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THE LEAST SEPARATIVE CONGRUENCE ON A WEAKLY COMMUTATIVE SEMIGROUP

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In [1], a relation $\pi$ on an arbitrary semigroup $S$ has been defined. For elements $a$ and $b$ of $S$, $a \pi b$ if and only if $ab^n = b^{n+1} = b^n a$ and $ba^n = a^{n+1} = a^n b$ for a positive integer $n$. It has been proved that, if $S$ is a weakly commutative semigroup, then $\pi$ is a separative congruence on $S$. The author has proved that, if $S$ is a duo semigroup (i.e., every one-sided ideal of $S$ is two-sided), then $S/\pi$ is a maximal separative homomorphic image of $S$. See Theorem 5 of [1].

In this note we shall extend this result on duo semigroups to weakly commutative semigroups.

**Definition 1.** A semigroup $S$ is called weakly commutative if, for any $a, b \in S$, we have $(ab)^k = xa = by$ for some $x, y \in S$ and a positive integer $k$. See Definition 6.4 of [2].

**Definition 2.** We define a relation $\pi$ on a semigroup $S$ as follows: $a \pi b$ if and only if $ab^n = b^{n+1} = b^n a$ and $ba^n = a^{n+1} = a^n b$ for a positive integer $n$. See [1].

**Remark 1.** Let $S$ be a semigroup, $a, b \in S$ and $\varphi$ a congruence on $S$. If $ab^{n+1} \varphi b^{n+2}$ and $(ab^n)^m \varphi (b^{n+1})^m$ for positive integers $n$ and $m$, then $(ab^n)^M \varphi (b^{n+1})^M$ for any positive integer $M > m$.

Similarly, if $b^{n+1}a \varphi b^{n+2}$ and $(b^n a)^m \varphi (b^{n+1})^m$ for positive integers $n$ and $m$, then $(b^n a)^M \varphi (b^{n+1})^M$ for any positive integer $M > m$.

**Proof.** We prove only the first part of the remark, because the second part can be proved in a similar way. Let us suppose $ab^{n+1} \varphi b^{n+2}$, $(ab^n)^m \varphi (b^{n+1})^m$ for some $a, b \in S$ and positive integers $n$ and $m$. Let $M$ be an arbitrary positive integer with $M > m$. Then $(ab^n)^M = (ab^n)^{M-m} (ab^n)^m \varphi (b^{n+1})^{M-m} (b^{n+1})^m = (ab^n)^{M-m-1} a b^n b^{n+1} (b^{n+1})^{m-1} \varphi (ab^n)^{M-m-1} (b^{n+1})^{m+1} \varphi \ldots \varphi (b^{n+1})^{m+M-m} = (b^{n+1})^M$.

**Lemma 1.** (B. Pondělíček [1].) If $S$ is a weakly commutative semigroup, then $\pi$ is a separative congruence on $S.$
Theorem 1. If $S$ is a weakly commutative semigroup, then $S/\pi$ is a maximal separative homomorphic image of $S$.

Lemma 2. Let $S$ be a weakly commutative semigroup and $\varrho$ a separative congruence on $S$. Let $a, b \in S$. If $ab^n \varrho b^{n+1} \varrho b^a a$ and $ba^n \varrho a^{n+1} \varrho a^b$ for a positive integer $n$, then $a \varrho b$.

Proof. Since $\varrho$ is a separative congruence, the result is true for $n = 1$. Assume now that the assertion holds for $n \geq 1$. Let $ab^{n+1} \varrho b^{n+2} \varrho b^{n+1} a$ and $ba^{n+1} \varrho a^{n+2} \varrho \varrho a^{n+1} b$. Since $S$ is weakly commutative, $(ab^n)^k = by$ for some $y \in S$ and a positive integer $k$. Thus $(ab^n)^{k+1} = ab^{n+1} y \varrho b^{n+2} y = b^{n+1} (ab^n)^k = b^{n+1} ab^n (ab^n)^{k-1} \varrho \varrho (b^{n+1})^2 (ab^n)^{k-1} \varrho \ldots \varrho (b^{n+1})^{k+1}$. Similarly, $(b^a)^{t} = ub$ for some $u \in S$ and a positive integer $t$. Thus

$$
(b^a)^{t+1} = ub^{n+1} a \varrho ub^{n+2} = (b^a)^{t+1} a = (b^a)^{t-1} b^a ab^{n+1} \varrho \varrho (b^a)^{t-1} (b^{n+1})^2 \varrho \ldots \varrho (b^{n+1})^{t+1} .
$$

Consequently, $(ab^n)^{k+1} \varrho (b^{n+1})^{k+1}$ and $(b^{n+1})^{k+1} \varrho (b^a)^{t+1}$. By Remark 1, it follows that

$$
(ab^n)^m \varrho (b^{n+1})^m \varrho (b^a)^m \quad \text{for a positive integer } m .
$$

Let $m_1 = \min \{ m : (ab^n)^m \varrho (b^{n+1})^m \varrho (b^a)^m \}$. We prove that $m_1 = 1$. Let us suppose that $m_1 \neq 1$ and let

$$
m_2 = \begin{cases} m_1 & \text{if } m_1 \text{ is an even number}, \\ m_1 + 1 & \text{if } m_1 \text{ is an odd number}. \end{cases}
$$

Then, by Remark 1,

$$
(ab^n)^{m_2} \varrho (b^{n+1})^{m_2} \varrho (b^a)^{m_2} .
$$

Let $m_3 = m_2 / 2$. Then $m_3 > m_1$ and

$$
((ab^n)^{m_3})^2 = (ab^n)^{2m_3} = (ab^n)^{m_2} \varrho (b^{n+1})^{m_2} = ((b^{n+1})^{m_3})^2 = (b^{n+1})^{m_3} \varrho (b^a)^{m_2} = (b^a)^{2m_3} = ((b^a)^{m_3})^2 .
$$

Moreover,

$$
(ab^n)^{m_3} (b^{n+1})^{m_3} = (ab^n)^{m_3-1} ab^n b^{n+1} (b^{n+1})^{m_3-1} \varrho \varrho (ab^n)^{m_3-1} (b^{n+1})^{m_3-1} \varrho \ldots \varrho (b^{n+1})^{2m_3} = ((b^{n+1})^{m_3})^2
$$

and

$$
(b^a)^{m_3} (b^{n+1})^{m_3} = (b^a)^{m_3-1} b^a ab^n b^{n+1} (b^{n+1})^{m_3-1} \varrho \varrho (b^a)^{m_3-1} (b^{n+1})^{m_3-1} \varrho \ldots \varrho (b^{n+1})^{2m_3} = ((b^{n+1})^{m_3})^2 .
$$

Thus we have $((ab^n)^{m_3})^2 \varrho (ab^n)^{m_3} (b^{n+1})^{m_3} \varrho ((b^{n+1})^{m_3})^2$ and

$$
((b^a)^{m_3})^2 \varrho (b^a)^{m_3} (b^{n+1})^{m_3} \varrho ((b^{n+1})^{m_3})^2 .
$$

Since $\varrho$ is a separative congruence, it follows that

$$
(ab^n)^{m_3} \varrho (b^{n+1})^{m_3} \varrho (b^a)^{m_3} .
$$
Since this result contradicts \( m_3 < m_1 \), we have \( m_1 = 1 \). Consequently, \( ab^n \vartriangleleft b^{n+1} \vartriangleleft a^b \vartriangleleft a^n b \). We can prove \( ba^n \vartriangleleft a^{n+1} \vartriangleleft a^b \) in a similar way. Hence we get \( a \vartriangleleft b \). The result therefore follows by induction. Thus the lemma is proved.

The proof of Theorem 1. Let \( \vartriangleleft \) be an arbitrary separative congruence on a weakly commutative semigroup \( S \). If \( a \vartriangleleft b \) \((a, b \in S)\), then \( ab^n = b^{n+1} = b^a \) and \( ba^n = a^{n+1} = a^b \) for a positive integer \( n \). Thus \( ab^n \vartriangleleft b^{n+1} \vartriangleleft b^a \) and \( ba^n \vartriangleleft a^{n+1} \vartriangleleft a^b \). By Lemma 2, it follows that \( a \vartriangleleft b \). Consequently \( \pi \subseteq \vartriangleleft \).

**Corollary 1.** If \( S \) is a duo semigroup, then \( S/\pi \) is a maximal separative homomorphic image of \( S \).

**Corollary 2.** If \( S \) is a normal semigroup (i.e. \( aS = Sa \) for any \( a \in S \)), then \( S/\pi \) is a maximal separative homomorphic image of \( S \).

**Corollary 3.** If \( S \) is a quasicommutative semigroup (i.e. for any \( a, b \in S \), we have \( ab = b^a \) for a positive integer \( r \)), then \( S/\pi \) is a maximal separative homomorphic image of \( S \).

**References**


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