pour some fine gold powder from her resources onto the pan of the scales of the game referee (she can pour as much as she wants: even all that she has, or nothing). Then Bob – who saw how much Alice poured into the pan – can give away some of his gold to the referee, who then moves the figure by one unit: if Alice paid more, then towards her and in every other case, towards Bob. The players see each other's wealth all through the game.

At the moment the figure is on square nr 3. Show that if Alice still has more than one and a half times the quantity of gold that Bob currently has, then she has a winning strategy, but if she has just one and a half times (or less) the quantity of gold that Bob has, then he has a winning strategy!

10. Let A and B be two symmetric real matrices of size $n \times n$. Prove that $\det(A^2 + B^2) \geq \det(A)^2 + \det(B)^2$, and decide if this holds even if we drop the requirement that these matrices must be symmetric!