— Corrected answer sheets are available for review to all participants in office H666 from 15:15 till 17:00 on 15 May. All interested are cordially invited to the ceremonial announcement of results and to the following reception whose time and place will be announced later.
— Each exercise is worth 10 points. Partial solutions are also considered. In some instances even more than 10 points are awarded on a problem (e.g. by providing interesting generalizations or multiple solutions).
— **Each problem should be answered on a separate sheet of paper** carrying the exercise number as well as the name and NEPTUN code of the participant.

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1. Let $q, n > 1$ be integers. We would like to fix some $a_0 \le a_1 \le \ldots \le a_n$ real values such that every natural number $m < q^{n+1}$ could be expressed as

$$m = \sum_{k=0}^{n} c_k a_k$$

with the coefficients c_k ($k = 0, 1, \ldots, n$) being non-negative integers, all (strictly) smaller than q. How many choice of $(a_0, \ldots, a_n)$ will make this happen?

2. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a differentiable function. Can it be that $f' = f \circ f$? First show that if it was so, then there would exist an $x_0 > 0$ such that $f(x_0) - x_0 > 1$, then using this (or in any other way) prove that f cannot satisfy the equation in question!

3. Let A, B be two real symmetric matrices of size 77×77. Prove that there must exist a vector $\underline{v} \neq 0$ on which A and B commute; i.e. for which $AB\underline{v} = BA\underline{v}$!

4. Prove that for any natural number n,

$$\sum_{k=1}^{n} \frac{k \cdot \binom{n}{k} \cdot k!}{n^{k+1}} = 1$$

holds!

5. Someone drew a square on the Euclidean plane with a pencil and then marked a point on each of its four sides in red. Unfortunately, the square faded over time; now only the 4 red dots are visible. For what configuration of the 4 dots is it true that there is only a single square whose sides contain all 4 marked points with each side containing at least one? Give a Euclidean construction that reproduces the faded square whenever its position is uniquely determined by the red dots!

6. Five competitors of equal skill shoot on a circular target; each of them 10 times. We know that no shot will completely miss and moreover that the shots hit the target in an independent and uniform manner. How many shots are expected to be closer to the center of the target than the best shot of the youngest competitor? (The answer should preferably be not some hard-to-evaluate formula involving integrals or sums but an explicitly given number!)

7. Let $\underline{u}_1, \ldots, \underline{u}_k$ be k points of the unit sphere $S = \{\underline{x} \in \mathbb{R}^n \,|\, x_1^2 + \ldots + x_n^2 = 1\}$. We would like to move them to some $\underline{u'}_1, \ldots, \underline{u'}_k \in S$ points so that the distance would be halved between them; i.e. to have $d(\underline{u'}_j, \underline{u'}_l) = \frac{1}{2} d(\underline{u}_j, \underline{u}_l)$ for all j, l, where d is the Euclidean distance:

$$d(\underline{v}, \underline{w}) \equiv \|\underline{v} - \underline{w}\| \equiv \sqrt{(\underline{v} - \underline{w}) \cdot (\underline{v} - \underline{w})} \equiv \sqrt{(v_1 - w_1)^2 + \ldots + (v_n - w_n)^2}.$$

Show that this can be done if and only if $\underline{u}_1, ..., \underline{u}_k$ lie on a hyperplane; i.e. if there exist a vector $\underline{w} \neq 0$ and scalar c such that $\underline{w} \cdot \underline{u}_j = c$ for all $j = 1, \ldots, k$ indices!

8. The numbers $\alpha_1, \alpha_2, \ldots, \alpha_{2n} \in \mathbb{C}$ satisfy the equation

$$\alpha_1^{2k-1} + \alpha_2^{2k-1} + \ldots + \alpha_{2n}^{2k-1} = 0$$

for every $k = 1, \ldots, n$. Prove that they can be arranged into pairs of the form $(\beta, -\beta)$!

9. Let $\mathcal{I}$ be a finite collection of closed bounded intervals of $\mathbb{R}$ and $|\mathcal{I}| \geq k \geq 3$. Suppose that from any k members of $\mathcal{I}$, one can choose 3 that have a common point. Prove that the whole collection can be "blocked" by $k-2$ points; i.e. that one can choose $k-2$ points in such a way that every element of $\mathcal{I}$ will contain at least one!

10. Let $S \subset \mathbb{R}$ be a set having a positive Lebesgue measure. Without assuming the continuum-hypothesis, prove: the cardinality of S cannot be smaller than that of $\mathbb{R}$!