BME Mathematics Contest **10 May 2023** 14^{15}–18^{00}, **Q-I**

— Corrected answer sheets are available for review to all participants in office H666 from 14:15 till 16:00 on 18 May. All interested are cordially invited to the ceremonial announcement of results and to the following reception whose time and place will be announced later.
— Each exercise is worth 10 points. Partial solutions are also considered. In some instances even more than 10 points are awarded on a problem (e.g. by providing interesting generalizations or multiple solutions).
— **Each problem should be answered on a separate sheet of paper** carrying the exercise number as well as the name and NEPTUN code of the participant.

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1. Decide if H can be written as the disjoint union of two nonempty sets with both being closed under addition if

a) $H = \{q \mid q \in \mathbb{Q},\, q > 0\}$,
b) $H = \{x \mid e^x \in \mathbb{Q},\, x > 0\}$.

(These are two separate questions that together count as an exercise.)

2. There are n circles on the plane with radii $r_1, \dots r_n$. Their placement is such that if a line does not intersect any of them, then all circles must fall to the same side of the line. Prove that the circles can be covered with a single disc of radius R where $R = r_1 + \dots r_n$.
If no solution is found – although for less points – one can try show the (considerably weaker) statement that the given circles can be covered with a disc of radius $\sqrt{2}R$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a $2n$-times continuously differentiable, periodic function with period length L (i.e. $f(x + L) = f(x)$ for all $x \in \mathbb{R}$) that has a root (i.e. $\exists x_0 \in \mathbb{R} :\ f(x_0) = 0$). Prove the

$$\max_{x\in\mathbb{R}} |f(x)| \le \left(\frac{L^2}{8}\right)^n \max_{x\in\mathbb{R}} |f^{(2n)}(x)|.$$

inequality!

Remark: the above inequality is not optimal; the number 8 in it could be replaced by 16. However, the proof of the given form is already enough for obtaining the 10 points.

4. Consider the composite numbers: $4, 6, 8, 9, 10, 12, 14, 15, \dots$ Take the reciprocal of the first n and add them: $S_n := (1/4) + (1/6) + \dots + (1/r_n)$ where r_n is the n^{th} composite number (in order of magnitude). Can S_n be an integer?

5. For a given $m \in \mathbb{N}$, the functions $f_1, \dots f_n$ and $g_1, \dots g_n$ satisfy the equation

$$\sum_{k=1}^{n} f_k(x) g_k(y) = (x+y)^m$$

for all pairs of reals $x, y \in \mathbb{R}$. Can n be smaller than or equal to m?

6. Numbers are drawn in an independent and uniform manner from the interval $(0,1)$ until their sum exceeds 1. What is the expected number of draws and the expected value of the obtained sum?

7. A subset $H \subset \mathbb{N}$ is called *arithmetic triple free*, if for any $n, d \in \mathbb{N}, d \neq 0$, at least one of the numbers $n, n+d, n+2d$ is not in H. We further say that H is a *maximal* arithmetic triple free subset if it is arithmetic free but cannot be enlarged in a way such that this property is preserved: for any $n \in \mathbb{N} \setminus H$, the set $H \cup \{n\}$ is *no longer* arithmetic triple free. Give an infinite arithmetic triple free subset of $\mathbb{N}$ in an explicit manner, and prove that there exists a maximal arithmetic triple free subset $H \subset \mathbb{N}$ which is "exponentially rare": i.e. for which there exists an $a > 1$ such that, denoting the n^{th} element (in order of magnitude) of H by h_n, we have $\forall n \in \mathbb{N}: \quad h_n > a^n$.
Hint: if in the first part of the exercise we construct a <u>suitable</u> H_0 set, then we will be able to "grow it" into a maximal such set which is still exponentially rare.

8. Let A be an $n \times n$ real matrix, whose trace – i.e. the sum of its diagonal entries – is zero. Prove that there exists an orthogonal matrix O such that *every* diagonal entry of OAO^T is zero!

9. Let $f : [0,1] \to \mathbb{R}$ be a monotonously increasing function and

$$I_1 = \int_0^1 \int_0^1 |f(x) - f(y)|\, dx\, dy, \quad I_2 = \int_0^1 \int_0^1 (f(x) - f(y))^2\, dx\, dy.$$

Show that $I_1^2 \leq (2/3) I_2$ and that in case f is a polynomial of degree ≤ 1, then the given relation is satisfied as an equality!

10. There are n children sitting around a large table. In front of each, there is a card with its face down. We want to have all cards face up. The children are obedient, but each takes any request made for a friend upon himself or herself. Therefore, if one of them is asked to turn over the card in front of him or her, he/she and all his/her friends will turn over theirs. Prove that by asking appropriately chosen children to turn over their cards, it is always possible to achieve the state with all cards facing up! (Note: friendship is symmetric, but not necessarily transitive.)