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Special Classes of Semigroups

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Preface

Semigroups are generalizations of groups and rings. A group is a semigroup in which the operation is invertible; a ring is a multiplicative semigroup in wich the operation together with an additive operation satisfies certain conditions. In the beginning of the development of semigroup theory investigations were strongly motivated by this fact. Semigroups in which every element has an inverse were in focus, and the results of ring theory were adapted for semigroups. In algebra, congruences play a central role. In this respect, there is a difference between semigroups and groups or rings. The congruences of a group are uniquely determined by its normal subgroups, and there is a bijection between the congruences and the ideals of a ring. In semigroup theory the situation is more complicated. Although an ideal of a semigroup defines a special congruence, there are no subsemigroups which uniquely determine the congruences of semigroups. This problem envolves many difficulties. Thus semigroup theory has developed special methods and new semigroup classes have come into the center of interest.

In semigroup theory there are certain kinds of band decompositions which are very useful in the study of the structure of semigroups. There is a number of special semigroup classes in which these decompositions can be used very successfully, because the semigroups belonging to them are decomposable into special bands of left archimedean or right archimedean, archimedean semigroups. The structure of these different types of archimedean semigroups is thoroughly studied in these semigroup classes. In this book, we focus our attention on such classes of semigroups. Some of them are partially discussed in earlier books, but in the last thirthy years new semigroup classes have appeared and a fairly large body of material has been published on them. In this book we provide a systematic review on this subject.

In the first chapter of the book we present notions and results of semigroup theory needed in the sequel. This chapter also contains theorems and lemmas (with proof) which are used throughout the book. The other chapters are devoted to special semigroup classes. These are Putcha semigroups, commutative semigroups, weakly commutative semigroups, $\mathcal{R}$-commutative semigroups, $\mathcal{L}$-commutative semigroups, $\mathcal{H}$-commutative semigroups, conditionally commutative semigroups, $\mathcal{RC}$-commutative semigroups, quasicommutative semigroups, medial semigroups, right commutative semigroups, externally commutative semigroups, Em semigroups, exponential semigroups, WE-m semigroups, weakly exponential semigroups, (m,n)-commutative semigroups and $n_{(2)}$-permutable semigroups. In any of these semigroup classes we deal with different kinds of band decompositions, describe the structure of simple semigroups and that of archimedean semigroups, characterize regular semigroups, inverse semigroups, study the embedding of semigroups into groups and into semigroups which are unions of groups, construct least left (right) separative and weakly separative congruences, determine subdirect irreducible semigroups and describe semigroups whose lattice of congruences is a chain with respect to inclusion.

In this book we also present theorems stated and proved in other books. Other theorems, lemmas and corollaries are fully proved. In general, we present the original proofs, but in a number of cases we give a new and shorter one.

Finally, I would like to express my hearty thanks to *Professor Jenő Szép* for his assistance in every phase of writing this book. I would also like to thank *Mrs. Éva Németh* for helping me in preparing the camera-redy version of the LaTeX-file. I further acknowledge the encouragement and support of the publisher in producing the book.

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Attila Nagy

Chapter 1

Preliminaries

In this chapter we present those basic notions and results of semigroup theory which are used in this book. This chapter contains further theorems and lemmas. There are several assertions corresponding to different semigroup classes examined in this book whose proofs are similar to each other and based on common ideas. The common parts of these proofs are formulated as theorems and lemmas, and they are presented and proved in this chapter.

Semigroups

Definition 1.1 *Let S be a nonempty set. By a binary operation on S we mean a function $*$ from $S \times S$ into S. The image in S of the elements $(a, b) \in S \times S$ is denoted by $a * b$. Frequently, we write ab for $a * b$.*

Definition 1.2 *A binary operation on a set S is said to be associative if $a(bc) = (ab)c$ is satisfied for all $a, b, c \in S$. If $ab = ba$ holds for every $a, b \in S$ then we say that the operation is commutative.*

Definition 1.3 *A set together with an associative binary operation is called a semigroup. A semigroup having only one element is said to be trivial. A semigroup is said to be a commutative semigroup if the operation is commutative.*

Subsemigroups

Definition 1.4 *A nonempty subset A of a semigroup S is called a subsemigroup of S if A is closed under the operation, that is, $ab \in A$ for every $a, b \in A$.*

Definition 1.5 *A subset X of a semigroup S is called a set of generators of S (or S is generated by X) if, for every element $s \in S$, there are elements $x_1, \ldots, x_n \in X$ such that $s = x_1 \ldots x_n$. In such a case, we write $S = \langle X \rangle$. A semigroup is said to be finitely generated if it has a finite set of generators. We* *say that a semigroup is a cyclic semigroup if it is generated by a single element. An element a of a semigroup S is called periodic if the cyclic subsemigroup $\langle a\rangle$ of S generated by a is finite. A semigroup is called a periodic semigroup if its every element is periodic.*

Definition 1.6 *We say that a semigroup S has the permutation property P_n if, for every sequence $(x_1,\dots,x_n)$ of elements of S, there is a non-identity permutation σ of the set $\{1,2,\dots,n\}$ such that $x_1x_2\dots x_n = x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(n)}$. We say that a semigroup has the permutation property if it has the permutation property P_n for some positive integer $n\geq 2$.*

Theorem 1.1 *([84]) A finitely generated semigroup is finite if and only if it is periodic and has the permutation property.*

Free semigroups

Definition 1.7 *Let X be a non-empty set and let $\mathcal{F}_X$ denote the set of all finite sequences of elements of X. If $(x_1,\dots,x_n)$ and $(y_1,\dots,y_m)$ are elements of $\mathcal{F}_X$ then we define their product by simple juxtaposition:*

$$(x_1,\dots,x_n)(y_1,\dots,y_m)=(x_1,\dots,x_n,y_1,\dots,y_m);$$

this product is associative. The semigroup $\mathcal{F}_X$ is called the free semigroup over the set X. The elements of $\mathcal{F}_X$ is called words. As $(x_1,\dots,x_n)=(x_1)\dots(x_n)$, the set X is a set of generators of $\mathcal{F}_X$.

Identities

Definition 1.8 *An element e of a semigroup S is called a left (right) identity element of S if $ea=a$ ($ae=a$) holds for every $a\in S$. We say that $e\in S$ is an identity element of a semigroup S if e is both a left and a right identity element of S.*

It is easy to see that every semigroup has at most one identity element. Moreover, if a semigroup has a right identity element and a left identity element then it contains an identity element.

Definition 1.9 *A semigroup containing an identity element is called a monoid.*

If S is a semigroup then let S^1 denote the semigroup $S\cup\{1\}$ arising from S by the adjunction of an identity element 1 unless S already has an identity element, in which case $S^1=S$.

For example, it is often convenient to work with the free monoid $\mathcal{F}_X^1$ rather than the free semigroup $\mathcal{F}_X$. The adjoined identity element 1 may be regarded as the "empty word".

Bicyclic semigroup

Definition 1.10 *A monoid S (with the identity element e) is called a bicyclic semigroup if it is isomorphic to a semigroup C generated by two elements a, b with the single generating relation $ab = e$.*

Theorem 1.2 *([19]) Let e, a, b be elements of a semigroup S such that $ae = ea = a$, $be = eb = b$ and $ab = e$, $ba \neq e$. Then every element of the subsemigroup $\langle a, b\rangle$ of S generated by a and b is uniquely expressible in the form $b^m a^n$ (m and n are non-negative integers), and hence $\langle a, b\rangle$ is a bicyclic semigroup.*

Zeros

Definition 1.11 *An element f of a semigroup S is called a left (right) zero element of S if $fa = f$ ($af = f$) for every $a \in S$. An element of a semigroup S is called a zero element of S if it is both a left and a right zero element of S.*

For an arbitrary non-empty set S, we can define an operation by $ab = a$ for every $a, b \in S$. It is easy to see that S is a semigroup in which every element is a left zero element. A semigroup with this property is called a *left zero semigroup.* A semigroup in which every element is a right zero element is called a *right zero semigroup.*

It is easy to see that every semigroup has at most one zero element. Moreover, if a semigroup has a left zero element and a right zero element then it has a zero element.

If S is a semigroup then let S^0 denote the semigroup $S \cup \{0\}$ arising from S by the adjunction of a zero element 0 unless S already has a zero element, in which case $S^0 = S$.

For any non-empty set S and an arbitrary element $a \in S$, we can define an operation $*$ on S by $x * y = a$ for all $x, y \in A$. It is easy to see that $(S, *)$ is a semigroup with a zero element a. In this semigroup the product of any two elements is the zero element of S. A semigroup with this property is called a *null semigroup.*

An element s of a semigroup S with zero is called a *left (right) divisor of zero* if there is an element $x \neq 0$ in S such that $sx = 0$ ($xs = 0$). An element is called a *divisor of zero* if it is a left divisor or a right divisor of zero.

Definition 1.12 *A semigroup S with a zero element 0 is called a nil semigroup if, for every $a \in S$, there is a positive integer n such that $a^n = 0$.*

Idempotents

Definition 1.13 *An element e of a semigroup is called an idempotent element if $e^2 = e$.*

The set E_S of all idempotent elements of a semigroup S is partially ordered by $e \leq f$ if and only if $ef = fe = e$. If $e \leq f$ and $e \neq f$ then we write $e < f$. It is easy to see that if a semigroup S has a zero element 0 then $0 \leq e$ for every $e \in E_S$. A non-zero idempotent element f of a semigroup S is called a *primitive idempotent* if $e \leq f$ implies $e = 0$ or $e = f$ for every $e \in E_S$.

Definition 1.14 *A semigroup in which every element is an idempotent element is called a band.*

The classification of bands can be faund in [75]. In this book we need only some of them listed in Definition 1.15.

Definition 1.15 *A commutative band is called a semilattice. A band satisfying the identity $aba = a$ is called a rectangular band. We say that a band is a left (right) normal band if it satisfies the identity $axy = ayx$ ($xya = yxa$). A dand satisfying the identity $axya = ayxa$ is called a normal band. A band is called a left (right) regular band if it satisfies the identity $axa = ax$ ($axa = xa$).*

We note that a left (right) zero semigroup is a left (right) regular band.

Theorem 1.3 *([19]) A semigroup is a rectangular band if and only if it is a direct product of a left zero semigroup and a right zero semigroup.*

Cancellation and separativity of semigroups

Definition 1.16 *A semigroup S is called a left (right) cancellative semigroup if $ax = ay$ ($xa = ya$) implies $x = y$ for every $a, x, y \in S$. We say that S is a cancellative semigroup if it is both left and right cancellative. S said to be a weakly cancellative semigroup if $ax = ay$ and $xa = ya$ together imply $x = y$ for every $a, x, y \in S$.*

Lemma 1.1 *A semigroup S is weakly cancellative if and only if it satisfies the condition that, for every $a, b, x, y \in S$, $ax = ay$ and $xb = yb$ together imply $x = y$.*

Proof. Let S be a weakly cancellative semigroup and $a, b, x, y \in S$ be arbitrary elements with $ax = ay$ and $xb = yb$. Then $bax = bay$ and $xba = yba$ which imply $x = y$.

Conversely, assume that a semigroup S satisfies the condition that, for every $a, b, x, y \in S$, $ax = ay$ and $xb = yb$ imply $x = y$. Let $a, x, y \in S$ be arbitrary elements with $ax = ay$ and $xa = ya$. Then, for $b = a$, we get $ax = ay$ and $xb = yb$ and so $x = y$. □

Definition 1.17 *A semigroup S is said to be a left (right) separative semigroup if $ab = a^2$ and $ba = b^2$ ($ab = b^2$ and $ba = a^2$) imply $a = b$ for every $a, b \in S$. A semigroup is said to be a separative semigroup if it is both left and right separative. S is called a weakly separative semigroup if $a^2 = ab = b^2$ implies $a = b$ for every $a, b \in S$.*

It is easy to see that every left (right, weakly) cancellative semigroup is left (right, weakly) separative.

Lemma 1.2 *([13]) If S is a weakly separative semigroup then $ab^{n+1} = b^{n+1}a$ and $ab^n = b^n a$ together imply $ab = ba$ for every $a, b \in S$ and every integer $n > 1$.*

Proof. Let S be a weakly separative semigroup and $a, b \in S$ be arbitrary elements satisfying $ab^{n+1} = b^{n+1}a$, $ab^n = b^n a$ for an integer $n > 1$. Then

$$(bab^{n-1})^2 = bab^n ab^{n-1} = b^{n+1}a^2 b^{n-1} = a^2 b^{2n} = (ab^n)^2$$

and

$$(ab^n)(bab^{n-1}) = a^2 b^{2n} = (ab^n)^2.$$

Thus, by weakly separativity, it follows that

$$bab^{n-1} = ab^n.$$

In the same way we obtain

$$b^{n-1}ab = b^n a.$$

Hence

$$bab^{n-1} = ab^n = b^n a = b^{n-1}ab.$$

From these we get

$$(ab^{n-1})^2 = ab^{n-1}ab^{n-1} = ab^{2n-2}a,$$

$$(b^{n-1}a)^2 = b^{n-1}ab^{n-1} = ab^{2n-2}a,$$

$$(ab^{n-1})(b^{n-1}a) = ab^{2n-2}a$$

and, by weakly separativity,

$$ab^{n-1} = b^{n-1}a.$$

At this point we may conclude that $ab = ba$. □

Homomorphisms

Definition 1.18 *A mapping ϕ of a semigroup $(S, *)$ into a semigroup $(T, \circ)$ is called a homomorphism if $\phi(a * b) = \phi(a) \circ \phi(b)$ for every $a, b \in S$. If ϕ is one-to-one then it is called an isomorphism or an embedding of S into T. If ϕ is also maps S onto T then we say that ϕ is an isomorphism of S onto T and S is isomorphic to T.*

Congruences

Definition 1.19 *By a left (right) congruence on a semigroup S we mean an equivalence relation α of S if $(a, b) \in \alpha$ implies $(sa, sb) \in \alpha$ ($(as, bs) \in \alpha$) for every $a, b, s \in S$. An equivalence relation of S is called a congruence if it is both a left and a right congruence of S.*

It is easy to see that an equivalence relation α is a congruence on a semigroup S if and only if $(a, b) \in \alpha$ and $(c, d) \in \alpha$ imply $(ac, bd) \in \alpha$ for every $a, b, c, d \in S$.

Definition 1.20 *A non-empty subset H of a semigroup S is called a normal complex of S if $xHy \cap H \neq \emptyset$ implies $xHy \subseteq H$ for every $x, y \in S^1$.*

Lemma 1.3 *([40]) If H is a normal complex of a semigroup S then the relation α_H defined by $a \ \alpha_H \ b$ if and only if there is a positive integer n and there are elements $x_i, y_i \in S^1$ and $p_i, q_i \in H$ $(i = 1, 2, \ldots, n)$ such that*

$$a = x_1 p_1 y_1, \ x_1 q_1 y_1 = x_2 p_2 y_2, \ldots, x_n q_n y_n = b$$

is the least congruence on S such that H is a congruence class.

The set $\mathcal{L}(S)$ of all congruences of a semigroup S is partially ordered (by the inclusion of relations) such that any two elements have a greatest lower bound and a least upper bound. With other word, $\mathcal{L}(S)$ is a lattice which is called the *congruence lattice* of the semigroup S.

Let α be a congruence on a semigroup S and denote $[a]_\alpha$ the α-class of S containing $a \in S$. Then $S/\alpha = \{[a]_\alpha : a \in S\}$ form a semigroup under the operation $[a]_\alpha [b]_\alpha = [ab]_\alpha$. This semigroup is called the *factor semigroup* of S modulo α. The mapping $a \mapsto [a]_\alpha$ $(a \in S)$ is called the *canonical homomorphism* of S onto S/α. Conversely, if ϕ is a homomorphism of semigroup S onto a semigroup T then the equivalence relation σ on S induced by ϕ, defined by $(a, b) \in \sigma$ if and only if $\phi(a) = \phi(b)$, is a congruence on S and S/σ is isomorphic to T.

$\mathcal{C}$-decompositions

Let $\mathcal{C}$ be a class of semigroups. A congruence α of S is called a $\mathcal{C}$-congruence of S if S/α belongs to $\mathcal{C}$. The meet of all $\mathcal{C}$-congruences of a semigroup, if it is a

$\mathcal{C}$-congruence, is called the *least $\mathcal{C}$-congruence* of S. If α is the least $\mathcal{C}$-congruence of a semigroup S then the factor semigroup S/α is the *greatest $\mathcal{C}$-homomorphic image* of S.

If S is a semigroup and α is a band congruence on S, that is, $B = S/\alpha$ is a band then the α-classes S_i $(i \in B)$ of S are subsemigroups of S. In this case we say that S is a band B of semigroups S_i $(i \in B)$. With other words, S is decomposable into the band B of semigroups S_i $(i \in B)$. A semigroup is called band indecomposable if the universal relation of S is the only band congruence of S.

Theorem 1.4 *([75]) Every semigroup is decomposable into a semilattice of semilattice indecomposable semigroups. With other words, every semigroup has a least semilattice congruence η and the η-classes of S are semilattice indecomposable.*

Theorem 1.5 *([75]) Every band is decomposable into a semilattice of rectangular bands.*

Archimedean semigroups

Let S be a semigroup and $a, b \in S$. Cosider the following notations.

(1) $a|b$ iff $b \in S^1aS^1$.

(2) $a|_lb$ $(a|_rb)$ iff $b \in S^1a$ $(b \in aS^1)$.

(3) $a|_tb$ iff $a|_lb$ and $a|_rb$.

(4) $a - b$ iff $a|b^i$ and $b|a^j$ for some positive integers i and j.

(5) $a -_l b$ iff $a|_lb^i$ and $b|_la^j$ for some positive integers i and j.

(6) $a -_r b$ iff $a|_rb^i$ and $b|_ra^j$ for some positive integers i and j.

(7) $a -_t b$ iff $a|_tb^i$ and $b|_ta^j$ for some positive integers i and j.

Definition 1.21 *A semigroup S is called archimedean (left archimedean, right archimedean, t-archimedean) if, for every $a, b \in S$, we have $a - b$ $(a -_l b$, $a -_r b$, $a -_t b)$.*

With other words, a semigroup S is called a *left (right) archimedean semigroup* if, for every $a, b \in S$, there are positive integers m and n such that $a^m \in S^1b$ and $b^n \in S^1a$ $(a^m \in bS^1$ and $b^n \in aS^1)$. A semigroup is called a *t-archimedean semigroup* if it is both left and right archimedean. We say that a semigroup S is an *archimedean semigroup* if, for every $a, b \in S$, there are positive integers m and n such that $a^m \in S^1bS^1$ and $b^n \in S^1aS^1$.

Theorem 1.6 *([81]) A semigroup S is a band of left (right) archimedean semigroups if and only if, for all $a \in S$ and $x, y \in S^1$, $xay -_l xa^2y$ ($xay -_r xa^2y$).*

Theorem 1.7 *([81]) A semigroup S is a band of t-archimedean semigroups if and only if, for all $a \in S$ and $x, y \in S^1$, $xay -_t xa^2y$. In such a case the corresponding band congruence is equal to the relation $-_t$ and is the finest band congruence on S.*

Theorem 1.8 *([50]) If a semigroup satisfies the identity $(ab)^3 = a^2b^2(ab) = (ab)a^2b^2$ then it is a band of t-archimedean semigroups.*

Proof. Let S be a a semigroup satisfying the identity $(ab)^3 = a^2b^2(ab) = (ab)a^2b^2$. By Theorem 1.7, it is sufficient to show that $xay -_t xa^2y$ for every $a, x, y \in S$. Since

$$(xay)^4 = x(ayx)^3ay = xa^2(yx)^2ayxay = (xa^2y)xyxayxay$$

then

$$xa^2y \mid_r (xay)^4.$$

We can prove, in a similar way, that

$$xa^2y \mid_l (xay)^4.$$

Hence

$$xa^2y \mid_t (xay)^4.$$

Since

$$\begin{aligned}
(xa^2y)^7 &= x(a^2(yxa^2))y(xa^2y)^5 \\
&= x(a^2(yxa^2))yxa(ayxa)^3(ayxa)ay \\
&= x(a^2(yxa^2))yxa(a^2(yxa)^2)(ayxa)^2ay \\
&= xa^2(yxa^2)^2ayxayxa(ayxa)^2ay \\
&= x(a^2(yxa^2)^2)ayx(ayxa)^3ay \\
&= x(a^2(yxa^2)^2)ayxa^2(yxa)^2(ayxa)ay \\
&= x(a^2(yxa^2)^2ayxa^2)(yxa)^2ayxa^2y \\
&= x(ayxa^2)^3(yxa)^2ayxa^2y \\
&= (xay)xa^2(ayxa^2)^2(yxa)^2ayxa^2y
\end{aligned}$$

then

$$xay \mid_r (xa^2y)^7.$$

We can prove, in a similar way, that

$$xay \mid_l (xa^2y)^7.$$

Hence

$$xay \mid_t (xa^2y)^7$$

and so

$$xay \;-_t\; xa^2y.$$

□

Theorem 1.9 *([81]) If a semigroup is a band of archimedean semigroups then it is a semilattice of archimedean semigroups.*

Theorem 1.10 *([80]) A semigroup S is a semilattice of left (right) archimedean semigroups if, for every $a, b \in S$, the assumption $b \in aS$ ($b \in Sa$) implies $b^n \in Sa$ ($b^n \in aS$) for some positive integer n.*

Theorem 1.11 *([80]) A semigroup S is a semilattice of archimeden semigroups if and only if, for every $a, b \in S$, the assumption $a \in S^1bS^1$ implies $a^n \in S^1a^2S^1$ for some positive integer n.*

We remark that a little bit more complete version of Theorem 1.11. will be proved later (see Theorem 2.1.).

Strong semilattice of semigroups

Definition 1.22 *Let a semigroup S be a semilattice of semigroups S_i, $i \in I$. Assume that, for every $i, j \in I$ with $i \geq j$, there is a homomorphism $(\ )f_{i,j}$ of S_i into S_j such that the following are satisfied.*

(1) If $i > j > k$ then $f_{i,j}f_{j,k} = f_{i,k}$.

(2) For each $i \in I$, $f_{i,i}$ is the identity mapping of S_i.

(3) If $a \in S_i$ and $b \in S_j$ then $ab = (a)f_{i,ij}(b)f_{j,ij}$.

In such a case S is called a strong semilattice of semigroups S_i ($i \in I$). The family $\{f_{i,j}\}_{i \geq j}$ is said to be a transitive system of homomorphisms which determines the multiplication in S.

Direct product, subdirect product

Definition 1.23 *Let $\{S_i\}$, $i \in I$ be a family of semigroups. The Cartesian product $\prod_{i \in I} S_i$ is a semigroup under the "componentwise" multiplication; this semigroup is called the direct product of semigroups $\{S_i\}$, ($i \in I$). The homomorphisms $\pi_i : a \mapsto a_i \in S_i$ ($a \in \prod_{j \in I} S_j$, $i \in I$) are called projection homomorphisms.*

Definition 1.24 *We say that a semigroup S is a subdirect product of semigroups S_i $(i \in I)$ if S is isomorphic to a subsemigroup T of the direct product $\prod_{i \in I} S_i$ of semigroups S_i $(i \in I)$ such that the restriction of the projection homomorphisms to T are surjective.*

Theorem 1.12 *([75]) If α_i $(i \in I)$ are congruences on a semigroup S and $\cap_{i \in I} \alpha_i = id_S$, the equality relation on S, then S is a subdirect product of the factor semigroups S/α_i. Conversely, if a semigroup is a subdirect product of semigroups S_i $(i \in I)$ and α_i is the congruence on S induced by the projection homomorphism π_i $(i \in I)$ then $\cap_{i \in I} \alpha_i = id_S$.*

Theorem 1.13 *([75]) If a semigroup S is a strong semilattice of semigroups S_i $(i \in I)$ then S is a subdirect product of semigroups S_i with a zero possibly adjoined.*

Definition 1.25 *Let S_1 and S_2 be semigroups having Y as their common greatest semilattice homomorphic image. Let $\phi_1 : S_1 \mapsto Y$ and $\phi_2 : S_2 \mapsto Y$ be the canonical homomorphisms. Let*

$$S = \{(a, b) \in S_1 \times S_2 : \phi_1(a) = \phi_2(b)\}.$$

S is a subdirect product of S_1 and S_2 which is called the spined product of S_1 and S_2.

Ideals, Green's relations

Definition 1.26 *A nonempty subset A of a semigroup S is called a left (right) ideal of S if $sa \in A$ $(as \in A)$ for every $a \in A$ and $s \in S$. A subset which is both left and right ideal of a semigroup S is called a two-sided ideal (briefly, an ideal) of S. The left (right, two-sided) ideals of a semigroup S different from S are called proper left (right, two-sided) ideals. An ideal of a semigroup S is called a minimal ideal if it does not properly contain any ideal of S. An ideal M of a semigroup S containing a zero 0 is called a 0-minimal ideal if $M \neq \{0\}$ and $\{0\}$ is the only ideal of S properly contained in M.*

Let S be a semigroup and A be an ideal of S. It is easy to see that $\rho_A = \{(x, y) \in S \times S : a = b \text{ or } a, b \in A\}$ is a congruence on S. This congruence is called the *Rees congruence* of S modulo A. S is called an *ideal extension* of A by S/ρ_A.

Theorem 1.14 *([19]) If A is an ideal of a semigroup S and B is an ideal of A such that $B^2 = B$ then B is an ideal of S.*

Definition 1.27 *A semigroup is called a left (right) simple semigroup if it has no proper left (right) ideal. If a semigroup has no proper two-sided ideal then it is said to be a simple semigroup.*

Definition 1.28 *A semigroup with zero 0 is called a 0-simple semigroup if $S^2 \neq \{0\}$ and only S and $\{0\}$ are the ideals of S.*

The intersection of all left (right, two-sided) ideals of a semigroup containing a non-empty subset X of S is called the ideal of S generated by X. In case $X = \{a\}$, this left (right, two-sided) ideal is said to be the *principal left (right, two-sided) ideal* of S generated by the element a of S and is denoted by $L(a)$ ($R(a)$, $J(a)$).

It is easy to see that $L(a) = S^1a = a \cup Sa$, $R(a) = aS^1 = a \cup aS$, $J(a) = S^1aS^1 = a \cup Sa \cup aS \cup SaS$. We note that sometimes we write a instead of the one-element subset $\{a\}$ of a semigroup. Moreover, $aX = \{ax : x \in X\}$.

Definition 1.29 *In an arbitrary semigroup S, we define the following Green's equivalences. $a \mathcal{L} b$ ($a \mathcal{R} b$, $a \mathcal{J} b$) if and only if a and b generate the same principal left (right, two-sided) ideal of S. We define $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$, the smallest equivalence of S containing both $\mathcal{L}$ and $\mathcal{R}$.*

It is easy to see that $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence of an arbitrary semigroup.

Definition 1.30 *A semigroup S is called a $\mathcal{J}$-trivial semigroup if the Green's equivalence $\mathcal{J}$ is the equality relation on S, that is, $J(a) = J(b)$ if and only if $a = b$ for every $a, b \in S$.*

Definition 1.31 *Let S be a semigroup and $a, b \in S$ be arbitrary elements. We say that a divisible by b if $b|a$, that is, $a \in S^1bS^1$.*

Theorem 1.15 *On a semigroup S the following are equivalent.*

(i) S is $\mathcal{J}$-trivial.

(ii) The divisibility on S is an ordering.

Proof. (i) implies (ii). Let S be a $\mathcal{J}$-trivial semigroup. It is clear that the divisibility is reflexive and transitive. To show that it is also antisymmetric, assume $a|b$ and $b|a$ for some $a, b \in S$. Then $a \in J(b)$ and $b \in J(a)$ from which it follows that $J(a) = J(b)$. As S is $\mathcal{J}$-trivial, we have $a = b$.

(ii) implies (i). Let S be an arbitrary semigroup in which the divisibility is an ordering. Assume $J(a) = J(b)$ for arbitrary $a, b \in S$. Then $a \in J(b)$ and $b \in J(a)$, that is, $b|a$ and $a|b$ and so $a = b$. Hence S is $\mathcal{J}$-trivial. □

Theorem 1.16 *On a semigroup S the following are equivalent.*

(i) S is $\mathcal{J}$-trivial and the principal ideals form a chain with respect to inclusion.

(ii) *The divisibility relation is an ordering on S and S is a chain with respect to the divisibility ordering.*

Proof. By Theorem 1.15, it is obvious. □

Theorem 1.17 *Every nil semigroup is $\mathcal{J}$-trivial.*

Proof. If a and b are elements of a nil semigroup S such that $J(a) = J(b)$ then $a = xby$ and $b = uav$ for some $x, y, u, v \in S^1$ and so $a = (xu)^n a(vy)^n$ for every positive integer n. As S is a nil semigroup, we can conclude that $a = b$. □

We note that Theorem 1.17 implies that there is no 0-simple nil semigroup, because $J(a) = J(b)$ holds for every non-zero elements a and b of a 0-simple nil semigroup S.

Theorem 1.18 *Let S be a nil semigroup which is a chain with respect to the divisibility ordering. Then every congruence of S is a Rees congruence.*

Proof. Let S be a nil semigroup which is a chain with respect to the divisibility ordering. Let ρ be an arbitrary congruence on S. If $\rho = id_S$ then it is regarded as the Rees congruence modulo $\{0\}$. Assume $\rho \neq id_S$. Then there are elements $a, b \in S$ such that $a \neq b$ and $(a, b) \in \rho$. As S is a chain with respect to the divisibility ordering, we have either $a|b$ or $b|a$. Assume $b|a$. Then there are elements $x, y \in S^1$ such that $a = xby$ and so $(b, xby) \in \rho$. From this we get $(b, x^n by^n) \in \rho$ for every positive integer n. As S is a nil semigroup, we have $(b, 0) \in \rho$, and also $(a, 0) \in \rho$. Consequently, for every $a, b \in S$, $a = b$ or $a \neq b$ and $(a, 0) \in \rho$, $(b, 0) \in \rho$. Let $I = \{a \in S : (a, 0) \in \rho\}$. Then I is an ideal and ρ is a Rees congruence modulo I. □

Regular semigroups, inverse semigroups

Definition 1.32 *An element a of a semigroup S is called a left regular (right regular, regular, intra-regular) element of S if $xa^2 = a$ ($a^2x = a$, $axa = a$, $xa^2y = a$) for some $x, y \in S$. A semigroup is said to be a left regular (right regular, regular, intra-regular) semigroup if its every element is left regular (right regular, regular, intra-regular). We say that a semigroup S is a completely regular semigroup if, for every element $a \in S$, there is an element $x \in S$ such that $a = axa$ and $ax = xa$.*

If $a = axa$ for some elements a and x of a semigroup S then ax and xa are idempotent elements of S.

Definition 1.33 *We say that the elements a and y of a semigroup are inverses of each other if $aya = a$ and $yay = y$.*

It is clear that if a is a regular element of a semigroup S, say $a = axa$ for some $x \in S$, then $aya = a$ and $yay = y$, where $y = xax$. Thus every regular element of a semigroup has at least one inverse.

Definition 1.34 *A regular semigroup in which every element has exactly one inverse is called an inverse semigroup.*

Theorem 1.19 *([19]) On an arbitrary semigroup S the following are equivalent.*

(i) S is an inverse semigroup.

(ii) S is a regular semigroup, and any two idempotent elements of S are commutable with each other.

(iii) Every principal right ideal and every principal left ideal of S has an unique idempotent generator.

We note that if a regular semigroup S is a semilattice Y of semigroups S_α ($\alpha \in Y$) then each S_α is regular. Indeed, if $a \in S_\alpha$ and $a = axa$ then $y = xax \in S_\alpha$. It is clear that if S is an inverse semigroup then each S_α is an inverse semigroup.

Definition 1.35 *A semigroup is called a Clifford semigroup if it is regular and the idempotent elements of S are central, that is, $ex = xe$ for every $x \in S$ and every idempotent element e of S.*

Subgroups

A subsemigroup G of a semigroup $(S, *)$ is called a *subgroup* of S if G is a group under the restriction of the operation $*$ to G.

Let S be a monoid with identity e. If a and b are elements of S such that $ab = e$ then a is called a *left inverse* of b, and b is called a *right inverse* of a. A *left (right) unit* in S is defined to be an element of S having a left (right) inverse in S. By a *unit* in S we mean an element of S having both a left and a right inverse in S.

Let S be a monoid with identity element e. The set U of all units of S is a subgroup of S. Each unit has a unique two-sided inverse in U, and has no other left or right inverse in S. Moreover, every subgroup of S containing e is contained in U.

We note that if f is an idempotent element of a semigroup S then fSf is the maximal submonoid in S in which f is the identity element.

Theorem 1.20 *([19]) Let f be an idempotent element of a semigroup S, and let H_f be the group of units of fSf. Then H_f contains every subgroup G of S that meets H_f.*

We note that, by Theorem 1.20, a semigroup is a union of subgroups if and only if it is a union of disjoint subgroups. The next theorem characterizes a semigroup which is a semilattice of subgroups.

Theorem 1.21 *([31]) On an arbitrary semigroup S the following are equivalent.*

(i) S is a semilattice of groups.

(ii) S is a strong semilattice of groups.

(iii) S is a Clifford semigroup.

Completely simple (0-simple) semigroups, Rees matrix semigroups

Definition 1.36 *A simple (0-simple) semigroup is called a completely simple (0-simple) semigroup if it contains a primitive idempotent.*

Theorem 1.22 *([19]) If e is a non-zero idempotent of a 0-simple semigroup S which is not completely 0-simple then S contains a bicyclic subsemigroup having e as identity element.*

Theorem 1.23 *A semigroup is completely simple if and only if it is a rectangular band of groups.*

Proof. If S is a completely simple semigroup then, by Corollary 2.52b of [19], S is a rectangular band of its subgroups $H_{i,j} = R_i \cap L_j$, where $\{R_i;\ i \in I\}$ and $\{L_j :\ j \in J\}$ are the minimal right ideals and the minimal left ideals of S, respectively.

Conversely, assume that a semigroup is a rectangular band $B = I \times J$ of groups $G_{i,j}$ (I is a left zero semigroup, J is a right zero semigroup, $i \in I$, $j \in J$). Let K be an ideal of S. Assume $K \cap G_{i_0,j_0} \neq \emptyset$ for some $i_0 \in I$ and $j_0 \in J$. Then $G_{i_0,j_0} \subseteq K$. Thus, for every $i \in I$ and $j \in J$, $G_{i,j_0}G_{i_0,j_0}G_{j_0,j} \subseteq K$ and so $G_{i,j} \cap K \neq \emptyset$, because $G_{i,j_0}G_{i_0,j_0}G_{j_0,j} \subseteq G_{i,j}$. Then $G_{i,j} \subseteq K$ and so $K = S$. Hence S is simple. We prove that the identity element $e_{n,m}$ of $G_{n,m}$ is a primitive idempotent ($n \in I$, $m \in J$). Assume $e_{i,j} \leq e_{n,m}$, that is, $e_{i,j}e_{n,m} = e_{n,m}e_{i,j} = e_{i,j}$, where $e_{i,j}$ is the identity element of $G_{i,j}$. Then $G_{i,m} \cap G_{n,j} \neq \emptyset$ and so $i = n$, $j = m$. Hence $e_{i,j} = e_{n,m}$ which implies that $e_{n,m}$ is a primitive idempotent. Consequently, S is completely simple □

Definition 1.37 *Let G (G^0) be a group (a group with a zero adjoined), I and J be non-empty sets and P be a function from $J \times I$ into G (G^0) with value $p_{j,i}$ at (j,i). On $S = I \times G \times J$ ($S' = I \times G^0 \times J$) define a multiplication by*

$$(i,g,j)(k,h,l) = (i, gp_{j,k}h, l).$$

It is easy to see that S and S' are semigroups, and $A = \{(i,0,j),\ i \in I,\ j \in J\}$ is an ideal of S'. The semigroup S (the Rees factor semigroup S'/A) is called the Rees matrix semigroup over the group G (the group with a zero adjoined G^0) with the sandwich matrix P, and denoted by $\mathcal{M}(I,G,M;P)$ ($\mathcal{M}^0(I,G,J;P)$). The sandwich matrix P is called regular if no row or column of P consists wholly of zeros. If this is the case then the Rees matrix semigroup is regular.

Theorem 1.24 *([19]) Two Rees matrix semigroups $\mathcal{M}=(I,G,J;P)$ and $\mathcal{M}'=(I,G,J;P')$ over the same group G are isomorphic if there exists a mapping $i \mapsto u_i$ of I into G and a mapping $j \mapsto v_j$ of J into G such that $p'_{j,i} = v_j p_{j,i} u_i$ for all $i \in I$ and $j \in J$.*

We note if $\mathcal{M} = \mathcal{M}(I,G,J;P)$ is a Rees matrix semigroup then the previous theorem enables us to replace P by $P' = VPU$ with diagonal matrices U and V over G. For example, we may "normalize" P so that all the elements in a given row and a given column are the identity element of G. In our investigation, we allways will suppose that P is normalized.

Theorem 1.25 *(Rees, [19]) A semigroup is completely simple (0-simple) if and only if it is isomorphic to a (regular) Rees matrix semigroup over a group (a group with zero).*

Left (right) groups, rectangular groups

Definition 1.38 *A direct product of a rectangular band and a group is called a rectangular group. A direct product of a left (right) zero semigroup and a group is called a left (right) group.*

Theorem 1.26 *([32]) A semigroup is a rectangular group if and only if it is a completely simple semigroup in which the idempotents form a subsemigroup.*

Orthodox union of groups, orthodox band of groups

Definition 1.39 *We say that a semigroup is an orthodox union of groups (or an orthogroup) if it is a union of groups and the idempotents of S form a subsemigroup.*

Theorem 1.27 *([20]) A semigroup S is an orthogroup if and only if it is a semilattice of rectangular groups.*

Proof. Let S be an orthogroup. Then it is a union of disjoint subgroups and so, by Theorem 4.5 of [19], it is a semilattice Y of completely simple semigroups S_α, $\alpha \in Y$. By Theorem 1.26, every S_α is a rectangular group.

Conversely, assume that a semigroup S is a semilattice Y of rectangular groups $S_\alpha = L_\alpha \times G_\alpha \times R_\alpha$ (L_α is a left zero semigroup, G_α is a group, R_α is a right zero semigroup, $\alpha \in Y$). Clearly, S is a union of groups, and all that remains is to show that the product of two idempotent elements of S is idempotent. Let $e \in S_\alpha$ and $f \in S_\beta$ be arbitrary idempotent elements. Then $a = cf$ and $b = fc$ both belong to $S_{\alpha\beta}$. Let $a = (i_{\alpha\beta}, g_{\alpha\beta}, m_{\alpha\beta})$, $b = (j_{\alpha\beta}, h_{\alpha\beta}, n_{\alpha\beta})$. Then

$$(j_{\alpha\beta}, h^2_{\alpha\beta}, n_{\alpha\beta}) = (j_{\alpha\beta}, h_{\alpha\beta}, n_{\alpha\beta})^2 = b^2 = fefe = (fe)(ef)(fe) = bab$$

$$= (j_{\alpha\beta}, h_{\alpha\beta}, n_{\alpha\beta})(i_{\alpha\beta}, g_{\alpha\beta}, m_{\alpha\beta})(j_{\alpha\beta}, h_{\alpha\beta}, n_{\alpha\beta}) = (j_{\alpha\beta}, h_{\alpha\beta} g_{\alpha\beta} h_{\alpha\beta}, n_{\alpha\beta})$$

from which we get that $g_{\alpha\beta}$ is the identity element of $G_{\alpha\beta}$ and so a is an idempotent element. □

Let S be a semigroup and X be a set. By a *left (right) representation* of S by transformations of X we mean a homomorphism of S into the semigroup $\mathcal{T}_X$ of all transformations of X regarded as left (right) mappings.

Theorem 1.28 *(Preston's Theorem; [20]) Let E be a band and $E = \cup_{\alpha \in Y} E_\alpha$ be the decomposition of E into a semilattice Y of rectangular bands $E_\alpha = L_\alpha \times R_\alpha$ ($\alpha \in Y$). For each α in Y, let G_α be a group, 1_α be the identity element of G_α, $S_\alpha = L_\alpha \times G_\alpha \times R_\alpha$, and $S = \cup_{\alpha \in Y} S_\alpha$. Identify $1_\alpha \times E_\alpha$ with E_α.*

For each pair of elements $\alpha, \beta \in Y$ with $\alpha > \beta$, let $\psi_{\alpha,\beta}$ be a homomorphism of G_α into G_β, and let $t_{\alpha,\beta}$ ($\tau_{\alpha,\beta}$) be a left (right) representation of S_α by transformations of L_β (R_β) such that if $e_\alpha = (i_\alpha, \kappa_\alpha) \in E_\alpha$ and $f_\beta = (j_\beta, \lambda_\beta) \in E_\beta$ then

$$e_\alpha f_\beta = ((t_{\alpha,\beta} e_\alpha) j_\beta, \lambda_\beta),$$

$$f_\beta e_\alpha = (j_\beta, \lambda_\beta(e_\alpha \tau_{\alpha,\beta})).$$

Define $\psi_{\alpha,\alpha}$, $t_{\alpha,\alpha}$ and $\tau_{\alpha,\alpha}$ ($\alpha \in Y$) as follows. Let $\psi_{\alpha,\alpha}$ be the identity automorphism of G_α. For $A = (i_\alpha, a_\alpha, \kappa_\alpha) \in S_\alpha$, let $t_{\alpha\alpha} A$ map every element of L_α onto i_α, and let $A\tau_{\alpha,\alpha}$ map every element of R_α onto κ_α.

Define the product AB of any two elements $A, B \in S$ as follows. Suppose $A = (i_\alpha, a_\alpha, \kappa_\alpha) \in S_\alpha$ and $B = (j_\beta, b_\beta, \lambda_\beta) \in S_\beta$. Let $\gamma = \alpha\beta$ (product in Y), and let

$$(k_\gamma, \mu_\gamma) = (i_\alpha, \kappa_\alpha)(j_\beta, \lambda_\beta)$$

be the given product of $(i_\alpha, \kappa_\alpha)$ and (j_β, λ_β) in the band E. Then define

$$AB = ((t_{\alpha,\gamma} A) k_\gamma, (a_\alpha \psi_{\alpha,\gamma})(b_\beta \psi_{\beta,\gamma}), \mu_\gamma(B\tau_{\beta,\gamma})).$$

This definition is consistent with the given products in E and the various S_α ($\alpha \in Y$). When $\alpha \geq \beta$, the product AB simplifies to

$$AB = ((t_{\alpha,\beta} A) j_\beta, (a_\alpha \psi_{\alpha,\beta}) b_\beta, \lambda_\beta),$$

$$BA = (j_\beta, b_\beta(a_\alpha \psi_{\alpha,\beta}), \lambda_\beta(A\tau_{\alpha,\beta})).$$

Assume furthermore that the following conditions hold for all $\alpha, \beta, \gamma \in Y$ such that $\alpha > \beta > \gamma$, and for all $A \in S_\alpha$, $B \in S_\beta$:

$$\psi_{\alpha,\beta}\psi_{\beta,\gamma} = \psi_{\alpha,\gamma};$$

$$t_{\beta,\gamma}(AB) = (t_{\alpha,\gamma} A)(t_{\beta,\gamma} B),$$

$$t_{\beta,\gamma}(BA) = (t_{\beta,\gamma} B)(t_{\alpha,\gamma} A),$$

$$(AB)\tau_{\beta,\gamma} = (A\tau_{\alpha,\gamma})(Bt_{\beta,\gamma}),$$

$$(BA)\tau_{\beta,\gamma} = (B\tau_{\beta,\gamma})(A\tau_{\alpha,\gamma}).$$

Then S becomes an orthogroup, and, conversely, every orthogroup can be constructed in this way.

Definition 1.40 *We say that a semigroup is an orthodox band of groups if S is a band of groups and the idempotents of S form a subsemigroup. If a semigroup S is an orthodox band B of groups such that B is a normal (left regular, right regular) band then we say that S is an orthodox normal (left regular, right regular) band of groups.*

Theorem 1.29 *([20]) An orthogroup S is an orthodox band of groups if and only if the Green's equivalence $\mathcal{H}$ is a congruence on S.*

Proof. By Theorem 4.3 of [19], a semigroup S is a union of groups if and only if every H-class of S is a group (a maximal subgroup of S). Thus the assertion of the theorem is obvious. □

Let S be an orthogroup. Then, by Theorem 1.27, it is a semilattice Y_S of rectangular groups $S_\alpha = L_\alpha \times G_\alpha \times R_\alpha$, $\alpha \in Y_S$. By Preston's theorem, there are objects $\psi_{\alpha,\beta}$, $t_{\alpha,\beta}$, $\tau_{\alpha,\beta}$ ($\alpha \geq \beta$ in Y_S) which determines the product in S. Let $Q_S = \cup_{\alpha \in Y_S} G_\alpha$, and define a product $*$ in Q_S by

$$g_\alpha * h_\beta = (g_\alpha \psi_{\alpha,\alpha\beta})(h_\beta \psi_{\beta,\alpha\beta}),$$

$g_\alpha \in G_\alpha$ and $h_\beta \in G_\beta$, $\alpha, \beta \in Y_S$. Using the condition for the class of homomorphisms $\psi_{\alpha,\beta}$ of Preston's theorem, we can see that Q_S is a semigroup under the operation $*$, and in fact an inverse semigroup which is a semilattice of groups.

Let Y be a semilattice, E a band and Q a semilattice of groups such that Y is the greatest semilattice homomorphic image of both E and Q. Let $\mathcal{C}(Y, E, Q)$ denote the class of all orthogroups S such that the greatest semilattice homomorphic image Y_S of S is isomorphic to Y, $E_S \cong E$ and $Q_S \cong Q$. Let $E = \cup_{\alpha \in Y} E_\alpha$ and $Q = \cup_{\alpha \in Y} G_\alpha$ be the decomposition of E and Q into the semilattice Y of rectangular groups E_α and of groups G_α ($\alpha \in Y$), respectively. Let $Q \times_Y E$ denote the spined product of Q and E. It is clear that $Q \times_Y E$ is a union of rectangular groups $G_\alpha \times E_\alpha$, hence of groups, and the product of the two idempotents $(1_\alpha, e_\alpha)$ and $(1_\beta, f_\beta)$ is the idempotent $(1_{\alpha\beta}, e_\alpha f_\beta)$, where 1_δ denotes the identity element of G_δ for all $\delta \in Y$. Hence $Q \times_Y E \in \mathcal{C}(Y, E, Q)$. The $\mathcal{H}$-classes (maximal subgroups) of $Q \times_Y E$ are the sets $G_\alpha \times e_\alpha$ ($\alpha \in Y$, $e_\alpha \in E_\alpha$), and

$$(G_\alpha \times e_\alpha)(G_\beta \times e_\beta) \subseteq G_{\alpha\beta} \times e_\alpha f_\beta.$$

Hence $\mathcal{H}$ is a congruence and so, by Theorem 1.29, $Q \times_Y E$ is an orthodox band of groups.

Theorem 1.30 *(Yamada's Theorem; [20]) Every orthodox band of groups is a spined product of a band and a semilattice of groups.*

More precisely, each class $\mathcal{C}(Y, E, Q)$ of orthogroups contains (to within isomorphism) precisely one member which is an orthodox band of groups, namely the spined product $Q \times_Y E$.

Proof. Let S be an orthodox band of groups, and let $S = \cup_{\alpha\in Y} S_\alpha$ be its decomposition into a semilattice Y of rectangular groups $S_\alpha = G_\alpha \times E_\alpha$ (G_α is a group, E_α is a rectangular band, $\alpha \in Y$). Let $E_S = \cup_{\alpha\in Y} E_\alpha$ be the band of idempotents of S, identifying $1_\alpha \times E_\alpha$ with E_α, where 1_α denotes the identity element of G_α.

Let us apply the converse half of Preston's Theorem to S, but for simplicity let us represent the elements of S by pairs (a_α, e_α) instead of triples $(i_\alpha, a_\alpha, m_\alpha)$. If $A = (a_\alpha, e_\alpha)$ and $B = (b_\beta, f_\beta)$ then the product AB of A and B has the form

$$AB = ((a_\alpha \psi_{\alpha,\alpha\beta})(b_\beta \psi_{\beta,\alpha\beta}), g_{\alpha\beta}),$$

where $g_{\alpha\beta}$ is some element of $E_{\alpha\beta}$. Let $Q_S = \cup_{\alpha\in Y} G_\alpha$, with multiplication $*$ defined above. Q_S is a semilattice of groups, and $S \in \mathcal{C}(Y, E_S, Q_S)$. The above product AB becomes

$$AB = (a_\alpha * b_\beta, g_{\alpha\beta}).$$

Now A and B are $\mathcal{H}$-equivalent respectively to the idempotents $(1_\alpha, e_\alpha)$ and $(1_\beta, f_\beta)$. Since S is a band of groups, $\mathcal{H}$ is a congruence, and so AB is $\mathcal{H}$-equivalent to their product, namely $(1_{\alpha\beta}, e_\alpha f_\beta)$. But this requires $g_{\alpha\beta} = e_\alpha f_\beta$, hence

$$AB = (a_\alpha * b_\beta, e_\alpha f_\beta).$$

Thus the Preston representation of S reduces to the spined product $Q_S \times_Y E_S$ when $\mathcal{H}$ is a congruence. If $S \in \mathcal{C}(Y, E, Q)$ then we can identify Y_S with Y, and since $E_S \cong E$ and $Q_S \cong Q$, we have

$$S = Q_S \times_Y E_S \cong Q \times_Y E.$$

□

Normal band of groups

Theorem 1.31 *(Th. IV.2.3; [75]) A semigroup is a normal band of groups if and only if it is a strong semilattice of completely simple semigroups.*

Theorem 1.32 *(Th. IV.2.6; [75]) The following conditions on a semigroup S are equivalent.*

1. *S is an orthodox normal band of groups.*
2. *S is a strong semilattice of rectangular groups.*
3. *S is a spined product of a normal band and a semilattice of groups.*

We note that if a semigroup S is a disjoint union of abelian groups then all subgroups of S are commutative. Thus Theorem 1.32 is true if we change expression "groups" for expression "abelian groups".

Translations, translational hull

Definition 1.41 *A transformation (single-valued mapping) $\lambda(\ )$ $((\ )\mu)$ of a semigroup S into itself is called a left (right) translation of S if $\lambda(xy) = (\lambda x)y$ $((xy)\mu = x(y)\mu)$ for every $x, y \in S$. A left translation λ and a right translation μ are said to be linked if $x(\lambda y) = (x\mu)y$ for every $x, y \in S$. The set of all pairs (λ, μ) of linked left and right translations λ and μ of S forms a semigroup under the operation $(\lambda_1(\ ), (\ )\mu_1)(\lambda_2(\ ), (\ )\mu_2) = (\lambda_1 \circ \lambda_2(\ ), (\ )\mu_1 \circ \mu_2)$. This semigroup $\Omega(S)$ is called the translational hull of S.*

It is easy to see that, for every element a of a semigroup S, the mappings $\lambda_a : x \mapsto ax$ and $\mu_a : x \mapsto xa$ $(x \in S)$ are left and right translations of S, respectively, such that they are linked. The pairs (λ_a, μ_a), $a \in S$ form a subsemigroup in $\Omega(S)$. This subsemigroup is called the inner part of $\Omega(S)$.

It is easy to check that $a \mapsto (\lambda_a, \mu_a)$ $(a \in S)$ is a homomorphism of S into the inner part of $\Omega(S)$.

Theorem 1.33 *([21]) Let $S = \mathcal{M}(I, G, J; P)$ be a Rees matrix semigroup over a group G with normalized sandwich matrix $P = (p_{j,i})$, and let $\mathcal{T}_I$ and $\mathcal{T}_J$ denote the semigroup of all transformations of I (acting on the left) and J (acting on the right), respectively. Then*

$$\Omega(S) = \{(k, a, h) \in \mathcal{T}_I \times G \times \mathcal{T}_J : (\forall i \in I, j \in J)\ p_{j,k(i)} a p_{(j_0)h,i} = p_{j,k(i_0)} a p_{(j)h,i}\}.$$

The product of two elements (k, a, h) and (f, b, g) of $\Omega(S)$ is given by:

$$(k, a, h)(f, b, g) = (k \circ f, a p_{(j_0)h, f(i_0)} b, h \circ g).$$

A bitranslation $(k, a, h) \in \Omega(S)$ is inner if and only if k and h are constant transformations. Identifying S with the inner part of $\Omega(S)$, for every $(k, a, h) \in \Omega(S)$ and $(i, g, j) \in S$,

$$(k, a, h)(i, g, j) = (k(i), a p_{k(j_0), i} g, j),$$

$$(i, g, j)(k, a, h) = (i, g p_{j,(i_0)h} a, (j)h).$$

Theorem 1.34 *Let G be a group, and let L and R be a left zero and a right zero semigroup, respectively. Then $\Omega(L \times G \times R) = \mathcal{T}_L \times G \times \mathcal{T}_R$. Especially, $\Omega(L \times R) = \mathcal{T}_L \times \mathcal{T}_R$, $\Omega(L) = \mathcal{T}_L$, $\Omega(R) = \mathcal{T}_R$.*

Proof. By Theorem 1.33, it is obvious.

Weakly reductive semigroups

Definition 1.42 *A semigroup S is called a weakly reductive semigroup if, for every $a, b \in S$, the assumption that $xa = xb$ and $ax = bx$ hold for all x in S implies $a = b$.*

It is clear that if S is a weakly reductive semigroup then $a \mapsto (\lambda_a, \mu_a)$ $(a \in S)$ is an isomorphism of S onto the inner part of $\Omega(S)$.

Theorem 1.35 *(Lemma 1.2 of [19]) Let S be a weakly reductive semigroup, and let us identify S with the inner part of the translational hull $\Omega(S)$ of S. Then S is an ideal of $\Omega(S)$ such that $(\lambda, \mu)a = \lambda(a)$ and $a(\lambda, \mu) = (a)\mu$ for every $a \in S$ and every $(\lambda, \mu) \in \Omega(S)$.*

Theorem 1.36 *(Theorem 4.20 of [19]) Let S be a weakly reductive semigroup and T be an arbitrary semigroup with zero 0. Let $T^* = T - \{0\}$, $F = T^* \cup S$ and $F' = T^* \cup \Omega(S)$. Let $(F', \circ)$ be an ideal extension of $\Omega(S)$ by T. Then $(F, \circ)$ is an ideal extension of S by T if and only if $(F, \circ)$ is a subsemigroup of $(F', \circ)$, and this is the case if and only if $a \circ b \in S$ for every $a, b \in T$ satisfying $ab = 0$ in T.*

Conversely, let $(F, \circ)$ be an ideal extension of S by T. Then there is an ideal extension $(F', \circ)$ of $\Omega(S)$ by T such that $(F, \circ)$ is a subsemigroup of $(F', \circ)$.

Dense ideals

Definition 1.43 *An ideal K of a semigroup S is called a dense ideal of S if $\alpha|K = id_K$ implies $\alpha = id_S$ for every congruence α of S, where $\alpha|K$ denotes the restriction of α to K.*

Theorem 1.37 *If K is a dense ideal of a semigroup S such that K is weakly reductive then S is isomorphic to a subsemigroup of $\Omega(K)$.*

Proof. Let K be a dense ideal of a semigroup S such that K is weakly reductive. For an arbitrary $s \in S$, let λ_s and ρ_s be transformations of K defined by $\lambda_s(k) = sk$ and $(k)\rho_s = ks$. It is easy to see that λ_s and ρ_s are left and right translations of K, respectively, such that they are linked. Let ϕ be a mapping of S into $\Omega(K)$ defined by $\phi(s) = (\lambda_s, \rho_s)$, $s \in S$. Since

$$(\lambda_{st}, \rho_{st})k = \lambda_{st}k = (st)k = s(tk) = (\lambda_s \circ \lambda_t)k$$

and

$$k(\lambda_{st}, \rho_{st}) = k\rho_{st} = k(st) = (ks)t = k(\rho_s \circ \rho_t)$$

for every $s, t \in S$ and $k \in K$, we have

$$\phi(st) = (\lambda_{st}, \rho_{st}) = (\lambda_s \circ \lambda_t, \rho_s \circ \rho_t) = (\lambda_s, \rho_s)(\lambda_t, \rho_t) = \phi(s)\phi(t)$$

and so ϕ is a homomorphism. Assume $\phi(s) = \phi(t)$ for some $s, t \in S$. Then $sk = tk$ and $ks = kt$ for every $k \in K$. It is easy to see that

$$\alpha = \{(a, b) \in S \times S : \ (\forall k \in K) \ ak = bk, \ ka = kb\}$$

is a congruence on S and $(s, t) \in \alpha$. Since K is weakly reductive, $\alpha|K = id_K$. Since K is a dense ideal of S, we get $\alpha = id_S$ and so $s = t$. Hence ϕ is an isomorphism. □

Retract ideals

Definition 1.44 *An ideal K of a semigroup S is called a retract ideal if there is a homomorphism of S onto K which leaves the elements of K fixed. Such a homomorphism is called a retract homomorphism of S onto K. In this case we say that S is a retract (ideal) extension of K.*

Definition 1.45 *Let $W = ((w_i, w_i'))_{i\in I}$ be a family of pairs of words of a free semigroup generated by two letters. We suppose that W satisfies the following conditions.*

(i) I is an ordered set.

(ii) If S is a semigroup with zero 0 and $x, y \in S$ such that if $w_i(x,y) = 0$ ($w_i'(x,y) = 0$) then $w_j(x,y) = 0$ ($w_j'(x,y) = 0$) for all $j \geq i$.

We say that a semigroup S is a W-semigroup if, for every $x, y \in S$ and every $i \in I$, there is a $j \geq i$ such that $w_j(x,y) = w_j'(x,y)$.

Theorem 1.38 *([41]). Let $W = ((w_i, w_i'))_{i\in I}$ be a family which satisfies conditions (i) and (ii). Then a retract extension of a W-semigroup by a W-semigroup with zero is a W-semigroup.*

Proof. Let $S = T \cup N^*$ be a retract extension of a semigroup T by a semigroup N with zero 0 (here $N^* = N - \{0\}$) and $f : S \mapsto T$ be a retraction. Let $x, y \in S$ and $i \in I$ be arbitrary. Since T is a W-semigroup, $w_j(f(x), f(y)) = w_j'(f(x), f(y))$ for some $j \geq i$. Suppose that $x \in T$ or $y \in T$. Then

$$w_j(x,y) = f(w_j(x,y)) = w_j(f(x), f(y)) = w_j'(f(x), f(y)) =$$

$$f(w_j'(x,y)) = w_j'(x,y).$$

Now suppose that $x, y \in N^*$. We define a subset of I:

$$J = \{j \in I : \ w_j(x,y) = w_j'(x,y) \text{ in } N\}.$$

Since N is a W-semigroup, J is non-empty. If $w_j(x,y) \neq 0$ in N for some $j \in J$ such that $j \geq i$ then $w_j(x,y) = w_j'(x,y)$ in S. Suppose that $w_j(x,y) = 0$ in N for any $j \in J$. Consider two elements $j \in J$ and $k \in I$ such that $i \leq j \leq k$ and $w_k(f(x), f(y)) = w_k'(f(x), f(y))$. By condition (ii) of Definition 1.45, we have $w_k(x,y) = w_k'(x,y) = 0$ in N. Then, in S,

$$w_k(x,y) = f(w_k(x,y)) = w_k(f(x), f(y)) = w_k'(f(x), f(y)) =$$

$$f(w_k'(x,y)) = w_k'(x,y).$$

Consequently, S is a W-semigroup. □

Variety of semigroups

Definition 1.46 *Let $\mathcal{F}$ be a nonempty family of identities. The class $\mathcal{V}$ of all semigroups satisfying each identity in $\mathcal{F}$ is called the variety determined by the identities of $\mathcal{F}$, or simply a variety. In such a case $\mathcal{F}$ is called the family of defining identities for $\mathcal{V}$.*

Theorem 1.39 *([75]) A class of semigroups is a variety if and only if it is closed under direct product, subemigroup and homomorphic image.*

Theorem 1.40 *([41]) A variety of semigroups is closed with respect to retract extension.*

Proof. It is enough to consider the case when I has only one element and so the assertion is obvious by Theorem 1.38. □

Group or group with zero congruences of a semigroup

Definition 1.47 *We say that a subset H of a semigroup S is a reflexive subset in S if $ab \in H$ implies $ba \in H$ for every $a, b \in S$.*

Definition 1.48 *A subset H of a semigroup S is called a left (right) unitary subset if $h, hs \in H$ $(h, sh \in H)$ implies $s \in H$ for ever $h \in H$ and $s \in S$. A subset H is called a unitary subset if it is both left and right unitary.*

Definition 1.49 *Let S be a semigroup and H be a subset of S. The right congruence $\mathcal{R}_H = \{(a,b) \in S \times S : (\forall x \in S)\ ax \in H \text{ iff } bx \in H\}$ and the congruence $\mathcal{P}_H = \{(a,b) \in S \times S : (\forall x, y \in S)\ xay \in H \text{ iff } xby \in H\}$ is called the principal right congruence and the principal congruence on S, respectively, defined by H.*

It is easy to see that if H is a reflexive unitary subsemigroup of S then $\mathcal{R}_H = \mathcal{P}_H$. Next theorem gives further informations about this case.

Theorem 1.41 *([19]) If H is a reflexive unitary subsemigroup of a semigroup S then $\mathcal{R}_H$ is a group or a group with zero congruence on S such that H is an identity element of $S/\mathcal{R}_H$.*

Conversely, is α is a group or a group with zero congruence on a semigroup S and H denotes the α-class of S which is the identity of S/α then H is a reflexive unitary subsemigroup of S and $\alpha = \mathcal{R}_H$. The right residue $W_H = \{x \in S : (\forall a \in S)\ xa \notin H\}$ of H is not empty if and only if S/α has a zero element. In this case the zero of S/α equals W_H.

Theorem 1.42 *Let S be a semigroup satisfying the identity $(ab)^2 = a^2b^2$. Then, for arbitrary $a \in S$,*

$$S_a = \{\ x \in S:\ a^ixa^j = a^k \text{ for some positive integers } i,j,k\ \}$$

is the least reflexive unitary subsemigroup of S that contains a. If S is also archimedean then the principal right congruence $\mathcal{R}_{S_a}$ defined by S_a is a group congruence of S.

Proof. It is clear that $a \in S_a$. To show that S_a is a subsemigroup of S, let $x, y \in S_a$ be arbitrary elements. Then $a^ixa^j = a^h$ and $a^mya^n = a^k$ for some positive integers i, j, h, m, n, k. Let $p = h + k$. Then

$$a^{2p} = (a^mya^na^ixa^j)^2 = a^{2m}(ya^{n+i}x)^2a^{2j}$$

$$= a^m(a^mya^n)a^ixya^n(a^ixa^j)a^j = a^{m+k+i}xya^{n+h+j}$$

and so $xy \in S_a$. We show that S_a is unitary. Assume $x, xy \in S_a$ for some $x, y \in S$. Then $a^ixa^j = a^h$ and $a^mxya^n = a^k$. Choose integer $r \geq max\{j-k, i-m, 0\}$. Then

$$a^{2(r+k)} = (a^{r+m}xya^n)^2 = (a^{r+m}x)^2(ya^n)^2$$

$$= a^{r+m}x(a^{r+m}xya^n)ya^n = a^{r+m}xa^{r+k}ya^n = a^{2r+m-i+h+k-j}ya^n.$$

Hence $y \in S_a$. We can prove, in a similar way, that $y, xy \in S_a$ implies $x \in S_a$. Hence S_a is unitary. S_a is reflexive, because it is unitary and $(xy)^3 = x(yx)^2y = xy^2x^2y = xy(yx)xy$ holds in S. If B is a reflexive unitary subsemigroup of S such that $a \in B$ then, for an arbitrary element $x \in S_a$, there are positive integers i, j, k such that $a^ixa^j = a^k \in B$. Then $x \in B$ and so $S_a \subseteq B$. Since S_a is reflexive and unitary then, by Theorem 1.41, the principal right congruence $\mathcal{R}_{S_a}$ of S determined by S_a is a group or a group with zero congruence of S. As S is archimedean, the right residue W_{S_a} is empty and so $\mathcal{R}_{S_a}$ is a group congruence on S. Thus the theorem is proved. □

Subdirectly irreducible semigroups

Definition 1.50 *We say that a semigroup S is a subdirectly irreducible semigroup if whenever S is written as a subdirect product of a family of semigroups $\{S_i\}_{i\in I}$ then, for at least one $j \in I$, the projection homomorphism π_j maps S onto S_j isomorphically. A semigroup which is not subdirectly irreducible is called subdirectly reducible.*

Theorem 1.43 *([75]) A semigroup S is subdirect irreducible if and only if, for any family $\{\alpha_i\}_{i\in I}$ of congruences of S, $\cap_{i\in I}\alpha_i = id_S$ if and only if $\alpha_j = id_S$ for some $j \in I$.*

Corollary 1.1 *A non-trivial semigroup is subdirectly irreducible if and only if it has a least non-identity congruence.*

Theorem 1.44 *([40]) Every semigroup is a subdirect product of subdirectly irreducible semigroups.*

Theorem 1.45 *([85]) Semigroups S and S^0 (S and S^1) are simultaneously subdirectly irreducible or reducible.*

The least non-empty ideal of a semigroup S (if it exists) is called the *kernel* of S. The kernel of a semigroup with zero is trivial. We call an ideal a non-trivial ideal if it contains at least two elements. The least non-trivial ideal of a semigroup S (if it exists) is called the *core* of S. If K is the core of a semigroup S, then K is either a minimal ideal or a 0-minimal ideal of S. Then either $K^2 = K$ or $K^2 = \{0\}$, where 0 denotes the zero of S. In the first case K is either simple or 0-simple (see Corollary 2.30 and Theorem 2.29 of [19]). In this case K is called a *globally idempotent core*. In the second case K is called a *nilpotent core*. A core is called a *primitive core* if it has two elements.

Theorem 1.46 *([85]) Every non-trivial subdirectly irreducible semigroup has a core.*

Proof. Let $\mathcal{A}$ denote the set of all non-trivial ideals of a subdirectly irreducible semigroup S. If $\cap_{A\in\mathcal{A}}A$ is empty or a trivial ideal of S then $\cap_{A\in\mathcal{A}}\rho_A$ is the identity congruence on S which is impossible (here ρ_A denotes the Rees congruence of S induced by A). □

Definition 1.51 *A semigroup is called a homogroup if it contains a kernel which is a group.*

Theorem 1.47 *([85]) A subdirectly irreducible homogroup without zero is a group.*

Proof. Let S be a subdirectly irreducible homogroup without zero. Let G denote the kernel of S and let e be the identity of the group G. It is easy to see that α defined by $a \ \alpha \ b$ $(a, b \in S)$ if and only if $ea = eb$ is a congruence on S and $\alpha \cap \rho_G = id_S$, where ρ_G denotes the Rees congruence of S induced by the ideal G. As S is subdirect irreducible without zero, we can cunclude $\alpha = id_S$ and so $S = G$. □

In Theorem 4.7 of [85], Schein proved that a band with zero 0 is subdirectly irreducible if and only if $S - \{0\}$ is a subsemigroup which is subdirectly irreducible and (if $|S| > 2$) contains no zero. Thus we can consider the subdirectly irreducible bands without zero. In the next theorem, $\theta(x, y)$ denotes the smallest congruence which identifies a and b. Moreover, S^* denotes the dual of a semigroup S. The dual $(S^*, *)$ of a semigroup $(S, \circ)$ is defined by $S^* = S$ and $a * b = b \circ a$ for any $a, b \in S$.

Theorem 1.48 *([25]) A band S without zero is subdirectly irreducible if and only if S or S^* is isomorphic to a semigroup T which satisfies the following two conditions.*

(i) $C(X) \subseteq T \subseteq X^X$, where X^X is the semigroup of all mappings of X into itself, $C(X)$ is the set of all constant mappings of X.

(ii) There exist $k, k' \in C(X)$ such that $\theta(k, k') \subseteq \theta(c, d)$ for all $c, d \in C(X)$ with $c \neq d$.

Especially, a left (right) zero semigroup is subdirectly irreducible if and only if it has at most two elements.

Proof. Let S be a subdirectly irreducible band without zero. Then, by Theorem 4.7 of [85], S satisfies one of the following conditions.

(1) $K = \{k \in S : \ ks = k \text{ for all } s \in S\}$ is a two-sided ideal of S and, for any $x, y \in S$, $xk = yk$ for all $k \in K$ implies $x = y$.

(2) $K = \{k \in S : \ sk = k \text{ for all } s \in S\}$ is a two-sided ideal of S and, for any $x, y \in S$, $kx = ky$ for all $k \in K$ implies $x = y$.

It is clear that S satisfies (1) if and only if the dual S^* of S satisfies (2). Assume that S satisfies condition (1). Define $\varphi : \ S \to K^K$ by $\varphi(s)(k) = sk$, for all $s \in S$ ($k \in K$). It is easy to check that φ is a homomorphism and that φ is one-one. The monomorphism establishes (i) and (ii) for $\varphi(S)$, since $\varphi(K) = C(K)$. If S satisfies condition (2) then the above argument shows that S^* is isomorphic to a semigroup T satisfying (i) and (ii).

To establish the converse, it is enough to show that if T satisfies (i) and (ii) then T is subdirectly irreducible. By (ii) it is enough to show that if $s, t \in T$, $s \neq t$ then there exist $c, d \in C(X)$, $c \neq d$ such that $\theta(c, d) \subseteq \theta(s, t)$. Since $s \neq t$, there exists $k \in C(X)$ such that $sk \neq tk$. Since $sk, tk \in C(X)$ and $\theta(sk, tk) \subseteq \theta(s, t)$, the proof is complete.

Since every equivalence of a left (right) zero semigroup is a congruence, the assertion of the theorem for the special case is obvious. □

Definition 1.52 *If S is a semigroup with zero 0 then the ideal*

$$A_S^l = \{a \in S : \ (\forall s \in S) \ as = 0\}, \ (A_S^r = \{a \in S : \ (\forall s \in S) \ sa = 0\})$$

of S is called the left (right) annihilator of S. The meet of A_S^l and A_S^r is called the annihilator of S. The annihilator of S is denoted by A_S.

Definition 1.53 *An element s of a semigroup S is called a disjunctive element if the congruence*

$$C_{\{s\}} = \{(a, b) \in S \times S : \ (\forall x, y \in S^1) \ xay = s \Longleftrightarrow xby = s\}$$

equals id_S.

It is known that, for an arbitrary element s of a semigroup S,

$$r(s) = \{a \in S : (\forall x, y \in S^1)\ xay \neq s\}$$

is either empty or a $C_{\{s\}}$-class and an ideal of S.

Lemma 1.4 *([85]) Every non-trivial subdirectly irreducible semigroup has at least two different disjunctive elements.*

Proof. Consider the congruence $\cap_{s \in S} C_{\{s\}}$ on a subdirectly irreducible semigroup S. Each of $\{s\}$ is a $C_{\{s\}}$-class so our congruence equals id_S. Since S is subdirectly irreducible, $C_{\{s_1\}} = id_S$ for some $s_1 \in S$. It is easy to see that $\cap_{s_1 \neq s \in S} C_{\{s\}} = id_S$ and so $C_{\{s_2\}} = id_S$ for some $s_2 \neq s_1$. Hence s_1 and s_2 are two different disjunctive elements of S. □

Lemma 1.5 *([85]) If a semigroup S with a zero has a non-zero disjunctive element then S has a core and every disjunctive element of S is in the core.*

Proof. Assume that a semigroup S with a zero has a non-zero disjunctive element k. Since $r(k) = \{s \in S : (\forall x, y \in S^1)\ xsy \neq k\}$ is a $C_{\{k\}}$-class and an ideal of S, it follows that $r(k) = \{0\}$ (because k is disjunctive). Let I be an arbitrary non-trivial ideal of S. Then, for every non-zero element a of I, there are elements $x, y \in S^1$ such that $xay = k$ (because $r(k) = \{0\}$). So $k \in I$. Consequently, S has a core K and $k \in K$. □

Lemma 1.6 *([85]) A semigroup (with zero) which has a primitive core is subdirectly irreducible if and only if its zero is disjunctive.*

Proof. If S is a subdirectly irreducible semigroup with a primitive core then, by Lemma 1.4, the zero of S is disjunctive.

Conversely, assume that a semigroup S has a primitive core, and the zero of S is disjunctive. Then, by Theorem 3.7 of [85], S is subdirectly irreducible. □

Theorem 1.49 *([85]) A semigroup S with a zero and a non-trivial annihilator is subdirectly irreducible if and only if it has a non-zero disjunctive element.*

Proof. Let S be a subdirectly irreducible semigroup with a zero and a non-trivial annihilator. By Lemma 1.4, S has a non-zero disjunctive element.

Conversely, assume that S is a semigroup with a zero and a non-trivial annihilator A_S such that S contains a non-zero disjunctive element k. By Lemma 1.5, S has a core K, and every disjunctive element of S is in K. Let g be an arbitrary element of A_S. Then, for all $u \in S$, $ug = gu = 0$. So $g \in r(k) \cup \{k\}$. As $r(k) = \{0\}$, it follows that $g \in K$. Consequently $A_S \subseteq K$ and so $A_S = K$. Let a be an arbitrary non-zero element of A_S. Since $\{a, 0\}$ is an ideal of S, it follows that $A_S = K \subseteq \{a, 0\} \subseteq A_S$ which implies that K has exactly two elements, that is, K is primitive. We prove that the zero of S is disjunctive. Let e and f be arbitrary elements of S with $e \neq f$. As k is a disjunctive element of S, we have $(e,f) \notin C_{\{k\}}$. So there are elements $x, y \in S^1$ such that, for example, $xey = k$ and $xfy \neq k$. If $xfy = 0$ then $(e,f) \notin C_{\{0\}}$. If $xfy \neq 0$ then there are elements $u, v \in S^1$ such that $uxfyv = k$ (using $r(k) = \{0\}$). As $xfy \neq k$, we have $u \neq 1$ or $v \neq 1$. So $ukv = 0$ (because $k \in A_S$) from which we get $uxeyv = ukv = 0$. This and $uxfyv = k$ (see above) together imply that $(e,f) \notin C_{\{0\}}$. Consequently $(e,f) \notin C_{\{0\}}$ for any elements $e, f \in S$ with $e \neq f$. So the zero of S is disjunctive. Since the core of S is primitive then, by Lemma 1.6, S is subdirectly irreducible. □

Example Let S be a semigroup defined by the following Cayley-table:

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	a

It can be easily verified that S is subdirectly irreducible in which a is the zero element and $A_S = \{a, b\}$. Moreover b is a non-zero disjunctive element.

Δ-semigroups

Definition 1.54 *A semigroup S is called a Δ-semigroup if the lattice $\mathcal{L}(S)$ of all congruences of S is a chain (with respect to inclusion).*

Remark 1.1 If S^1 or S^0 is a Δ-semigroup then S is also a Δ-semigroup. The converse statement is not true, in general. For example, $S = \{a, e, 0 : a^2 = ae = 0, e^2 = e, ea = a\}$ is a Δ-semigroup, but S^1 is not a Δ-semigroup.

Theorem 1.50 *A left (right) zero semigroup is a Δ-semigroup if and only if it has at most two elements.*

Proof. As every equivalence relation of a left (right) zero semigroup is a congruence, the assertion is obvious. □

Theorem 1.51 *([100]) Every homomorphic image of a Δ-semigroup is also a Δ-semigroup.*

Proof. Let T be a homomorphic image of a semigroup S. Denote ϕ the corresponding homomorphism of S onto T. Let α_1 and α_2 be arbitrary congruences on T. Then $\alpha_i^* = \{(a,b) \in S \times S : (\phi(a), \phi(b)) \in \alpha_i\}$ $(i = 1, 2)$ is a congruence on S. If S is a Δ-semigroup then $\alpha_1^* \subset \alpha_2^*$ or $\alpha_1^* = \alpha_2^*$ or $\alpha_2^* \subset \alpha_1^*$ and so $\alpha_1 \subset \alpha_2$ or $\alpha_1 = \alpha_2$ or $\alpha_2 \subset \alpha_1$ which implies that T is a Δ-semigroup. □

Remark 1.2 In Chapter 3, it will be proved that a semilattice is a Δ-semigroup if and only if it has at most two elements. Theorem 1.4 and Theorem 1.51 together imply that if a semigroup S is a Δ-semigroup then it is either semilattice indecomposable or a semilattice of two semilattice indecomposable semigroups S_0 and S_1 $(S_0 S_1 \subseteq S_0)$.

Theorem 1.52 *([100]) If a Δ-semigroup S contains a proper ideal I then neither S nor I has a non-trivial group homomorphic image.*

Proof. Suppose there is a homomorphism f of S onto a non-trivial group G. Since G contains no ideal except G, $f(I) = G$. Hence $|I| > 1$. Denote α the congruence on S induced by f. For each $a \in S - I$, there is an element $b \in I$ such that $(a, b) \in \alpha$. Hence $\alpha \not\subseteq \rho_I$, where ρ_I denotes the Rees congruence on S defined by I. Since $|G| > 1$, there are elements $x, y \in I$ such that $(x, y) \notin \alpha$. As $(x, y) \in \rho_I$, we have $\alpha \not\supset \rho_I$ which contradicts our assumption. Next, suppose that I has a non-trivial group homomorphic image G. Then, by Lemma 8 of [100], there is a homomorphism of S onto G which is impossible. $\square$

Theorem 1.53 *If S is a Δ-semigroup then all the ideals of S form a chain with respect to inclusion.*

Proof. As the Rees congruences of a Δ-semigroup form a chain with respect to inclusion, the assertion is obvious. $\square$

Theorem 1.54 *([89]) The ideals of a semigroup S form a chain with respect to inclusion if and only if the principal ideals of S do it.*

Proof. Assume that the principal ideals of a semigroup S form a chain with respect to inclusion. Let A and B be two arbitrary ideals of S with $A \neq A \cap B \neq B$. Then there are elements $x \in A$ and $y \in B$ such that $x \notin B$ and $y \notin A$. Clearly, $J(x) \subseteq A$ and $J(y) \in B$. By the assumption, $J(x) \subseteq J(y)$ or $J(y) \subseteq J(x)$. Then $x \in B$ or $y \in A$ which is a contradiction. Consequently, $A \subseteq B$ or $B \subseteq A$. Hence the ideals of S form a chain with respect to inclusion. As the converse is obvious, the theorem is proved. $\square$

Theorem 1.55 *([107]) Let S be a Δ-semigroup and σ be a non-identity congruence of S which is not a Rees congruence. Then, for some $a \in S$,*

$$[b]_\sigma = I_a, \ \textit{if } J(b) \subset J(a),$$
$$[b]_\sigma \subseteq J_a, \ \textit{if } J(b) = J(a),$$
$$[b]_\sigma = \{b\}, \ \textit{if } J(b) \supset J(a),$$

where J_a denotes the $\mathcal{J}$-class of S containing a and $I_a = J(a) - J_a$.

Proof. For some $a \in S$ assume $|[a]_\sigma| > 1$ and that $[a]_\sigma$ is not an ideal of S. If $c, d \in [a]_\sigma$ and $J(c) \subset J(d)$ then, since σ is comparable with the Rees congruence $\rho_{J(c)}$, $\sigma \supseteq \rho_{J(c)}$. But then $J(c) \subseteq [a]_\sigma$ and $[a]_\sigma$ is an ideal. Hence $[a]_\sigma \subseteq J_a$. Since $[a]_\sigma \not\subseteq I_a$, if $I_a \neq \emptyset$ then σ contains the Rees congruence modulo I_a. Hence if $J(b) \subset J(a)$ then $b \in I_a$ and so $[b]_\sigma \supseteq I_a$. But then $[b]_\sigma$ is an ideal of S. Ideals of S are chain ordered and $[a]_\sigma \not\subseteq [b]_\sigma$ so $[b]_\sigma \subseteq I_a$. Thus $[b]_\sigma = I_a$. If $J(b) \supseteq J(a)$ and $[b]_\sigma \neq \{b\}$ then $[b]_\sigma$ is not an ideal; otherwise $[b]_\sigma \supseteq J(b) \supseteq [a]_\sigma$ and $[a]_\sigma$ is an ideal. Hence as above, $[b]_\sigma \subseteq J_b$ and either $I_b = \emptyset$ or I_b is a σ-class and an ideal of S. In either case $I_b \not\supseteq [a]_\sigma$ so $I_b = I_a$ and then $J_a = J_b$. $\square$

Theorem 1.56 *On a nil semigroup S, the following are equivalent.*

(i) S is a Δ-semigroup.

(ii) The principal ideals of S form a chain with respect to inclusion

(iii) S is a chain with respect to the divisibility ordering.

Proof. (i) implies (ii). It is obvious for arbitrary semigroups.

(ii) implies (iii). Let S be a nil semigroup in which the principal ideals form a chain with respect to inclusion. By Theorem 1.17, S is $\mathcal{J}$-trivial. Then, by Theorem 1.16, S is a chain with respect to the divisibility ordering.

(iii) implies (i). Let S be a nil semigroup which is a chain with respect to the divisibility ordering. Then, by Theorem 1.18, every congruence on S is a Rees congruence. To prove (i), we may prove that the ideals form a chain. Let I and J be ideals of S. Suppose $I \not\subseteq J$. There is an element $a \in I$ but $a \notin J$. Let x be an arbitrary element of J. Clearly, $a \neq x$. By (iii), either $a \in S^1xS^1$ or $x \in S^1aS^1$. In the first case $a \in J(x) \subseteq J$ which is a contradiction. So $x \in S^1aS^1$ from which we get $x \in J(a) \subseteq I$. Hence $J \subseteq I$. □

Theorem 1.57 *If a Δ-semigroup S is a semilattice of a nil semigroup S_1 and an ideal S_0 of S then $|S_1| = 1$.*

Proof. If S_1 is a nil semigroup (with zero 0) then $I = S_0 \cup \{0\}$ is an ideal of S. As $S_0 \subseteq I$, we have $\eta \subseteq \rho_I$, where ρ_I is the Rees congruence on S modulo I, and η is the semilattice congruence on S (the η-classes are S_1 and S_0). Hence S_1 has only one element. □

Definition 1.55 *Let S be a Δ-semigroup which is a semilattice of a semigroup P and a non-trivial nil semigroup N such that $NP \subseteq N$. Then S is called*

- *a T1 semigroup if P has only one element,*
- *a T2L semigroup if P is a two-element left zero semigroup,*
- *a T2R semigroup if P is a two-element right zero semigroup.*

Theorem 1.58 *([63]) Let S be a semigroup which is a disjoint union $S = P \cup N$ of a one-element subsemigroup $P = \{e\}$ of S and an ideal N of S such that N is a nil semigroup. Then S is a Δ-semigroup if and only if N is a Δ-semigroup and $S^1eS^1 = S$.*

Proof. Assume that S is a Δ-semigroup. Then $S^1eS^1 = S$ and, by lemma 2.7 of [63], $J(a) = N^1aN^1$ for every $a \in N$. As the principal ideals of S are chain ordered, the principal ideals of N are chain ordered. Then, by Theorem 1.56, N is a Δ-semigroup.

Conversely, assume that N is a Δ-semigroup and $S^1eS^1 = S$. Let α be a non-identity congruence on S. Assume $(e, a) \in \alpha$ for some $a \in N$. Then

$(e, ae) \in \alpha$ which implies that $(e, a^m e) \in \alpha$ for every positive integer m. As N is a nil semigroup, we get $(e, 0) \in \alpha$, where 0 is the zero element of N. Then $(xey, 0) \in \alpha$ for every $x, y \in S^1$. As $S^1 e S^1 = S$, we get $(s, 0) \in \alpha$ for every $s \in S$. Consequently, α is the universal relation of S. This means that $\{e\}$ is a β-class for every non-universal congruence β of S. Thus the congruences of S form a chain, because N is a Δ-semigroup. □

Corollary 1.2 *A nil semigroup with an identity adjoined N^1 is a Δ-semigroup if and only if N is a Δ-semigroup.*

Theorem 1.59 *([107]) If a Δ-semigroup S is a semilattice of a subgroup P of a quasicyclic p-group (p is a prime) and a nil semigroup N, $NP \subseteq N$ then either $|N| = 1$ or $|P| = 1$.*

Proof. It is sufficient to show that $|P| > 1$ implies $|N| = 1$.

Part 1: In this part we show that $J_b = PbP$, bP, Pb or $\{b\}$ for $b \in PSP$, $SP - PSP$, $PS - PSP$ or $S - (SP \cup PS)$, respectively (here J_b denotes the $\mathcal{J}$-class of S containing b). If $J_b = J_a$ then $rbs = a$, $paq = b$ for some $r, s, p, q \in S^1$. Hence $prbsq = b$ and $rpaqs = a$. Assume $b \neq a$. As S is a nil semigroup, $r, s, p, q \in P$. Then $a \in PbP$ or bP or Pb. It is easy to see that elements of these sets are $\mathcal{J}$-related.

Part 2: Let H be the subgroup of P of order p with generator g. In this part we prove that if $b \in PSP$ then $Hb \subseteq bP$ or $\{b\} = bH$. Let e denote the identity of P and define $P' = P - \{e\}$.

Case 1: If $b \in P'bP$ then $b = hbk$ for some $h \in P'$ and $k \in P$. Thus $b = h^m b k^m$ for every positive integer m. It is clear that H is contained by the subgroup of P generated by h. Thus, for each integer j, there is an m such that $g^{-j} = h^m$. Hence $Hb \subseteq bP$.

Case 2: If $b \in bP'$ then $b = bk$ for some $k \in P'$. Thus $b = bk^m$ for every positive integer m. Hence $\{b\} = bH$.

Case 3: Suppose $b \notin P'bP \cup PbP'$. As $b \in PSP$, we have $J_b = PbP$ (as it was proved in Part 1). It is easy to see that $xJ_b y \cap J_b \neq \emptyset$ $(x, y \in S^1)$ if and only if $x, y \in P \cup \{1\}$. We show that Hb is a normal complex of S. Assume $xHby \cap Hb \neq \emptyset$ for some $x, y \in S^1$. As $Hb \subseteq PbP = J_b$, we get $x, y \in P \cup \{0\}$. Let $u, v \in H$ such that $xuby = vb$. Then $b = v^{-1}xuby$. As $b \notin P'bP$, we have $x \in H \cup \{1\}$ and $y \in \{1, e\}$. Hence $xHby \subseteq Hb$. Consequently, Hb is a normal complex. Let α be the least congruence on S with Hb for an α-class. It can be easily shown that in J_b, α has classes $rHbs$ $(r, s \in P)$. Likewise, there is a congruence β on S with classes $rbHs$ $(r, s \in P)$ in J_b. Since $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$ then $rHbS \subseteq bH$ or $bH \subseteq rHbs$ for some $r, s \in P$. But then $b \in P'bP \cup PbP'$ which is a contradiction.

Part 3: Now we complete the proof. Since H is normal in P, there is a least congruence ρ on S with H as a ρ-class. By Theorem 1.55, $[a]_\rho = N$ for any $a \in N$. Thus, by Lemma 1.3, there are $x_i, y_i \in S^1$, $p_i, q_i \in H$ $(i = 1, 2, \ldots, n)$ such that

$$a = x_1 p_1 y_1,\ x_1 q_1 y_1 = x_2 p_2 y_2,\ \ldots, x_n q_n y_n = 0.$$

Let $z_i = ey_ie$. Then

$$ae = x_1p_1z_1,\ x_1q_1z_1 = x_2p_2z_2, \dots, x_nq_nz_n = 0.$$

As $z_i \in PSP$, by Part 2, $\{z_i\} \neq z_iH$ or $Hz_i \subseteq z_iP$. If $\{z_i\} \neq z_iH$ then $Hz_i \subseteq z_iP$. But then $x_ip_iz_i = x_iz_iu$, $x_iq_iz_i = x_iz_iv$ for some $u, v \in P$. Then, by Part 1, $x_ip_iz_i\ \mathcal{J}\ x_iq_iz_i$. Let j be the least integer so that $\{z_j\} = z_jH$; if there is no such j then $0 \in J_{ae}$ and $ae = 0$. Since $ae = 0$ or $x_jp_jz_j \in J_{ae}$ then, by Part 1, $\{ae\} = aeH$. So $\{xe\} = xH$ for all $x \in N$. Then $x_i \in P \cup \{1\}$ or $x_ip_i = x_iq_i$. In either case $x_ip_iy_i\ \mathcal{J}\ x_iq_iy_i$, $1 \leq i \leq n$. Thus $a = 0$, that is, $|N| = 1$. □

Theorem 1.60 *Let S be a semigroup in which $\alpha \cap \beta = id_S$ implies $\alpha = id_S$ or $\beta = id_S$ for every congruences α and β on S. If S is an ideal extension of a rectangular group K by a semigroup with zero then K is either a subgroup or a left zero subsemigroup or a right zero subsemigroup of S.*

Proof. Let S be a semigroup satisfying the condition of the theorem. Assume that it is an ideal extension of a rectangular group K by a semigroup with zero. We can suppose that $|K| > 1$. Then K is a dense ideal of S. As K is weakly reductive, by Theorem 1.37, S can be embedded into the translational hull $\Omega(K)$ of K. We suppose that S is a subsemigroup of $\Omega(K)$. Since K is a rectangular group, it is a direct product $L \times G \times R$ of a left zero semigroup L, a group G and a right zero semigroup R. Let $\mathcal{T}_L$ denote the semigroup of all transformations of L acting on the left. By Theorem 1.34, $\Omega(L) = \mathcal{T}_L$ and so L is isomorphic to a subsemigroup of $\mathcal{T}_L$, because L is weakly reductive. Hence we can suppose that L is a subsemigroup of $\mathcal{T}_L$. Let $\mathcal{T}_R$ be the semigroup of all transformations of R acting on the right. We can suppose that R is a subsemigroup of $\mathcal{T}_R$. By Theorem 1.34, $\Omega(K) \cong \mathcal{T}_L \times G \times \mathcal{T}_R$. Let α_L, α_G and α_R denote the congruence on $\Omega(K)$ induced by the projection homomorphism of $\Omega(K)$ onto $\mathcal{T}_L$, G and $\mathcal{T}_R$, respectively. Then $\alpha_L \cap \alpha_G \cap \alpha_R = id_{\Omega(K)}$. Let α'_L, α'_G and α'_R denote the restriction of α_L, α_G and α_R to S, respectively. Then $\alpha'_L \cap \alpha'_G \cap \alpha'_R = id_S$. By the condition for S, we have $\alpha'_L = id_S$ or $\alpha'_G = id_S$ or $\alpha'_R = id_S$. As $i_0 \times G \times R \subseteq \alpha'_L$, $L \times g_0 \times R \subseteq \alpha'_G$ and $L \times G \times j_0 \subseteq \alpha'_R$ for fixed $i_0 \in L$, $g_0 \in G$ and $j_0 \in R$, we get that K is isomorphic to either L or G or R. Thus the theorem is proved. □

Corollary 1.3 *If a Δ-semigroup S is an ideal extension of a rectangular group K by a semigroup with zero then K is either a subgroup or a left zero subsemigroup or a right zero subsemigroup of S.*

Corollary 1.4 *If a subdirectly irreducible semigroup S is an ideal extension of a rectangular group K by a semigroup with zero then K is either a subgroup or a left zero subsemigroup or a right zero subsemigroup of S.*

Theorem 1.61 *([107]) A non-trivial band is a Δ-semigroup if and only if it is isomorphic to either R or R^1 or R^0, where R is a two-element right zero semigroup, or L or L^1 or L^0, where L is a two-element left zero semigroup, or F, where F is a two-element semilattice.*

Proof. Let S be a non-trivial Δ-band. Then, by Theorem 1.5 and Remark 1.1, S is either a rectangular band or a disjoint union $S = S_1 \cup S_0$ of two rectangular bands S_0 and S_1 such that S_0 is an ideal of S. If S is a rectangular band then it is a direct product of a left zero semigroup L and a right zero semigroup R. As S is a Δ-semigroup, S is isomorphic to either L or R. Hence, by Theorem 1.52, S is either a two-element left zero semigroup or a two-element right zero semigroup.

Assume that S is a disjoint union $S = S_1 \cup S_0$ of two rectangular bands S_1 and S_0, where S_0 is an ideal of S. By Theorem 1.51, S_1^0 and so S_1 is a Δ-semigroup. Then S_1 is either a one-element semigroup or a two-element left zero semigroup L or a two-element right zero semigroup R. If $|S_0| = 1$ then either S is a two-element semilattice or $S = L^0$ or $S = R^0$. Assume that $|S_0| > 1$. First, consider the case when $S_1 = \{e\}$. We show that the Green's left congruence $\mathcal{R}$ is a congruence on S. Let $a, b \in S$ be arbitrary elements with $(a, b) \in \mathcal{R}$ and $a \neq b$. Then $a, b \in S_0$ and $a = bx$, $b = ay$ for some $x, y \in S_0$. Let $s \in S$ be arbitrary. Then

$$asxys = bxsxys = bx(sx)xys = bxys = ays = bs,$$

because $x, sx \in S_0$ and S_0 is a rectangular band. Thus $bs \in asS^1$. We can prove, in a similar way, that $as \in bsS^1$. Hence $(as, bs) \in \mathcal{R}$ which implies that $\mathcal{R}$ is a congruence. We can prove, in a similar way, that $\mathcal{L}$ is a congruence on S. It is clear that aS_0 and S_0a are $\mathcal{R}$-classes and $\mathcal{L}$-classes, respectively, for every $a \in S_0$. Assume $\mathcal{R} \subseteq \mathcal{L}$. Then $aS_0 \subseteq S_0a$ and so $aS_0 \subseteq aS_0a = \{a\}$. Hence S_0 is a left zero semigroup. It is easy to see that $\alpha = \{(a, b) \in S \times S : \ a = b \text{ or } a, b \in e \cup eS_0\}$ is a congruence on S such that $e \cup eS_0$ is an α-class. Since $\{e\} \subseteq e \cup eS_0$ and since S is a Δ-semigroup, we get $\rho_{S_0} \subseteq \alpha$, where ρ_{S_0} is the Rees congruence of S modulo S_0. Hence $S_0 = eS_0$. Then, for $a \in S_0$, $ea = a$ and $ae = a(ae) = a$. Hence $S = S_0^1$. Since S_0^1 is a Δ-semigroup then S_0 is a Δ-semigroup. Then, by Theorem 1.50, S_0 is a two-element left zero semigroup. We can prove, in a similar way, that $S = S_0^1$ and S_0 is a two-element right zero semigroup if $\mathcal{L} \subseteq \mathcal{R}$.

Next, consider the case when S_1 is a two-element right zero semigroup. We prove that each element of S_0 is a right zero of S and that $|S_0| \leq 2$. Note that, for $a, b \in S_0$, $c \in S^1$ and $u \in S_1$, we have $acb = a(acb)b = ab$ and so if $ac = uac$ then $a = aca = uaca = ua$. Hence there is a congruence ρ on S with classes S_1, S_1S_0 and $S_0 - S_1S_0$. Since ρ must be comparable with the Rees congruence on S modulo S_0 then $S_0 = S_1S_0$. Hence $S_1x = \{x\}$ for all $x \in S$. Then there is a congruence

$$\sigma = \{(p, q) \in S \times S : \ p = qx \text{ for each } x \in S\}$$

with $[u]_\sigma = S_1$. Comparison of σ with the Rees congruence of S modulo S_0, we get $[a]_\sigma = S_0$ for all $a \in S_0$. Hence $ax = xx = x$ for $x \in S_0$ and so $S_0x = \{x\}$.

Since $S_1x = \{x\}$ then $Sx = \{x\}$ for all $x \in S_0$. Hence every element of S_0 is a right zero of S. For $a \neq b$ $(a, b \in S_0)$ we have $[a]_\sigma = [b]_\sigma$. Since σ is the least congruence on S with S_1 as a σ-class then there exist elements $x_i, y_i \in S^1$ and $p_i, q_i \in S_1$ $(i = 1, 2, \dots n)$ such that $a = x_1p_1y_1$, $x_1q_1y_1 = x_2p_2y_2$, ... , $x_nq_ny_n = b$. We may assume $y_i = 1$ since $p_iy_i = q_iy_i$ if $y_i \neq 1$. Then $a = ap_1 = bp_1$, $b = bq_n = aq_n$ and so $|S_0| \leq 2$. We get the same result if S_1 is a two-element left zero semigroup. Consequently, S_0 is either a two-element left zero semigroup or a two-element right zero semigroup.

Assume that $S_1 = \{u, v\}$ is a right zero semigroup and $S_0 = \{a, b\}$ is a left zero semigroup. Then $au = a(au) = a = a(av) = av$ and, similarly, $bu = bv$. If $ua = a$ then $ua = vua = va$. Similarly, $va = a$ implies $va = ua$. If $ua \neq a$ and $va \neq a$ then $ua = b = va$. Hence $au = av$ and $ua = va$. Similarly, $bu = bv$ and $ub = vb$. Thus the equivalence with classes S_1, $\{a\}$, $\{b\}$ is a congruence on S which is not comparable with the Rees congruence on S modulo S_0.

Assume that $S_1 = \{u, v\}$ and $S_0 = \{a, b\}$ are right zero semigroups. Then $ua = a = va$ and $ub = b = vb$. If $au = av$ and $bu = bv$ then the equivalence on S with classes S_1, $\{a\}$, $\{b\}$ is a congruence on S which is not comparable with the Rees congruence on S modulo S_0. Assume that $au \neq av$. If $au = a$ and $av = b$ then $bu = a$ and $bv = b$. Hence the equivalence on S with classes $\{u, a\}$, $\{v, b\}$ is a congruence on S wich is not comparable with the Rees congruence on S modulo S_0. If $au = b$ and $av = a$ then $bu = b$ and $bv = a$. Hence the equivalence on S with classes $\{v, a\}$, $\{u, b\}$ is a congruence on S which is not comparable with the Rees congruence on S modulo S_0. We also get that S is not a Δ semigroup if we suppose $bu \neq bv$.

We can prove, in a similar way, that S is not a Δ-semigroup if S_1 is a two-element left zero semigroup and S_0 is a two-element right zero semigroup or S_1 and S_0 are two-element left zero semigroups. Thus the first part of the theorem is proved. As the semigroups listed in the theorem are Δ-bands, the theorem is proved. $\square$

Chapter 2

Putcha semigroups

In [80], M.S. Putcha characterized semigroups which are decomposable into semilattice of archimedean semigroups. He showed that a semigroup S is a semilattice of archimeden semigroups if and only if, for every $a, b \in S$, the assumption $a \in S^1bS^1$ implies $a^n \in S^1a^2S^1$ for some positive integer n. Semigroups with this condition are called Putcha semigroups. In this chapter we also consider the left Putca semigroups and the right Putcha semigroups (Definition 2.1). It is proved that a semigroup is a simple left and right Putcha semigroup if and only if it is completely simple. By the help of this result, the retract extension of completely simple semigroups by nil semigroups are characterized. It is shown that a semigroup is a retract extension of a completely simple semigroup by a nil semigroup if and only if it is an archimedean left and right Putcha semigroup containing at least one idempotent element.

Definition 2.1 *A semigroup S is called a left (right) Putcha semigroup if, for every $x, y \in S$, the assumption $y \in xS^1$ ($y \in S^1x$) implies $y^m \in x^2S^1$ ($y^m \in S^1x^2$) for some positive integer m.*

A semigroup S is called a Putcha semigroup if, for every $x, y \in S$, the assumption $y \in S^1xS^1$ implies $y^m \in S^1x^2S^1$ for some positive integer m.

Lemma 2.1 *([41]) S is a left (right) Putcha semigroup if and only if, for any $x, y \in S$ and positive integer n, there is a positive integer m such that $(xy)^m \in x^nS^1$ ($(xy)^m \in S^1y^n$).*

Proof. Let S be a left Putcha semigroup. As $xy \in xS^1$, there is a positive integer t such that

$$(xy)^t \in x^2S^1.$$

From this it follows that, for every positive integer k, there is a positive integer p such that

$$(xy)^p \in x^{2k}S^1.$$

Let n be an arbitrary positive integer. Assume $2k \geq n$. Then, for some positive integer m,

$$(xy)^m \in x^n S^1.$$

Conversely, assume that a semigroup S satisfies the condition that, for every $x, y \in S$ and positive integer n, there is a positive integer m such that

$$(xy)^m \in x^n S^1.$$

Assume that

$$y \in xS^1$$

for some $x, y \in S$. Then

$$y^2 = xu$$

for some $u \in S$ and so, for $n = 2$, there is a positive integer m such that

$$y^{2m} = (xu)^m \in x^2 S^1$$

which implies that S is a left Putcha semigroup. The proof of the assertion for right Putcha semigroups is similar. □

Lemma 2.2 *([41]) A left (right) Putcha semigroup is a Putcha semigroup.*

Proof. Let S be a left Putcha semigroup and $a, b \in S$ be arbitrary elements with

$$b \in S^1 a S^1,$$

that is,

$$b = xay$$

for some $x, y \in S^1$. We can suppose that one of x and y is in S. Then, by Lemma 2.1, there is a positive integer m such that

$$(a(yx))^m \in a^2 S^1$$

and so

$$b^{m+1} = (xay)^{m+1} = x(ayx)^m ay \in S^1 a^2 S^1.$$

Hence S is a Putcha semigroup. We can prove, in a similar way, that a right Putcha semigroup is a Putcha semigroup. □

Theorem 2.1 *([80]) A semigroup S is a semilattice of archimedean semigroups if and only if it is a Putcha semigroup. In such a case the corresponding semilattice congruence equals*

$$\eta = \{(a, b) \in S \times S : \ a^m \in SbS, \ b^n \in SaS \text{ for some positive integers } m, n\}$$

and is the least semilattice congruence on S.

Proof. Let S be a Putcha semigroup. Define a relation η on S as follows.

$$\eta = \{(a,b) \in S \times S : \ a^m \in SbS, \ b^n \in SaS \text{ for some positive integers } m,n\}.$$

It is easy to see that η is reflexive and symmetric on an arbitrary semigroup. We show that η is transitive on S. Let $a,b,c \in S$ be arbitrary elements with

$$(a,b) \in \eta$$

and

$$(b,c) \in \eta,$$

that is,

$$a^m \in SbS, \ b^n \in SaS,$$

and

$$b^t \in ScS, \ c^k \in SbS$$

for some positive integers m,n,t,k. As S is a Putcha semigroup, for every positive integer r, there is a positive integer u such that

$$c^u \in Sb^{2^r}S.$$

Assume that $2^r \geq n$. Then

$$c^u \in Sb^{2^r}S \subseteq SaS.$$

Similarly,

$$a^v \in ScS$$

for some positive integer v. Hence η is transitive. We show that η is a congruence on S. Let $a,b,s \in S$ be arbitrary elements with

$$(a,b) \in \eta.$$

Then there are positive integers m,n and elements $x,y,u,v \in S$ such that

$$a^m = ubv$$

and

$$b^n = xay.$$

Let k be a positive integer such that $2^k \geq m$. Since S is a Putcha semigroup and since $sa \in S^1aS^1$ then

$$(sa)^t \in Sa^{2^k}S$$

for some positive integer t. Thus, for some $e,f \in S$,

$$(sa)^t = ea^{2^k}f = ea^m a^{2^k-m}f = eubva^{2^k-m}f$$

and so

$$(sa)^{t+1} = eu(bva^{2^k-m}fs)a \in Sbva^{2^k-m}fsS.$$

As S is a Putcha semigroup, we have

$$(sa)^p \in S(bva^{2^k-m}fs)^2S \subseteq SsbS$$

for some positive integer p. We can prove, in a similar way, that

$$(sb)^q \in SsaS$$

for some positive integer q. Thus

$$(sa, sb) \in \eta.$$

Hence η is left compatible on S. Similarly, η is right compatible on S. Thus η is a congruence on S. As $(a, a^2) \in \eta$ and $(bc, cb) \in \eta$ for every $a, b, c \in S$, the factor semigroup $Y = S/\eta$ is a semilattice. Hence S is a semilattice Y of the η-classes S_α. Let S_α be an η-class of S. Then, for every $a, b \in S_\alpha$, there are positive integers m, n and elements $x, y, u, v \in S$ such that

$$xay = b^m$$

and

$$ubv = a^n.$$

Assume $x \in S_\gamma$ and $y \in S_\delta$. Then

$$\alpha = \alpha\gamma\delta$$

in Y, that is,

$$S_\alpha = S_{\alpha\gamma\delta}.$$

As

$$(xayx)a(yxay) = b^{3m}$$

and

$$xayx, yxay \in S_{\alpha\gamma\delta} = S_\alpha,$$

we get

$$b^{3m} \in S_\alpha a S_\alpha.$$

Similarly,

$$a^{3n} \in S_\alpha b S_\alpha.$$

Hence S_α is an archimedean semigroup.

We show that η is the least semilattice congruence on S. Let σ be an arbitrary semilattice congruence on S. Assume $(a, b) \in \eta$ for some $a, b \in S$, that is, $xay = b^i$ and $ubv = a^j$ for some $x, y, u, v \in S$ and some positive integers i, j. Then

$$a \;\sigma\; a^j = ubv \;\sigma\; ub^{i+1}v = uxaybv \;\sigma\; xaubvy =$$

$$xa^{j+1}y \;\sigma\; xay = b^i \;\sigma\; b.$$

Hence $\eta \subseteq \sigma$. Thus the first part of the theorem is proved.

Conversely, assume that a semigroup S is a semilattice Y of archimedean semigroups S_α $(\alpha \in Y)$. Assume

$$b \in S^1 a S^1$$

for some $a, b \in S$. Then

$$xay = b^3$$

for some $x, y \in S$. It is clear that $xay = b^3$ and xa^2y are in the same semilattice component S_α. As S_α is archimedean,

$$b^{3k} \in Sxa^2yS \subseteq Sa^2S$$

for some positive integer k. Consequently, S is a Putcha semigroup. $\square$

Corollary 2.1 *Let G be a subgroup of a semigroup S. If S is a semilattice Y of archimedean semigroups S_α $(\alpha \in Y)$ and $G \cap S_\alpha \neq \emptyset$ then $G \subseteq S_\alpha$.*

Proof. It is obvious, because the elements of G are in the same η-class of S. $\square$

Corollary 2.2 *Every left (right) Putcha semigroup is decomposable into a semilattice of archimedean semigroups.*

Proof. By Lemma 2.2 and Theorem 2.1, it is obvious. $\square$

Theorem 2.2 *([16]) A semigroup S is archimedean and contains at least one idempotent element if and only if it is an ideal extension of a simple semigroup containing an idempotent by a nil semigroup.*

Proof. Let S be an archimedean semigroup containing an idempotent e. Let $K = SeS$. It is obvious that K is an ideal of S. As S is archimedean, K contains all idempotent elements of S. Let A be an arbitrary ideal of S and $a \in A$ be arbitrary. Then

$$e \in S^1 a S^1 \subseteq A$$

and so

$$K \subseteq A.$$

Hence K is the kernel of S and so, by Corollary 2.30 of [19], it is simple. As S is archimedean, the Rees factor semigroup S/K is nil.

Conversely, let S be a semigroup which is an ideal extension of a simple semigroup K containing an idempotent by a nil semigroup. Let $a, b \in S$ be arbitrary elements. Then

$$a^n,\ b^m \in K$$

for some positive integers m and n. As K is simple,

$$a^n \in Kb^nK \subseteq S^1 b S^1.$$

Similarly,

$$b^m \in S^1 a S^1 .$$

Hence S is an archimedean semigroup. $\square$

Theorem 2.3 *([41]) A semigroup is a simple left and right Putcha semigroup if and only if it is completely simple.*

Proof. Let S be a simple left and right Putcha semigroup. First we prove that if $x \in S$ and n is an integer with $n \geq 3$ then x^n is regular. So let $x \in S$ and $n \geq 3$. Because of S is simple, x^{n-2} belongs to Sx^nS and so

$$x^n = xux^nvx$$

for some $u, v \in S$. Then, for any positive integer m, we have

$$x^n = (xu)^m x^n (vx)^m.$$

By Lemma 2.1, there is a positive integer m such that

$$(xu)^m \in x^nS$$

and

$$(vx)^m \in Sx^n.$$

Hence

$$x^n \in x^nSx^n,$$

that is, x^n is regular. Consequently, S has an idempotent element. We show that S is completely simple. Assume, in an indirect way, that S is not completely simple. Then, by Theorem 1.22, S contains a bicyclic semigroup

$$C = \langle p, q; pq = e \rangle$$

having e as identity element. We have

$$qp \in S^1p.$$

Since S is a right Putcha semigroup,

$$qp = (qp)^m = xp^2$$

for some $x \in S^1$ and a positive integer m. Then we obtain

$$xe = xp^2q^2 = qpq^2 = q^2 \in C.$$

But

$$xep^2 = xp^2$$

and so we get

$$qp = xp^2 = xep^2 = q^2p^2.$$

This is a contradiction. Consequently, S is completely simple.

Conversely, let S be a completely simple semigroup. Then it is isomorphic to a Rees matrix semigroup $\mathcal{M} = \mathcal{M}(I, G, J; P)$ over a group G with a sandwich matrix P. We can identify S and $\mathcal{M}$. Let (i, a, j), $(k, b, n) \in S$ be arbitrary elements. Then

$$(i,a,j)(k,b,n) = (i,a,j)^2(i,(p_{j,i}ap_{j,i})^{-1}p_{j,k}b,n) \in (i,a,j)^2S$$

which implies that S is a left Putcha semigroup. We can prove, in a similar way, that S is a right Putcha semigroup. $\square$

Theorem 2.4 *([41]) A semigroup is an archimedean left and right Putcha semigroup containing at least one idempotent element if and only if it is a retract extension of a completely simple semigroup by a nil semigroup.*

Proof. Let S be an archimedean left and right Putcha semigroup with idempotent elements. Then, by Theorem 2.2, S is an ideal extension of a simple semigroup K containing all idempotents of S by the nil semigroup $N = S/K$. It is easy to see that an ideal of a left and right Putcha semigroup is also a left and right Putcha semigroup. So, by Theorem 2.3, K is completely simple and so it is isomorphic to a Rees matrix semigroup $\mathcal{M}(I, G, J; P)$ over a group G with a normalized sandwich matrix P. Since K is also weakly reductive then, by Theorem 1.35, it is an ideal of the translational hull $\Omega(K)$ of K, where, by Theorem 1.33,

$$\Omega(K) = \{(k, a, h) \in \mathcal{T}_I \times G \times \mathcal{T}_J : (\forall i \in I, j \in J) p_{j,k(i)} a p_{(j_0)h,i} = p_{j,k(i_0)} a p_{(j)h,i}\}.$$

The product of two elements (k, a, h) and (f, b, g) of $\Omega(K)$ is given by:

$$(k, a, h)(f, b, g) = (k \circ f, a p_{(j_0)h, f(i_0)} b, h \circ g).$$

Moreover, if $(i, g, j) \in K$ and $(k, a, h) \in \Omega(K)$ are arbitrary elements then

$$(k, a, h)(i, g, j) = (k(i), a p_{(j_0)h,i} g, j) \in K$$

and

$$(i, g, j)(k, a, h) = (i, g p_{j,k(i_0)} a, (j)h) \in K.$$

A bitranslation $(k, a, h) \in \Omega(K)$ is inner if and only if k and h are constant transformations. By Theorem 1.36, there is an ideal extension $(S', +)$ of $\Omega(K)$ by N such that S is a subsemigroup in S'. Let e denote the identity of $\Omega(K)$. Then

$$\phi : \; x \mapsto x + e$$

is a retract homomorphism of S' onto $\Omega(K)$. The operation on S' is determined by ϕ. If $x, y \in N^* = N - \{0\}$ and $s, t \in \Omega(K)$ then $x + t = \phi(x)t$, $t + x = t\phi(x)$, $s + t = st$, $x + y = xy$ in N if $xy \notin \Omega(K)$ and $x + y = \phi(x)\phi(y)$ if $xy \in \Omega(K)$. We prove that the restriction of ϕ to S is a retract homomorphism of S onto K. It is sufficient to show that, for every $s \in N^*$, we have $\phi(s) \in K$, that is, $\phi(s)$ is an inner bitranslation of K. Let s be an arbitrary element of N^*, and let $\phi(s) = (k, a, h) \in \Omega(K)$. As N is a nil semigroup,

$$s^n \in K$$

for some positive integer n. Thus

$$(k, a, h)^n = (k_0, b, h_0).$$

Let $(i, g, j) \in K$ be arbitrary. Because of S is a left Putcha semigroup, by Lemma 2.1, there is a positive integer m and an element $x \in S$ such that

$$(s(i, g, j))^m = s^n x.$$

Let $\phi(x) = (k_x, b_x, h_x) \in \Omega(K)$. Then

$$(k(i), (ap_{(j_0)h,i}gp_{j,k(i)})^{m-1}ap_{(j_0)h,i}g, j) = (k(i), ap_{(j_0)h,i}g, j)^m$$

$$= ((k,a,h)(i,g,j))^m = (\phi(s)(i,g,j))^m = (s(i,g,j))^m = \phi((s(i,g,j))^m) =$$

$$\phi(s^n x) = (k,a,h)^n\phi(x) = (k_0, b, h_0)(k_x, b_x, h_x) = (k_0, bp_{h_0,k_x(i_0)}b_x, (h_0)h_x).$$

From this we get

$$k(i) = k_0$$

for every $i \in I$, that is, k is a constant transformation. Using that S is also a right Putcha semigroup, we obtain that h is a constant transformation. Hence

$$\phi(s) \in K.$$

Thus the first part of the theorem is proved.

Conversely, since a completely simple semigroup is an archimedean left and right Putcha semigroup, it is easy to see that a retract extension of a completely simple semigroup by a nil semigroup has the same property. □

Chapter 3

Commutative semigroups

In 1984, A. Restivo and C. Reutenauer solved the Burnside problem for semigroups. They proved that a finitely generated semigroup is finite if and only if it is periodic and has the permutation property P_n for some integer $n \geq 2$. This fact drown the attention to semigroups satisfying some permutation properties. The semigroups satisfying the permutation property P_2 are exactly the commutative semigroups. All of semigroups considered in this book are generalized commutative semigroups and most of them have some permutation property. In their examinations the commutative semigroups are appeared in subcases. That is why we deal with them in a separated chapter. The literature of commutative semigroups is very rich, but we present only those results which will be used in the other chapters of this book.

In the first part of the chapter we deal with the semilattice decomposition of commutative semigroups. It is proved that every commutative semigroup is a semilattice of commutative archimedean semigroups. Moreover, a semigroup is a commutative archimedean semigroup containing at least one idempotent element if and only if it is an ideal extension of a commutative group by a commutative nil semigroup. It is proved that every commutative archimedean semigroup without idempotent element has a non-trivial group homomorphic image. It is proved that a commutative semigroup is separative if and only if its archimedean components are cancellative.

In the second part of the chapter we determine the subdirectly irreducible commutative semigroups. The following results are proved. A semigroup is a subdirectly irreducible commutative semigroup with a globally idempotent core if and only if it is isomorphic to G or G^0 or F, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime) and F is a two-element semilattice. A commutative semigroup with zero and a non-trivial annihilator is subdirectly irreducible if and only if it has a non-zero disjunctive element. A semigroup is a commutative subdirect irreducible semigroup with a nilpotent core and a trivial annihilator if and only if it contains an identity, a non-zero divisor of zero and a non-zero disjunctive element, and the set of all non-divisors of zero forms a subdirectly irreducible commutative group.

In the last part of the chapter we determine the commutative Δ-semigroups. It is shown that a semigroup S is a commutative Δ-semigroup if and only if it is isomorphic to either G or G^0, where G is a nontrivial subgroup of a quasicyclic p-group (p is a prime) or N or N^1, where N is a commutative nil semigroup satisfying the divisibility chain condition.

First of all we give a condition for commutative semigroups to be finite.

Theorem 3.1 *Every finitely generated periodic commutative semigroup is finite.*

Proof. By Theorem 1.1, it is obvious. □

Semilattice decomposition of commutative semigroups

Theorem 3.2 *Every commutative semigroup is a left and right Putcha semigroup.*

Proof. It is obvious. □

Theorem 3.3 *([95]) Every commutative semigroup is a semilattice of commutative archimedean semigroups.*

Proof. By Theorem 3.2 and Corollary 2.2, it is obvious. □

Definition 3.1 *A semigroup S is called a power joined semigroup if, for every $a, b \in S$, there are positive integers m, n such that $a^n = b^m$.*

It is clear that a power joined semigroup is a special archimedean semigroup.

Theorem 3.4 *([66]) Let S be a commutative semigroup. S is a semilattice of power joined semigroups if and only if every group and every group with zero homomorphic image of S is a periodic group and a periodic group with zero, respectively.*

Theorem 3.5 *([64]) The following conditions on a commutative semigroup S are equivalent.*

(i) S is power joined.

(ii) Every subsemigroup of S is archimedean.

(iii) Every finitely generated subsemigroup of S is archimedean.

Theorem 3.6 *([64]) Every proper subsemigroup of a commutative semigroup is archimedean if and only if it is either power joined or a two-element semilattice.*

Theorem 3.7 *([19]) A commutative semigroup is simple if and only if it is an abelian group.*

Proof. It is obvious. □

Theorem 3.8 *([95]) A semigroup is a commutative archimedean semigroup containing at least one idempotent element if and only if it is an ideal extension of a commutative group by a commutative nil semigroup.*

Proof. Let S be a commutative archimedean semigroup containing at least one idempotent element. Since S is a left and right Putcha semigroup, by Theorem 2.4, it is an ideal extension of a completely simple commutative semigroup G by the nil semigroup $N = S/G$. It is clear that G is a group and N is commutative.

Conversely, assume that a semigroup S is an ideal extension of a commutative group G (with the identity element e) by a commutative nil semigroup N (with the zero element 0). It is clear that

$$f:\ s \mapsto es\ (s \in S)$$

is a retract homomorphism of S onto G. Then, by Theorem 2.4, S is an archimedean semigroup with an idempotent. Since the commutative semigroups form a variety then, by Theorem 1.40, S is commutative. □

Theorem 3.9 *Every commutative archimedean semigroup without idempotent element has a non-trivial group homomorphic image.*

Proof. Let S be a commutative archimedean semigroup without idempotent element. Then, by Theorem 1.42,

$$S_a = \{x \in S;\ a^i = xa^j \text{ for some positive integers } i, j\}$$

is the least reflexive unitary subsemigroup of S that contains $a \in S$, and the principal right congruence $\mathcal{R}_{S_a}$ of S is a group congruence on S. Assume that $S_a = S$ (otherwise the result is immediate). If s is an arbitrary element of S then

$$a^i = sa^j$$

for some positive integers i and j. If

$$a^n = sa^m$$

also holds for some positive integers n and m then

$$a^{i+m} = sa^j a^m = a^{j+n}$$

and so

$$i + m = j + n,$$

that is,

$$j - i = m - n,$$

because S does not contain idempotent elements. Thus

$$s' = j - i$$

is defined for each $s \in S$. Define an equivalence

$$\alpha = \{(x, y) \in S \times S;\ x' = y'\}.$$

As $a' = 1$ and $(sx)' = s' + x'$ for every $s, x \in S$, we get that α is a congruence on S and S/α is isomorphic to the additive semigroup of the integers or the non-negative integers or the positive integers. These semigroups have non-trivial group homomorphic images. Thus the theorem is proved. □

Cancellation and separativity

Theorem 3.10 *([19]) A commutative semigroup can be embedded into a group if and only if it is cancellative.*

Proof. It is clear that the cancellativity is necessary for a semigroup to be embeddable into a group.

Conversely, assume that S is an arbitrary commutative cancellative semigroup. On the direct product $S \times S$, consider the following relation α. For arbitrary $a, b, c, d \in S$,

$$(a, b)\ \alpha\ (c, d) \text{ if and only if } ad = bc \text{ in } S.$$

It can be easily verified that α is a congruence on $S \times S$. Let G^{-1} denote the factor semigroup of $S \times S$ modulo α. For every $a, b \in S$, let $[a, b]$ denote the α-class of $S \times S$ containing (a, b). Then

$$G^{-1} = \{[a, b] :\ a, b \in S\}.$$

It is easy to see that $[a, a]$ is the identity element of G^{-1}, and $[b, a]$ is the inverse of $[a, b]$ $(a, b \in S)$, that is, G^{-1} is a group. It can be easily verified that

$$\phi :\ s \mapsto [s^2, s]$$

is an embedding of S into G^{-1}. □

Definition 3.2 *The group $G^{-1} = \{[a, b] :\ a, b \in S\}$ defined in the proof of the previous theorem is called the quotient group of a commutative semigroup S.*

Theorem 3.11 *Let S be a separative commutative semigroup and x, y be arbitrary elements of S such that $x^{n+1} = x^n y$ for some positive integer n. Then $x^2 = xy$.*

Proof. We can suppose that $n \geq 2$. Then

$$(x^n)^2 = x^{n-1}x^{n+1} = x^{n-1}x^n y = x^n x^{n-1} y$$

$$= x^{n+1}x^{n-2}y = x^n y x^{n-2} y = (x^{n-1}y)^2.$$

Thus

$$x^n = x^{n-1}y,$$

by the sparativity of S. Repeating this process $n-1$ times, we get

$$x^2 = xy.$$

□

Theorem 3.12 *([75]) On a semigroup S the following are equivalent.*

(i) S is commutative and separative.

(ii) S is a semilattice of commutative cancellative semigroups.

(iii) S is embeddable into a semilattice of abelian groups.

(iv) S is a subdirect product of commutative cancellative semigroups with a zero possibly adjoined.

Proof. (i) implies (ii). Let S be a commutative separative semigroup. Then, by Theorem 3.3, S is a semilattice Y of commutative archimedean semigroups S_α, $\alpha \in Y$. Let a, x, y be arbitrary elements in S_α, $\alpha \in Y$ with $ax = ay$. As S_α is archimedean, $x^n = as$ for some $s \in S$ and a positive integer n. Then

$$x^{n+1} = xas = yas = yx^n$$

and so, by Theorem 3.11, $x^2 = xy$. Similarly, $y^2 = xy$. Thus

$$x^2 = xy = y^2.$$

As S is separative, we get $x = y$. Hence S_α is cancellative.

(ii) implies (iii). Assume that a semigroup S is a semilattice Y of commutative cancellative semigroups S_α, $\alpha \in Y$. By Theorem 3.10, every S_α is embeddable into its quotient group $(G_\alpha, *_\alpha)$. The groups $(G_\alpha, *_\alpha)$ are commutative, $S_\alpha \subseteq G_\alpha$ and

$$G_\alpha = \{a_\alpha *_\alpha b_\alpha^{-1} : a_\alpha, b_\alpha \in S_\alpha\}.$$

(We note that the restriction $*_\alpha$ to S_α is the operation of the semigroup S.) We can suppose that $G_\alpha \cap G_\beta = \emptyset$ if $\alpha \neq \beta$. Let $G = \cup_{\alpha \in Y} G_\alpha$. We define an operation $*$ on G as follows:

$$(a_\alpha *_\alpha b_\alpha^{-1}) * (a_\beta *_\beta b_\beta^{-1}) = (a_\alpha a_\beta) *_{\alpha\beta} (b_\alpha b_\beta)^{-1}$$

$(\alpha, \beta \in Y)$. It is easy to see that $(G, *)$ is a semigroup which is a semilattice Y of commutative groups G_α $(\alpha \in Y)$ and S is a subsemigroup of G.

(iii) implies (iv). Let ψ be an isomorphism of S into a semigroup G which is a semilattice Y of abelian groups G_α. Let $\alpha \geq \beta$, $\alpha, \beta \in Y$. Define $f_{\alpha,\beta} : G_\alpha \to G_\beta$ by

$$f_{\alpha,\beta} a \mapsto a e_\beta,$$

where e_β is the identity element of G_β. It is easy to see that the family $\{f_{\alpha,\beta}\}_{\alpha \geq \beta}$ is a transitive system of homomorphisms which determines the operation in G. Thus G is a strong semilattice Y of abelian groups G_α, $\alpha \in Y$. By Theorem 1.13, G is a subdirect product of commutative groups G_α with a zero possible adjoined, that is, there is an embedding θ of G into the direct product $\prod_{\alpha \in Y} T_\alpha$, where $T_\alpha = G_\alpha$ or $T_\alpha = G_\alpha^0$. Then S is a subdirect product of the projections of $S\psi\theta$ in various T_α, each of which is a commutative cancellative semigroup with a zero possibly adjoined.

(iv) implies (i). It is obvious. □

Corollary 3.1 *If S is a commutative cancellative archimedean semigroup with an idempotent element then it is a commutative group.*

Proof. Let S be a commutative cancellative archimedean semigroup with an idempotent. Then, by Theorem 3.8, it is an ideal extension of a commutative group G by a commutative nil semigroup N. Assume $G \neq S$. Let a be an arbitrary element in $S - G$. Then $a^n \in G$ for some positive integer $n \geq 2$. Hence $a^n = a^n e$, where e denotes the identity element of G. As S is cancellative, we get $a = ae \in G$ which is a contradiction. Consequently, $G = S$. □

Definition 3.3 *A commutative cancellative archimedean semigroup without idempotent is called an $\mathcal{N}$-semigroup.*

Theorem 3.13 *([75]) Let N be the additive semigroup of non-negative integers, G be an abelian group, $I : G \to N$ be a function satisfying:*

(i) $I(\alpha, \beta) + I(\alpha\beta, \gamma) = I(\alpha, \beta\gamma) + I(\beta, \gamma)$ $(\alpha, \beta, \gamma \in G)$,

(ii) $I(\alpha, \beta) = I(\beta, \alpha)$ $(\alpha, \beta \in G)$,

(iii) $I(e, e) = 1$, where e is the identity of G,

(iv) for each $\alpha \in G$, there exists a positive integer m such that $I(\alpha^m, \alpha) > 0$.

On the set $S = N \times G$ define a multiplication by

$$(m, \alpha)(n, \beta) = (m + n + I(\alpha, \beta), \alpha\beta).$$

Then S with this multiplication is an $\mathcal{N}$-semigroup, to be denoted by (G, I). Conversely, every $\mathcal{N}$-semigroup is isomorphic to some semigroup (G, I).

Subdirectly irreducible commutative semigroups

For a prime p, let $\mathbf{Z}_{p^\infty}$ denote the multiplicative group of all complex p^n-roots of unity for $n = 1, 2, \ldots$. A group is called a *quasicyclic p-group* if it is isomorphic to $\mathbf{Z}_{p^\infty}$.

Theorem 3.14 *([85]) A semigroup is a subdirectly irreducible commutative semigroup with a globally idempotent core if and only if it is isomorphic to G or G^0 or F, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime) and F is a two-element semilattice.*

Proof. Let S be a subdidirectly irreducible commutative semigroup with a globally idempotent core K. First, assume that S does not contain zero element. Then K is a commutative simple semigroup and so, by Theorem 3.7, it is a commutative group. Then S is a homogroup without zero. By Theorem 1.47, S is a commutative group. Let A denote the least nonunit (normal) subgroup of S. Since A does not contain any proper nonunit subgroup, it is a cyclic group of a prime order p. Let s be an arbitrary element of S with $s \neq e$, where e is the identity element of S. Then A is contained by the cyclic subgroup $[s]$ of S generated by s and so the order of s is mp for some positive integer m. Then $[s]$ has a subgroup B with order m. If $m \neq 1$ then $A \subseteq B$ and so the order of B is np for some positive integer $n < m$. Continuing this procedure, we can conclude that the order of s is p^k for some positive integer k. Then S is a subgroup of a quasicyclic p-group.

Next, assume that S has a zero element 0 and $ab = 0$ for some element $a, b \in S$. It is easy to see that

$$A = \{b \in S : \; ab = 0\}$$

is a non-trivial ideal of S and so the core K of S is contained by A. Then

$$aK = \{0\}.$$

Let

$$B = \{a \in S : \; aK = \{0\}\}.$$

Then B is a non-trivial ideal of S and so

$$K \subseteq B$$

which implies that K is nilpotent. But this is a contradiction. Hence

$$S = G \cup \{0\}$$

where G is a subdirectly irreducible commutative semigroup. If G has a zero element 0^* then $\{0, 0^*\}$ is an ideal of S and so G has only one element. In this case S is a two-element semilattice. Assume that G does not contain a zero element. Then G has a globally idempotent core. Thus G is a subgoup of a quasicyclic p-group (p is a prime) and $S = G^0$. As the converse statement is trivial, the theorem is proved. $\square$

Theorem 3.15 *([85]) A commutative semigroup with zero and a non-trivial annihilator is subdirectly irreducible if and only if it has a non-zero disjunctive element.*

Proof. By Theorem 1.49, it is obvious. □

Theorem 3.16 *([85]) A semigroup is a commutative subdirect irreducible semigroup with a nilpotent core and a trivial annihilator if and only if it contains an identity, a non-zero divisor of zero and a non-zero disjunctive element, and the set of all non-divisors of zero forms a subdirectly irreducible commutative group.*

Proof. Let S be a commutative subdirectly irreducible semigroup with a nilpotent core K and a trivial annihilator. By Lemma 1.4, S has a non-zero disjunctive element. Let

$$F = \{f \in S;\ Kf = \{0\}\}.$$

Since $K^2 = \{0\}$,

$$K \subseteq F.$$

Clearly, F is an ideal. As the annihilator of S is trivial, K is not the annihilator of S and so $F \neq S$. Let $G = S - F$, $K_0 = K - \{0\}$, $F_0 = F - K$. So $\{\{0\}, K_0, F_0, G\}$ is a partition of S (F_0 may be empty). If $g \in G$ then

$$gK \neq \{0\},$$

that is,

$$gK = K,$$

because gK is an ideal of S. $I = \{s \in S;\ gs = 0\}$ is an ideal of S. K is not included in I, therefore

$$I = \{0\},$$

that is, g is not a divisor of zero. Since elements of F are divisors of zero, G is the set of all non-divisors of zero. S has a non-trivial annihilator and so, for every $k \in K_0$,

$$K = Sk = Gk \cup \{0\},$$

that is, there is an element $e \in S$ such that

$$ek = k.$$

Let

$$J = \{s \in S;\ es = s\}.$$

It is easy to see that J is a non-trivial ideal of S and so

$$K \subseteq J.$$

Hence

$$ek = k$$

for every $k \in K$. Then, for every positive integer n and every $k \in K$, we get

$$e^n k = k.$$

Let

$$\alpha = \{(a,b) \in S \times S;\ e^n a = e^m b \text{ for some positive integers n, m}\}.$$

Clearly, α is a congruence on S and $\alpha|K = id_K$. Then

$$\alpha \cap \rho_K = id_S,$$

where ρ_K denotes the Rees congruence on S modulo K. As S is subdirectly irreducible and $\rho_K \neq id_S$, we get

$$\alpha = id_S.$$

As $(es, s) \in \alpha$ for every $s \in S$, we get

$$es = s$$

for every $s \in S$, that is, e is the identity element of S. Let $g \in G$ and $k \in K_0$ be arbitrary. Then

$$Ggk = K_0$$

and so there exists $g_1 \in G$ such that

$$g_1 g k = k.$$

It can be proved, as above, that $g_1 g$ is the identity element of S and so g_1 is an inverse of g. Then G is a subgroup of S. Let $g_1 k = g_2 k$ for some $g_1, g_2 \in G$ and $k \in K_0$. Then

$$k = g_1^{-1} g_2 k$$

and so $g_1^{-1} g_2$ is the identity element of S, that is, $g_1 = g_2$. $Gk = K_0$, therefore the set G and K_0 have the same cardinality. Let η be a congruence on the group generated by a subgroup H of G. Let

$$\eta^* = \{(a,b) \in S \times S;\ a \in bH\}.$$

It is easy to verify that η^* is a congruence on S and its restriction to G equals η. Let η_i, $i \in I$ be a family of congruences on G with $\cap_{i \in I} \eta_i = id_G$ and η_i^* be the family of corresponding congruences on S. If $k_1, k_2 \in K_0$ then

$$k_1 = g k_2$$

for some $g \in G$. Therefore $(k_1, k_2) \in \eta_i^*$ means that $k_1 \in k_2 H_i$ or that there exists $g_i \in H_i$ such that $k_1 = k_2 g_i$, or that $k_2 g = k_2 g_i$, or $g = g_i$, or $g \in H_i$. So $(k_1, k_2) \in \cap_{i \in I} \eta_i^*$ if and only if $g \in \cap H_i$ if and only if $g = e$ if and only if $k_1 = k_2$. Therefore

$$\rho_K \cap (\cap_{i \in I} \eta_i^*) = id_S$$

and, since S is subdirectly irreducible, there exists $i \in I$ such that

$$\eta_i^* = id_S$$

and

$$\eta_i = id_G.$$

Hence G is subdirectly irreducible.

Conversely, let S be a commutative semigroup satisfying the conditions of the theorem. S has a trivial annihilator and contains a non-zero divisor of zero. We show that S is subdirectly irreducible. Let G be the set of all non-divisors of zero, k_0 be a non-zero disjunctive element and K be the core of S (By Lemma 1.5, K exists). If f is a non-trivial divisor of zero then

$$ff_1 = 0$$

for some $f_1 \neq 0$. For every $k \in K$, there exist $x, y \in S$ such that

$$xf_1y = k$$

and so

$$fk = xff_1y = 0$$

which means that k is an annihilating element for the set $F = S - G$, that is,

$$FK = \{0\}.$$

If G has only one element then $S = F^1$ and F is a semigroup with non-trivial annihilator and F contains a non-zero disjunctive element. Then, by Theorem 1.49, F and so, by Theorem 1.45, $S = F^1$ is subdirectly irreducible. Let G have more than one element. Since G is subdirectly irreducible, it has a least nonunit subgroup H. Let η be a non-identity congruence on S. The considered disjunctive element k_0 does not form an η-class, so there exists $s \in S$ such that $s \neq k_0$ and

$$(s, k_0) \in \eta.$$

If $s \notin K$ then, for some $x, y \in S$ (x and y are not both void),

$$xsy = k_0,$$

whence

$$(xk_0y, k_0) \in \eta.$$

If $xy \in G$ then

$$s = (xy)^{-1}k_2 \in K.$$

So

$$xy \notin G$$

and

$$xk_2y = 0.$$

Therefore,

$$(k_2, 0) \in \eta$$

and

$$(hk_0, 0) \in \eta$$

for every $h \in H$, that is,

$$\{(u,v) \in S \times S;\ u = v \text{ or } u,v \in Hk_0\} \subseteq \eta.$$

If $s \in K_0$ then

$$s = g_0 k_0$$

for some $g_0 \in G$, because $K = Sk_0 = Gk_0 \cup \{0\}$. In this case the set of all $g \in G$ with $(gk_0, k_0) \in \eta$ forms a nonunit subgroup of G (this subgroup contains $g_0 \neq e$). So H is included in this subgroup and

$$\{(u,v) \in S \times S;\ u = v \text{ or } u,v \in Hk_0\} \subseteq \eta.$$

Hence

$$\{(u,v) \in S \times S;\ u = v \text{ or } u,v \in Hk_0\} \subseteq \eta$$

is always valid. Let η_0 be the intersection of all non-identity congruences of S. By

$$\{(u,v) \in S \times S;\ u = v \text{ or } u,v \in Hk_0\} \subseteq \eta,$$

Hk_0 is not divisible by η_0, so $\eta_0 \neq id_S$ and, by Corollary 1.1, S is subdirectly irreducible. □

Corollary 3.2 *A semigroup S is commutative and cancellative if and only if it is a subdirect product of subdirectly irreducible abelian groups.*

Proof. See Corollary IV.7.4 of [75]. □

Corollary 3.3 *A semigroup S is commutative and separative if and only if it is a subdirect product of subdirectly irreducible abelian groups with a zero possibly adjoined.*

Proof. See Corollary IV.7.5 of [75]. □

Commutative Δ-semigroups

Theorem 3.17 *([87],[100]) The following statement on a group G are equivalent.*

(i) G is an abelian group which is a Δ-semigroup.

(ii) G is a group in which all subgroups form a chain.

(iii) For every two elements a and b of G, either $a = b^n$ or $b = a^n$ for some positive integer N.

(iv) G is a subgroup of a quasicyclic p-group for some prime p.

(v) G is a group in which all subsemigroups form a chain.

Proof. (i) implies (ii). It is obvious.

(ii) implies (iii). Let G be a group satisfying (ii). Then G is periodic and all cyclic subgroups form a chain, therefore we have (iii).

(iii) implies (iv). Immediately the periodicity of G follows from (iii). Also it follows that all cyclic subgroups of G form a chain with respect to inclusion. Accordingly the order of every element, hence of every cyclic subgroup is a power of a same prime number p. Let $C(x)$ denote the cyclic subgroup generated by x. Let F_n be the set of all elements of order p^n in G. We have a finite or infinite sequence $\{F_n\}$ and

$$G = \bigcup_{n=1}^{\infty} F_n.$$

Let $x, y \in F_n$. By (iii), either

$$x = y^n$$

or

$$y = x^m$$

for some positive integer m. Assuming $x = y^n$,

$$C(x) \subseteq C(y).$$

Since $|C(x)| = |C(y)| = p^n$, we have

$$C(x) = C(y).$$

The same for $y = x^n$. Since the converse is obvious, $C(x) = C(y)$ if and only if x and y are in the same F_n. Choose one element a_0 from each F_n. Then we have a finite or infinite sequence

$$C(a_1) \subset C(a_2) \subset \ldots \subset C(a_n) \subset \ldots,$$

where $|C(a_n)| = p^n$ and $F_n \subset C(a_n)$. By $G = \cup_{n=1}^{\infty} F_n$, we have

$$G = \bigcup_{n=1}^{\infty} C(a_n).$$

If the sequence $\{C(a_n)\}$ is finite,

$$G = C(a_n)$$

for some n, that is, G is a cyclic subgroup of order p^n. Thus we have (iv).

(iv) implies (v). Let G be a quasicyclic p-group for some prime p, that is, $G = \cup_{n=1}^{\infty} C(a_n)$, where $C(a_n)$ is a cyclic group of order p^n. Let H be a subsemigroup of G, and let

$$H'_n = F_n \cap H,$$

where F_n has been defined above. Clearly,

$$H = \bigcup_{n=1}^{\infty} H'_n.$$

Let $x \in H'_n$. By the definition of F_n, we have

$$C(a_n) = C(x) \subseteq H.$$

If the set $\{n_i : H'_{n_i} \neq \emptyset\}$ is infinite then

$$H = G;$$

if the set is finite, and if n_m is its maximum,

$$H = C(a_{n_m}).$$

Consequently G has no proper subsemigroup, hence no proper subgroup except $C(a_n)$. We have (v).

(v) implies (i). Assume that (v) is satified by a group G. Let $S(x)$ denote the cyclic subsemigroup of G generated by the element x of G. Then, for arbitrary elements $a, b \in G$, either $S(a) \subseteq S(b)$ or $S(b) \subseteq S(a)$. Then

$$ab = ba.$$

Hence G is an abelian group. As the subgroups of abelian groups are normal subgroups, (v) implies that all normal subgroups form a chain with respect to inclusion. Hence G is a Δ-semigroup. □

Theorem 3.18 *([87],[100]) A group G^0 with zero is a Δ-semigroup if and only if G is a Δ-semigroup.*

Proof. Let G be a group and G^0 be the group G with zero 0 adjoined. Let ρ be any congruence on G. A congruence ρ^0 on G^0 is associated with ρ as follows:

$$\rho^0 = \{(a,b) \in G \times G : \ a = b \text{ or } (a,b) \in \rho\}.$$

The mapping $\rho \to \rho^0$ is one-to-one; and $\rho \subset \sigma$ if and only if $\rho^0 \subset \sigma^0$. Let ω_G and ω_{G^0} denote the universal relations on G and G^0, respectively. We will prove that every congruence on G^0 is either ω_{G^0} or ρ^0, a congruence associated with ρ on G. Let σ be a congruence on G^0 such that $(a, 0) \in \sigma$ for some $a \in G$. Let x be an arbitrary element of G^0. Then

$$x = aa^{-1}x$$

and so

$$(x, 0) \in \sigma.$$

Therefore

$$\sigma = \omega_{G^0}.$$

□

Theorem 3.19 *([87],[100]) An abelian group G^0 with a zero adjoined is a Δ-semigroup if and only if G is a subgroup of a quasicyclic p-group for some prime p.*

Proof. By Theorem 3.17 and Theorem 3.18, it is obvious. □

Theorem 3.20 *A semilattice is a Δ-semigroup if and only if it has at most two elements.*

Proof. Let L be a semilattice of order ≥ 2. As usual we define the odering $x \leq y$ $(x, y \in L)$ by $x = xy$. Let a and b be distinct elements of L and let

$$I_a = \{x: \ x \leq a\}, \ I_b = \{x: \ x \leq b\}.$$

Then I_a and I_b are ideals of L. Let ρ_a and ρ_b denote the Rees congruences of L modulo the ideals I_a and I_b, respectively. Clearly,

$$I_a \neq I_b.$$

Suppose L is a Δ-semigroup. Then, by Theorem 1.53, either $I_a \subset I_b$ or $I_b \subset I_a$. For the first case, $a \in I_b$, namely $a < b$; for the second $b \in I_a$, namely $b < a$. Therefore L is a chain. Suppose that L contains at least three elements a, b, c with $a < b < c$. Let

$$\rho_{a,b} = \{(x, y) \in L \times L: \ a \leq x, y \leq b \text{ or } x = y\}.$$

It is clear that $\rho_{a,b}$ is an equivalence. We show that $\rho_{a,b}$ is a congruence. Let x, y, z be arbitrary elements of L. Assume $(x, y) \in \rho_{a,b}$. We can suppose that $x \neq y$. Then $a \leq x, y \leq b$, that is, $a = axay$, $x = xb$ and $y = yb$. If $z \leq a$ then $zx = z = zy$ and so $(zx, zy) \in \rho_{a,b}$. If $b \leq z$ then $x = zx$ and $y = zy$ and so $(zx, zy) \in \rho_{a,b}$. If $a < z < b$ then $a = ax = azx$, $a = ay = azy$, $zx = zxb$ and $zy = zyb$. Therefore, $a \leq zx, zy \leq b$, that is, $(zx, zy) \in \rho_{a,b}$. Consequently, $\rho_{a,b}$ is a congruence on L. Similarly, $\rho_{b,c}$ is a congruence on L. As L is a Δ-semigroup, we have either $\rho_{a,b} \subset \rho_{b,c}$ or $\rho_{a,b} = \rho_{b,c}$ or $\rho_{b,c} \subset \rho_{a,b}$. In the first case $b \leq a \leq c$; in the second case $a = b = c$; in the third case $a \leq c \leq b$. These contradist the assumption $a < b < c$. Thus L has at most two elements. The converse is obvious. □

Definition 3.4 *A semigroup S is said to be naturally totally ordered if*

(i) S is totally ordered $(\leq)$;

(ii) For every $a, b, c \in S$, $a \leq b$ implies $ab \leq ac$ and $ca \leq cb$;

(iii) For every $a, b \in S$, $a \leq b$ implies $b|a$.

Theorem 3.21 *([100]) Let R be the semigroup of all positive real numbers with addition. A commutative nil Δ-semigroup can be embedded into the Rees factor semigroup R/I modulo I, where I is defined by either $\{x \in R: \ x > 1\}$ or $\{x \in R: \ x \leq 1\}$, $\leq$ is the usual order.*

Proof. Let S be a commutative nil Δ-semigroup. Then, by Theorem 1.56, S satisfies the divisibility chain condition. A commutative nil semigroup satisfying the divisibility chain condition is naturally totally ordered. According to [18], S can be embedded into the Rees factor semigroup R/I modulo I, where R denotes the semigroup of all positive real numbers and I is defined by either $\{x \in R: \ x > 1\}$ or $\{x \in R: \ x \geq 1\}$, $\geq$ is the usual order. □

Theorem 3.22 *([87],[100]) A semigroup S is a commutative Δ-semigroup if and only if it is isomorphic to either G or G^0, where G is a nontrivial subgroup of a quasicyclic p-group (p is a prime) or N or N^1, where N is a commutative nil semigroup satisfying the divisibility chain conditions.*

Proof. Let S be a commutative Δ-semigroup. Then, by Remark 1.2 and Theorem 3.3, S is either archimedean or a disjoint union $S = S_0 \cup S_1$ of two archimedean semigroup S_0 and S_1, where S_0 is an ideal of S. First, assume that S is an archimedean semigroup. Consider the case when S has no idempotent element. Then, by Theorem 3.9, S has a non-trivial group homomorphic image. From Theorem 1.52 it follows that S does not contain proper ideals, that is, S is simple. As a commutative simple semigroup is a commutative group, by Theorem 3.17, S is a non-trivial subgroup of a quasicyclic p-group for some prime p. Next, consider the case when S has an idempotent element f. It is easy to see that $K = Sf$ is the kernel of S. If $|K| = 1$, that is, f is the zero of S then S is a (commutative) nil semigroup and so it satisfies the divisibility chain condition. Assume $|K| > 1$. Then K is simple and so it is a subgroup of S. As K is also an ideal of S, by Theorem 1.52, $K = S$ which implies that S is a quasicyclic p-group for a prime p.

Next, consider the case when S is a disjoin union $S = S_0 \cup S_1$ of two archimedean subsemigroups S_0 and S_1, where S_0 is an ideal of S. Since S_1^0 is isomorphic to the factor semigroup S/S_0 of S modulo S_0 then, by Theorem 1.51, S_1^0 and so S_1 is a Δ-semigroup. By the previous part of the proof, S_1 is either non-trivial subgroup of a quasicyclic p-group (p is a prime) or a commutative nil semigroup which satisfies the divisibility chain condition. In the second case $|S_1| = 1$ by Theorem 1.57. Thus S_1 is a subgroup G of a quasicyclic p-group (p is a prime). If $|S_0| = 1$ then $S = G^0$. We note that S is a two-element semilattice if $|G| = 1$. Assume $|S_0| > 1$. Recall that S_0 is a commutative archimedean semigroup. If S_0 did not contain idempotents then, by Theorem 3.9, it would have a non-trivial group homomorphic image, contradicting Theorem 1.52. Assume that S_0 has an idempotent f. Then $K_0 = fS_0$ is the kernel of S_0 which is a group. By Theorem 1.52, $|K_0| = 1$ and so S_0 is a nil semigroup. By Theorem 1.59, S_1 contains only one element e. As S is a Δ-semigroup, the ideals eS^1 and S_0 of S are comparable only in that case when $S = eS^1$. Let a be an arbitary element of S. Then $a = ex$ for some $x \in S^1$ and so $ea = eex = ex = a$. Hence e is the identity element of S, that is, $S = S_0^1$. By Corollary 1.2, S_0 is a Δ-semigroup. Thus the first part of the theorem is proved. As the semigroups listed in the theorem are commutative Δ-semigroups, the theorem is proved. □

Chapter 4

Weakly commutative semigroups

A semigroup S is called left (right) weakly commutative if, for every $a, b \in S$, there exist $x \in S$ and a positive integer n such that $(ab)^n = bx$ ($(ab)^n = xa$). A semigroup which is both left and right weakly commutative is called a weakly commutative semigroup. In this chapter we deal with left weakly commutative, right weakly commutative and weakly commutative semigroups. It is shown that a semigroup is a semilattice of left archimedean (right archimedean, t-archimedean) semigroups if and only if it is right weakly commutative (left weakly commutative, weakly commutative). It is proved that a weakly commutative 0-simple semigroup is a group with a zero adjoined. Moreover, a semigroup is weakly commutative archimedean and contains an idempotent element if and only if it is an ideal extension of a group by a nil semigroup. We get, as a consequence, that a semigroup is weakly commutative and regular if and only if it is a Clifford semigroup. We show that a right (left) weakly commutative semigroup is embeddable into a group if and only if it is cancellative. At the end of the chapter we deal with the least weakly separative congruence on weakly commutative semigroups. It is proved that if S is a left weakly commutative semigroup then σ defined by $a\ \sigma\ b$ if and only if $ab^n = b^{n+1}$ and $ba^n = a^{n+1}$ for a positive integer n is a weakly separative congruence on S. Similarly, if S is a right weakly commutative semigroup then τ defined by $a\ \tau\ b$ if and only if $b^n a = b^{n+1}$ and $a^n b = a^{n+1}$ for some positive integer n is a weakly separative congruence on S. Moreover, $\pi = \sigma \cap \tau$ is the least weakly separative congruence on a weakly commutative semigroup.

Definition 4.1 *A semigroup S is called a left (right) weakly commutative semigroup if, for every $a, b \in S$, there exist $x \in S$ and a positive integer n such that $(ab)^n = bx$ ($(ab)^n = xa$). We say that S is a weakly commutative semigroup if it is both left and right weakly commutative, that is, for every $a, b \in S$ there are $x, y \in S$ and a positive integer n such that $(ab)^n = xa = by$.*

Semilattice decomposition of weakly commutative semigroups

Theorem 4.1 *Every left (right) weakly commutative semigroup is a left (right) Putcha semigroup.*

Proof. Let S be a left weakly commutative semigroup and $a, b \in S$ be arbitrary elements with $a \in bS^1$, that is, $a = bx$ for some $x \in S^1$. As S is left weakly commutative, there exist $u \in S$ and a positive integer n such that

$$a^{n+1} = (bx)^{n+1} = b(xb)^n x = bbux \in b^2 S^1.$$

Thus S is a left Putcha semigroup. We can prove, in a similar way, that a right weakly commutative semigroup is a right Putcha semigroup. □

Theorem 4.2 *A semigroup is a semilattice of left (right) archimedean semigroups if and only if it is right (left) weakly commutative.*

Proof. By Theorem 1.10, a semigroup S is decomposable into a semilattice of left (right) archimedean semigroups if and only if, for every $a, b \in S$, the assumption $b \in aS$ ($b \in Sa$) implies $b^i \in Sa$ ($b^i \in aS$) for some positive integer i. Let S be a semilattice of left archimedean semigroups. As $ab \in aS$, we get

$$(ab)^i \in Sa,$$

that is,

$$(ab)^i = xa$$

for some $x \in S$ and a positive integer i. Hence S is right weakly commutative.

Conversely, let S be a right weakly commutative semigroup and $a, b \in S$ be arbitrary elements with $b \in aS$. Then

$$b = ax$$

and, for some $u \in S$ and a positive integer i, we have

$$b^i = (ax)^i = ua \in Sa.$$

Hence S is a semilattice of left archimedean semigroups.

As the dual assertion can be proved in a similar way, the theorem is proved. □

Corollary 4.1 *A semigroup is a semilattice of t-archimedean semigroups if and only if it is weakly commutative.*

Proof. Let S be a weakly commutative semigroup. Then S is right weakly commutative and so, by Theorem 4.2, it is a semilattice Y of left archimedean semigroups S_α ($\alpha \in Y$). We show that every S_α is also right archimedean. Let $\alpha \in Y$ and $a, b \in S_\alpha$ be arbitrary elements. Since S_α is left archimedean then

$a^m = xb$ and $b^n = ya$ for some $x, y \in S_\alpha$ and some positive integers m, n. Since S is left weakly commutative, there are positive integers t and r such that

$$(xb)^t = bv$$

and

$$(ya)^r = aw$$

for some $v \in S_\beta$, $w \in S_\gamma$ $(\beta, \gamma \in Y)$. It is clear that $\alpha = \alpha\beta = \alpha\gamma$ and so $vbv, waw \in S_\alpha$. As

$$a^{2mt} = (bv)^2 = b(vbv) \in bS_\alpha^1$$

and

$$b^{2nr} = (aw)^2 = a(waw) \in aS_\alpha^1,$$

we get that S_α is right archimedean. Thus every S_α $(\alpha \in Y)$ is t-archimedean.
The convese statement is obvious by Theorem 4.2. □

Theorem 4.3 *([78]) Every weakly commutative semigroup is a semilattice of weakly commutative archimedean semigroups.*

Proof. Let S be a weakly commutative semigroup. Then, by Theorem 4.2, S is a semilattice Y of left archimedean and so archimedean semigroups S_α $(\alpha \in Y)$. Let $a, b \in S_\alpha$ be arbitrary elements. There is a positive integer i such that

$$(ab)^i = xa = by$$

for some $x \in S_\beta$ and $y \in S_\gamma$ $(\beta, \gamma \in Y)$. It is clear that $\alpha\beta = \alpha\gamma = \alpha$ and so $abx, yab \in S_\alpha$. As

$$(ab)^{i+1} = (abx)a = b(yab),$$

we get that S_α is weakly commutative. □

Theorem 4.4 *A weakly commutative 0-simple semigroup is a group with a zero adjoined.*

Proof. Let S be a weakly commutative 0-simple semigroup. By Theorem 4.3, S is a semilattice of weakly commutative archimedean semigroups. By Theorem 2.1, it is easy to see that S has two archimedean components S_1 and S_0 $(S_0S_1 \subseteq S_0)$, and $S_0 = \{0\}$. Hence $S = S_1^0$, and S_1 is a simple semigroup. It is clear that S_1 is weakly commutative. Then, by Theorem 4.1, S_1 is a left and right Putcha semigroup and so, by Theorem 2.3, it is completely simple. By Theorem 1.25, S_1 is a Rees matrix semigroup $\mathcal{M}(I, G, J; P)$ over a group G with a sandwich matrix P. We can suppose that P is normalized, that is, there are $i_0 \in I$ and $j_0 \in J$ such that

$$p_{j_0,i} = p_{j,i_0} = e,$$

the identity of G, for every $i \in I$ and $j \in J$. Let $(i, g, j_0), (i_0, h, j) \in S_1$ be arbitrary elements $(i \in I,\ g, h \in G,\ j \in J)$. Then, for every positive integer n,

$$(i, (gh)^n, j_0) = (i, gh, j_0)^n = ((i, g, j)(i_0, h, j_0))^n.$$

As S is weakly commutative, there is a positive integer n such that

$$(i,(gh)^n,j_0)=(i_0,h,j_0)(m,x,k)=(i_0,hx,k)$$

and

$$(i,(gh)^n,j_0)=(t,y,r)(i,g,j)=(t,yp_{r,i}g,j)$$

for some $(m,x,k),(t,y,r)\in S_1$. Then we have

$$i=i_0,\ j=j_0$$

for every $i\in I$ and $j\in J$. Hence S_1 is isomorphic to G. Thus S is a group with a zero adjoined.

As every group is weakly commutative, the converse statement is obvious. □

Theorem 4.5 *A semigroup is weakly commutative archimedean and contains an idempotent if and only if it is an ideal extension of a group by a nil semigroup.*

Proof. Let S be a weakly commutative archimedean semigroup containing an idempotent f. Then, by Theorem 2.2, S is an ideal extension of a simple semigroup K by a nil semigroup N. Let $a,b\in K$ be arbitrary elements. As S is weakly commutative, there are $x,y\in S$ such that

$$(ab)^n=xa=by$$

for some positive integer n. As

$$(ab)^{n+1}=(abx)a=b(yab)$$

and

$$abx,yab\in K,$$

we get that K is weakly commutative. Then, by Theorem 4.4, K is a group.

Conversely, assume that a semigroup S is an ideal extension of a group G by a nil semigroup N. Let $a,b\in S$ be arbitrary elements. As $N=S/G$ is a nil semigroup, there is a positive integer n such that

$$(ab)^n\in G.$$

Let e denote the identity element of G. Then

$$(ab)^n=e(ab)^n=(e(ab))(ab)^{n-1}=(e(ab))e(ab)^{n-1}=\dots$$
$$=(e(ab))^n=((eab)e)^n=((ea)(be))^n.$$

As G is a group, there are elements $u,v\in G$ such that

$$((ea)(be))^n=beu=vea.$$

As

$$(ab)^n=((ea)(be))^n=b(eu)=(ve)a,$$

S is weakly commutative. By Theorem 2.2, S is archimedean and contains an idempotent. Thus the theorem is proved. □

Corollary 4.2 *A semigroup is weakly commutative and regular if and only if it is a Clifford semigroup.*

Proof. Let S be a weakly commutative regular semigroup. By Theorem 4.3, S is a semilattice Y of weakly commutative archimedean semigroups S_α. As S is regular, every S_α is regular and so contains at least one idempotent element. By Theorem 4.5, we can conclude that every S_α is a group. Then, by Theorem 1.21, S is a Clifford semigroup.

Conversely, assume that S is a Clifford semigroup. Then it is regular. By Theorem 1.21, S is a semilattice Y of groups G_α. If $a \in G_\alpha$, $b \in G_\beta$ are arbitrary element of S then $ab, ba \in G_{\alpha\beta}$. Then $ab = bax$ and $ab = yba$ for some $x, y \in G_{\alpha\beta}$. Hence S is weakly commutative. Thus the corollary is proved. □

Theorem 4.6 *A right (left) weakly commutative semigroup is embeddable into a group if and only if it is cancellative.*

Proof. The cancellation is necessary for any semigroup to be embeddable into a group. Conversely, let S be a right weakly commutative cancellative semigroup. Then, for every $a, b \in S$, there is a positive integer n such that

$$(ab)^n \in Sa \cap Sb$$

. Then, by Theorem 1.23 of [19], S can be embedded into a group. The proof is similar if S is left weakly commutative. □

The least weakly separative congruence on a weakly commutative semigroup

Lemma 4.1 *On an arbitrary semigroup S,*

$$\sigma = \{(a,b) \in S \times S : \ ab^n = b^{n+1}, \ ba^n = a^{n+1} \text{ for a positive integer } n\}$$

and

$$\tau = \{(a,b) \in S \times S : \ b^n a = b^{n+1}, \ a^n b = a^{n+1} \text{ for a positive integer } n\}$$

are equivalences on S.

Proof. Let S be an arbitrary semigroup. It is clear that σ is reflexive and symmetric. To show that σ is transitive, assume $(a,b) \in \sigma$, $(b,c) \in \sigma$ for some $a, b \in S$. Then

$$ab^n = b^{n+1}, \ ba^n = a^{n+1}$$

and

$$bc^n = c^{n+1}, \ cb^n = b^{n+1}$$

for a positive integer n. Then

$$a^{2n} = a^{n-1}a^{n+1} = a^{n-1}a^n b = a^{2n-1}b = \ldots = a^n b^n$$

and, similarly,

$$c^{2n} = b^n c^n.$$

Thus

$$ac^{2n} = ab^n c^n = b^{n+1} c^n = bb^n c^n = bc^{2n} = bc^n c^n = c^{n+1} c^n = c^{2n+1}$$

and, similarly,

$$ca^{2n} = a^{2n+1}.$$

Hence σ is transitive.

We can prove, in a similar way, that τ is an equivalence on S. □

Lemma 4.2 *([47]) If ρ is a congruence on a semigroup S and $ab^{n+1}\ \rho\ b^{n+2}$, $(ab^n)^r\ \rho\ (b^{n+1})^r$ for some positive integers n and r then $(ab^n)^m\ \rho\ (b^{n+1})^m$ for all positive integers $m \geq r$. Similarly, $b^{n+1}a\ \rho\ b^{n+2}$ and $(b^n a)^r\ \rho\ (b^{n+1})^r$, for some positive integers n and r, implies $(b^n a)^m\ \rho\ (b^{n+1})^m$ for all positive integers $m \geq r$.*

Proof. We prove only the first part of the lemma, because the second part can be proved in a similar way. Assume $ab^{n+1}\ \rho\ b^{n+2}$ and $(ab^n)^r\ \rho\ (b^{n+1})^r$ for some $a, b \in S$ and positive integers n, r. Let m be an arbitrary positive integer with $m \geq r$. We can suppose that $m > r$. Then

$$(ab^n)^m = (ab^n)^{m-r}(ab^n)^r\ \rho\ (ab^n)^{m-r}(b^{n+1})^r$$

$$= (ab^n)^{m-r-1} ab^n b^{n+1} (b^{n+1})^{r-1}\ \rho\ (ab^n)^{m-r-1}(n^{n+1})^{r+1}\ \rho \ldots\ \rho(b^{n+1})^m.$$

□

Lemma 4.3 *([78]) If S is a left (right) weakly commutative semigroup then, for arbitrary $a, b \in S$ and arbitrary positive integer n, there is a positive integer m and an element $x \in S$ such that $(ab)^m = b^n x$ ($(ab)^m = ya^n$).*

Proof. Let S be a left weakly commutative semigroup. Then, by Theorem 4.1, S is a left Putcha semigroup. Let $a, b \in S$ be arbitrary elements and n be a positive integer. Then

$$(ab)^k = bu$$

for some $u \in S$ and a positive integer k. By Lemma 2.1, there is a positive integer t such that

$$(bu)^t \in b^n S.$$

Let $m = kt$. Then

$$(ab)^m = b^n x$$

for some $x \in S$.

We can prove the result for right weakly commutative case in a similar way. □

Remark 4.1 If $ab^n = b^{n+1}$ ($b^n a = b^{n+1}$) holds for elements a and b of a semigroup and a positive integer n then $ab^k = b^{k+1}$ ($b^k a = b^{k+1}$) holds for all positive integers $k \geq n$. This fact will be used without comment.

Theorem 4.7 *([78]) If S is a left (right) weakly commutative semigroup then*

$$\sigma = \{(a,b) \in S \times S : \ ab^n = b^{n+1}, \ ba^n = a^{n+1} \text{ for a positive integer } n\}$$

$$(\tau = \{(a,b) \in S \times S : \ b^n a = b^{n+1}, \ a^n b = a^{n+1} \text{ for a positive integer } n\})$$

is a weakly separative congruence on S.

Proof. Let S be a left weakly commutative semigroup. By Lemma 4.1, σ is an equivalence on S. We shall show that σ is a congruence on S. Let $a, b \in S$ be arbitrary elements with $a \ \sigma \ b$. Then

$$ab^n = b^{n+1}$$

and

$$ba^n = a^{n+1}$$

for a positive integer n. Let s be an arbitrary element of S. It follows from Lemma 4.3 that

$$(sb)^m = b^n x$$

for some $x \in S$ and positive integers m. Hence

$$(as)(bs)^m = a(sb)^m s = a(b^n x)s = b(b^n x)s = b(sb)^m s = (bs)^{m+1}$$

and

$$(sa)(sb)^m = (sa)b^n x = s(ab^n)x = sb^{n+1}x = sb(b^n x) = (sb)^{n+1}.$$

Similarly, we obtain

$$(bs)(as)^k = (as)^{k+1}, \ (sb)(sa)^k = (sa)^{k+1}$$

for a positive integer k. Hence

$$as \ \sigma \ bs$$

and

$$sa \ \sigma \ sb.$$

Next we prove that σ is weakly separative. Let $a^2 \ \sigma \ ab \ \sigma \ b^2$. It follows from $ab \ \sigma \ b^2$ that

$$(ab)(b^2)^m = (b^2)^{m+1}$$

for a positive integer m, and so

$$ab^{2m+1} = b^{2m+2}.$$

Since σ is a congruence, we have

$$ba^3 \ \sigma \ (ba)a^2 \ \sigma \ (ba)b^2 \ \sigma \ b(ab)b \ \sigma \ b(b^2)b$$

$$= b^2 b^2 \ \sigma \ a^2 a^2 \ \sigma \ a^4.$$

This implies

$$ba^3(a^4)^k = (a^4)^{k+1}$$

for a positive integer k. Thus

$$ba^{4k+3} = a^{4k+4}$$

and so

$$a \ \sigma \ b.$$

Hence σ is a weakly separative congruence. We can prove, in a similar way, that τ is a weakly separative congruence on a right weakly commutative semigroup. □

Lemma 4.4 *([47]) Let S be a weakly commutative semigroup and ρ a weakly separative congruence on S. If $ab^n \ \rho \ b^{n+1} \ \rho \ b^n a$ and $ba^n \ \rho \ a^{n+1} \ \rho \ a^n b$ for elements $a, b \in S$ and some positive integer n then $a \ \rho \ b$.*

Proof. By the iduction for n. Since ρ is a weakly separative congruence, the result is true for $n = 1$. Assume that the assertion holds for some $n \geq 1$. We prove that the assertion also holds for $n+1$. Let $a, b \in S$ be elements with $ab^{n+1} \ \rho \ b^{n+2} \ \rho \ b^{n+1}a$ and $ba^{n+1} \ \rho \ a^{n+2} \ \rho \ a^{n+1}b$. Since S is weakly commutative,

$$(ab^n)^k = by$$

for some $y \in S$ and a positive integer k. Then

$$(ab^n)^{k+1} = ab^{n+1}y \ \rho \ b^{n+2}y = b^{n+1}(ab^n)^k$$

$$= b^{n+1}ab^n(ab^n)^{k-1} \ \rho \ (b^{n+1})^2(ab^n)^{k-1} \rho \ldots \rho \ (b^{n+1})^{k+1}.$$

Similarly,

$$(b^{n+1})^{t+1} \ \rho \ (b^n a)^{t+1}$$

for some positive integer t. By Lemma 4.2, it follows that

$$(ab^n)^m \ \rho \ (b^{n+1})^m \ \rho \ (b^n a)^m.$$

for some positive integer m. Let

$$m_1 = min\{m; \ (ab^n)^m \ \rho \ (b^{n+1})^m \ \rho \ (b^n a)^m\}.$$

We prove that $m_1 = 1$. Assume, in an indirect way, that $m_1 \neq 1$ and let

$$m_2 = \begin{cases} m_1 & \text{if } m_1 \text{ is even;} \\ m_1 + 1 & \text{if } m_1 \text{ is odd.} \end{cases}$$

Then, by Lemma 4.2,

$$(ab^n)^{m_2} \ \rho \ (b^{n+1})^{m_2} \ \rho \ (b^n a)^{m_2}.$$

Let
$$m_3 = \frac{m_2}{2}.$$
Then
$$m_3 < m_1$$
and
$$((ab^n)^{m_3})^2 = (ab^n)^{2m_3} = (ab^n)^{m_2} \;\rho\; (b^{n+1})^{m_2} = ((b^{n+1})^{m_3})^2$$
$$= (b^{n+1})^{m_2} \;\rho\; (b^n a)^{m_2} = (b^n a)^{2m_3} = ((b^n a)^{m_3})^2.$$
Moreover,
$$(ab^n)^{m_3}(b^{n+1})^{m_3}$$
$$= (ab^n)^{m_3-1} ab^n b^{n+1} (b^{n+1})^{m_3-1} \;\rho\; (ab^n)^{m_3-1}(b^{n+1})^2(b^{n+1})^{m_3-1}$$
$$= (ab^n)^{m_3-1}(b^{n+1})^{m_3+1} \;\rho\; \ldots \rho (b^{n+1})^{2m_3} = ((b^{n+1})^{m_3})^2.$$
and, similarly,
$$(b^n a)^{m_3}(b^{n+1})^{m_3} \;\rho\; ((b^{n+1})^{m_3})^2.$$
Since ρ is a weakly separative congruence, it follows that
$$(ab^n)^{m_3} \;\rho\; (b^{n+1})^{m_3} \;\rho\; (b^n a)^{m_3}$$
and so
$$m_1 \le m_3.$$
But this contradicts $m_3 < m_1$. Hence
$$m_1 = 1$$
and so
$$ab^n \;\rho\; b^{n+1} \;\rho\; a^n b.$$
We can prove
$$ba^n \;\rho\; a^{n+1} \;\rho\; a^n b$$
in a similar way. Then, by the condition for n, we can conclude
$$a \;\rho\; b.$$

□

Theorem 4.8 *([47])* $\pi = \sigma \cap \tau$ *is the least weakly separative congruence on a weakly commutative semigroup.*

Proof. Let S be a weakly commutative semigroup. Then, by Theorem 4.7, π is a weakly separative congruence on S. We show that π is the least weakly separative congruence on S. Let ρ be an arbitrary weakly separative congruence on S. If $a \;\pi\; b$ for some $a, b \in S$ then
$$ab^n = b^{n+1} = b^n a$$

and

$$ba^n = a^{n+1} = a^n b$$

for a positive integer. Then

$$ab^n \ \rho \ b^{n+1} \ \rho \ b^n a$$

and

$$ba^n \ \rho \ a^{n+1} \ \rho a^n b.$$

By Lemma 4.4, it follows that

$$a \ \rho \ b.$$

Hence

$$\pi \subseteq \rho.$$

□

Chapter 5

$\mathcal{R}$-, $\mathcal{L}$-, $\mathcal{H}$-commutative semigroups

In this chapter we deal with semigroups in which the Green equivalence $\mathcal{R}$ ($\mathcal{L}$, $\mathcal{H}$) is a congruence. These semigroups are called $\mathcal{R}$-commutative ($\mathcal{L}$-commutative, $\mathcal{H}$-commutative) semigroups. It is clear that a semigroup is $\mathcal{H}$-commutative if and only if it is $\mathcal{R}$-commutative and $\mathcal{L}$-commutative. We show that every $\mathcal{R}$-commutative semigroup is a semilattice of archimedean semigroups. We note that, in general, the archimedean components are not $\mathcal{R}$-commutative. At the end of the chapter we deal with left soluble (right soluble, soluble) semigroups of length n. A monoid, with the identity e, is called soluble (right soluble, left soluble) of length n if it is $\mathcal{H}$-commutative ($\mathcal{R}$-commutative, $\mathcal{L}$-commutative) and its n^{th} derived (right derived, left derived) semigroup equals e. We show that a cancellative semigroup is soluble of length n if and only if it is both right and left soluble of length n. Moreover, a cancellative soluble semigroup of length n can be embedded in a soluble group of length n.

Definition 5.1 *A semigroup S is called an $\mathcal{R}$-commutative ($\mathcal{L}$-commutative, $\mathcal{H}$-commutative) semigroup if, for every elements $a, b \in S$, there is an element $x \in S^1$ such that $ab = bax$ ($ab = xba$, $ab = bxa$).*

We note that, in [106], an $\mathcal{R}$-commutative ($\mathcal{L}$-commutative, $\mathcal{H}$-commutative) semigroup is called a right c-semigroup (left c-semigroup, c-semigroup).

Remark 5.1 Every $\mathcal{R}$-commutative ($\mathcal{L}$-commutative) semigroup is also a left (right) weakly commutative semigroup. Moreover, every $\mathcal{H}$-commutative semigroup is weakly commutative.

Theorem 5.1 *A semigroup is $\mathcal{H}$-commutative if and only if it is both $\mathcal{R}$-commutative and $\mathcal{L}$-commutative.*

Proof. Assume that S is an $\mathcal{L}$-commutative and $\mathcal{R}$-commutative semigroup. Let $a, b \in S$ be arbitrary. To show that $ab = bxa$ for some $x \in S^1$, we can suppose that $ab \neq ba$. As S is $\mathcal{R}$-commutative,

$$ab = bay$$

for some $y \in S$. As S is also $\mathcal{L}$-commutative,

$$ay = zya$$

for some $z \in S^1$. Hence

$$ab = bzya.$$

Consequently, S is $\mathcal{H}$-commutative.

Conversely, let S be a $\mathcal{H}$-commutative semigroup and $a, b \in S$ arbitrary elements. Then

$$ab = bxa$$

for some $x \in S^1$. We can suppose that $x \in S$. Then

$$bx = xyb$$

and

$$xa = azx$$

for some $y, z \in S^1$. Hence

$$ab = xyba$$

and

$$ab = bazx.$$

Thus S is both $\mathcal{R}$-commutative and $\mathcal{L}$-commutative. □

Theorem 5.2 *([55]) A semigroup is $\mathcal{R}$-commutative ($\mathcal{L}$-commutative, $\mathcal{H}$-commutative) if and only if the Green's equivalence $\mathcal{R}$ ($\mathcal{L}$, $\mathcal{H}$) on S is a commutative congruence on S.*

Proof. We deal with only the $\mathcal{R}$-commutative case. $\mathcal{L}$-commutative case can be proved in a similar way. The $\mathcal{H}$-commutative case then follows from Theorem 5.1 and the fact $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$.

Let S be an $\mathcal{R}$-commutative semigroup and $a, b, s \in S$ be arbitrary elements with $a \neq b$ and $(a, b) \in \mathcal{R}$. Then

$$aS^1 = bS^1$$

and so

$$a = by,$$

$$b = ax$$

for some $x, y \in S$. As $as = bys = bsyt$ and $bs = axs = asxt'$ for some $t, t' \in S^1$, we get

$$asS^1 = bsS^1,$$

that is,

$$(as, bs) \in \mathcal{R}.$$

Hence $\mathcal{R}$ is right compatible. As $\mathcal{R}$ is a left congruence on an arbitrary semigroup, it is a congruence on S. As $ab = bax$ and $ba = aby$ for some $x, y \in S^1$, we have

$$(ab, ba) \in \mathcal{R}.$$

Hence $\mathcal{R}$ is a commutative congruence on S.

Conversely, assume that S is a semigroup in which the Green equivalence $\mathcal{R}$ is a congruence. Then, for arbitrary elements $a, b \in S$,

$$(ab, ba) \in \mathcal{R}$$

and so

$$ab = bax$$

for some $x \in S^1$. Hence S is $\mathcal{R}$-commutative. □

Corollary 5.1 *Every $\mathcal{R}$-commutative ($\mathcal{L}$-commutative, $\mathcal{H}$-commutative) nil semigroup is commutative.*

Proof. It is easy to see that the Green equivalence $\mathcal{R}$ ($\mathcal{L}$, $\mathcal{H}$) is the identity relation on a nil semigroup S, that is, $S/\mathcal{R} \cong S$ ($S/\mathcal{L} \cong S$, $S/\mathcal{H} \cong S$). Thus, by Theorem 5.2, S is commutative if it is also $\mathcal{R}$-commutative ($\mathcal{L}$-commutative, $\mathcal{H}$-commutative). □

Theorem 5.3 *([55]) Every $\mathcal{R}$-commutative semigroup is decomposable into a semilattice of archimedean semigroups.*

Proof. Let S be an $\mathcal{R}$-commutative semigroup. Then, by Remark 5.1, it is left weakly commutative. Then, by Theorem 3.1, S is a left Putcha semigroup. By Corollary 2.2, S is a semilattice of archimedean semigroups. □

We note that the subsemigroups (and so the archimedean components) of an $\mathcal{R}$-commutative semigroup are not necessarily $\mathcal{R}$-commutative.

Lemma 5.1 *([55]) Every right ideal of an $\mathcal{R}$-commutative semigroup is a two-sided ideal.*

Proof. Let R be a right ideal of an $\mathcal{R}$-commutative semigroup. Then, for every $r \in R$ and $s \in S$, there is an element x in S^1 such that

$$sr = rsx \in R.$$

So

$$SR \subseteq R,$$

that is, R is also a left ideal of S. □

Lemma 5.2 *([55]) If K is an ideal of an $\mathcal{R}$-commutative semigroup such that K is simple, then K is an $\mathcal{R}$-commutative semigroup.*

Proof. Let k_1, k_2 be arbitrary elements of K. It is evident that k_2k_1K is a right ideal of S. By Lemma 5.1, k_2k_1K is a two-sided ideal of S and so

$$k_2k_1K = K.$$

Then there is an element k in K such that

$$k_1k_2 = k_2k_1k,$$

that is K is $\mathcal{R}$-commutative. □

Since every $\mathcal{R}$-commutative semigroup is left weakly commutative then, by Theorem 4.7,

$$\sigma = \{(a,b) \in S \times S : \ ab^n = b^{n+1}, \ ba^n = a^{n+1} \text{ for a positive integer } n\}$$

is a congruence on an $\mathcal{R}$-commutative semigroup S.

Lemma 5.3 *([55]) If S is an $\mathcal{R}$-commutative semigroup and I is an ideal of S such that $\sigma|I = id_I$ and I is archimedean, then I is right cancellative.*

Proof. Let I be an ideal of an $\mathcal{R}$-commutative semigroup S such that I is archimedean and the restriction $\sigma|I$ of σ to I equlas id_I. Assume $ac = bc$ for some $a, b, c \in I$. As I is archimedean, there are elements $x, y, u, v \in I$ and a positive integer n such that

$$a^n = xcy$$

and

$$b^n = ucv.$$

As S is $\mathcal{R}$-commutative,

$$xc = cxz$$

and

$$uc = cuw$$

for some $z, w \in S^1$. Thus

$$a^{n+1} = aa^n = axcy = acxzy = bcxzy = bxcy = ba^n$$

and

$$b^{n+1} = bb^n = bucv = bcuwv = acuwv = aucv = ab^n.$$

So

$$(a,b) \in \sigma$$

from which we get

$$a = b.$$

Thus I is right cancellative. □

Definition 5.2 *For an $\mathcal{H}$-commutative, $\mathcal{R}$-commutative, $\mathcal{L}$-commutative semigroup S, let α_S, β_S and γ_S denote the collection of all maps $K: S \times S \mapsto S$ such that $x_1x_2 = x_2K(x_1,x_2)x_1$, $x_1x_2 = x_2x_1K(x_1,x_2)$, $x_1x_2 = K(x_1,x_2)x_2x_1$, respectively for any $x_1, x_2 \in S$. For arbitrary $x_1, \ldots, x_m \in S$, we write*

$$K(x_1,\ldots,x_m) = K(K(x_1,\ldots,x_{m/2}), K(x_{m/2},\ldots,x_m)),$$

where $m = 2^n$. The subsemigroups of S defined by

$$S^{(n)} = \langle\{K(x_1,\ldots,x_m): \ x_1,\ldots,x_m \in S, m = 2^n, K \in \alpha_S\}\rangle,$$

$$S_R^{(n)} = \langle\{K(x_1,\ldots,x_m): \ x_1,\ldots,x_m \in S, m = 2^n, K \in \beta_S\}\rangle,$$

$$S_L^{(n)} = \langle\{K(x_1,\ldots,x_m): \ x_1,\ldots,x_m \in S, m = 2^n, K \in \gamma_S\}\rangle$$

are called the n^{th} derived semigroup, n^{th} right derived semigroup, n^{th} left derived semigroup, respectively.

Notice that if S is also cancellative then there is only one K in α_S, β_S and γ_S, respectively. Further, for a group S, $S^{(n)} = S_R^{(n)} = S_L^{(n)}$, and the above definition is the usual definition for a soluble group of length n.

Lemma 5.4 *([106]) If S is an $\mathcal{R}$-commutative ($\mathcal{L}$-commutative) semigroup then*

(i) S is left (right) reversible,

(ii) S is cancellative only if its firs right (left) derived semigroup is a group.

Proof. Let $x_1, x_2 \in S$ be arbitrary elements. Then, for some $K(x_1,x_2) \in S$, we have $x_1S \supseteq x_1x_2S = x_2x_1K(x_1,x_2)S \subseteq x_2S$. Hence (i) is satisfied.

The proof of (ii): For any $x, y \in S$, $xx = xxK(x,x)$ and by the cancellation law $x = xK(x,x) = K(x,x)x$. But then $xy = xK(x,x)y$ and $yx = yK(x,x)x$, so $yK(x,x) = K(x,x)y = y$. Thus S contains an identity element. Since S is cancellative, β_S contains only one element and so

$$x_1x_2 = x_2x_1K(x_1,x_2) = x_1x_2K(x_2,x_1)K(x_1,x_2)$$

and

$$x_2x_1 = x_2x_1K(x_1,x_2)K(x_2,x_1).$$

Therefore,

$$(K(x_1,x_2))^{-1} = K(x_2,x_1).$$

□

Lemma 5.5 *([106]) If S is a cancellative $\mathcal{H}$-commutative semigroup then $S^{(1)} = \langle S_R^{(1)}, S_L^{(1)}\rangle$.*

Proof. By Theorem 5.1, it is obvious. □

Definition 5.3 *A monoid S (with identity e) is called soluble (right soluble, left soluble) of length n if it is an $\mathcal{H}$-commutative ($\mathcal{R}$-commutative, $\mathcal{L}$-commutaive) semigroup and $S^{(n)} = \{e\}$ ($S_R^{(n)} = \{e\}$, $S_L^{(n)} = \{e\}$).*

Theorem 5.4 *([106]) A cancellative right soluble semigroup of length n can be embedded in a soluble group of length n.*

Proof. Let S be a cancellative right soluble semigroup of length n. By Lemma 5.4, S is left reversible and so, by Theorem 1.24 of [19], S is embeddable into the group G of right quotient of S. We know that the elements of $S_R^{(1)}$ satisfies the law $K(Y_1, \ldots, Y_m) = e$, where $m = 2^n$ and $K \in \beta_S$. Since β_S has only one element and $g \supseteq S$ then $K(a,b) = a^{-1}b^{-1}ab$ for any $a, b \in S$. We will see that the elements of G satisfy the law $K' = (x_1, \ldots, x_{2m}) = e$ for $K' \in \beta_G$, that is, $K'(y_1, \ldots, y_m) = e$, where $y_i = K'(x_{2i-1}, x_{2i})$, $m \geq i > 0$. Let $x_{2i-1} = ab^{-1}$ and $x_{2i} = cd^{-1}$ for some $a, b, c, d \in S$. Then

$$y_i = (ab^{-1})^{-1}(cd^{-1})^{-1}ab^{-1}cd^{-1} =$$
$$bd(d^{-1}a^{-1}da)(a^{-1}c^{-1}ac)(c^{-1}b^{-1}cb)b^{-1}d^{-1} =$$
$$bdX_ib^{-1}d^{-1},$$

where $X_i \in S_R^{(1)}$. Thus $b^{-1}d^{-1}y_idb = Y_i$, where $Y_i = K(b,d)X_i \in S_R^{(1)}$. We can therefore choose $p_i \in S$ for each integer $i, m \geq i > 0$, so that $p_i^{-1}y_ip_i = Y_i \in S_R^{(1)}$. Notice that, for $r \in S$ and $Y \in S_R^{(1)}$, $r^{-1}Yr = YK(Y,r) \in S_R^{(1)}$. Thus, writing $p = p_1p_2 \ldots p_{i-1}$, $q = p_{i+1}p_{i+2} \ldots p_m$ and $P = pp_iq$, we get

$$P^{-1}y_iP = q^{-1}p_i^{-1}p^{-1}y_ipp_iq = q^{-1}K(p_i,p)p^{-1}Y_ipK(p,p_i)q \in S_r^{(1)}.$$

But then

$$K'(y_1, \ldots, y_m) = PK'(P^{-1}y_1P, \ldots, P^{-1}y_mP)P^{-1} =$$
$$PK(P^{-1}y_1P, \ldots, P^{-1}y_mP)P^{-1} = PP^{-1} = e.$$

□

Lemma 5.6 *([106]) A cancellative semigroup S is soluble of length n if and only if it is both right and left soluble of length n.*

Proof. Let S be a soluble semigroup of length n. By Lemma 5.5, $S^{(1)} \supseteq S_R^{(1)}$. Proceeding by induction we assume that $S^{(r)} \supseteq S_R^{(r)}$. Then

$$S^{(r+1)} = (S^{(r)})^{(1)} \supseteq (S_R^{(r)})_R^{(1)} = S_R^{(r+1)}.$$

Thus, if $S^{(n)} = e$ then $S_R^{(n)} = e$. Similarly, $S_L^{(n)} = e$. Hence S is both right and left soluble of length n.

Conversely, if S is both left and right soluble of length n and G is its right quotient group then G also its left quotient group. By Theorem 5.4, G is soluble of length n. Since $G \supseteq S$ then $G^{(n)} = S^{(n)} = e$. □

Theorem 5.5 *([106]) A cancellative soluble semigroup of length n can be embedded in a soluble groups of length n.*

Proof. By Theorem 5.4 and Lemma 5.6, it is obvious. $\square$

Chapter 6

Conditionally commutative semigroups

In this chapter we deal with semigroups S in which, for any $a, b \in S$, the assumption $ab = ba$ implies $axb = bxa$ for all $x \in S$. These semigroups are called conditionally commutative semigroups. In the beginning of the chapter we present equivalent conditions for a conditionally commutative semigroup to be a semilattice of archimedean semigroups, a rectangular band of t-archimedean semigroups, or a semilattice of t-archimedean semigroups, respectively. We prove the followig results. A conditionally commutative semigroup is a semilattice of archimedean semigroups if and only if it is a band of t-archimedean semigroups. A conditionally commutative semigroup is a rectangular band of t-archimedean semigroups if and only if it is archimedean. A conditionally commutative semigroup S is a semilattice of t-archimedean semigroups if and only if, for every $a, b \in S$, there is a positive integer k such that $(ab)^k = (ba)^k$. We also present results about weakly separative conditionally commutative semigroups. It is shown that every weakly separative conditionally commutative semigroup is a disjoint union of commutative cancellative pover joined semigroups. It is also proved that a conditionally commutative semigroup is weakly separative and regular if and only if it is a normal band of abelian groups. It is shown that the simple conditionally commutative semigroups are exactly the Rees matrix semigroups over an abelian group. By the help of this result, the conditionally commutative archimedean semigroups containing at least one idempotent element are described. It is shown that S is a conditionally commutative archimedean semigroup containing at least one idempotent element if and only if it is a retract extension of a Rees matrix semigroup over an abelian group by a nil semigroup N such that if ϕ is the retract homomorphism of S and $\circ$ denotes the product in N then the relations $a \circ b = b \circ a$, $\phi(a)\phi(b) = \phi(b)\phi(a)$ imply $a \circ x \circ b = b \circ x \circ a$ for every $a, b, x \in N - \{0\}$. At the end of the chapter, it is shown that a semigroup is conditionally commutative and t-archimedean containing at least one idempotent element if and only if it is a retract extension of an abelian group by a conditionally commutative nil semigroup.

Definition 6.1 *A semigroup S is called a conditionally commutative semigroup if, for any $a, b \in S$, $ab = ba$ implies $axb = bxa$ for all $x \in S$.*

Lemma 6.1 *Every conditionally commutative semigroup satisfies the identity $aba^m = a^m ba$ for every positive integer m.*

Proof. It is obvious, because a and a^m are commutable with each other. □

Corollary 6.1 *Every conditionally commutative cancellative semigroup is commutative.*

Proof. As $aba^2 = a^2ba$ for every elements a and b of a conditionally commutative semigroup S, we get $ba = ab$ if S is also cancellative. □

Theorem 6.1 *([13]) For a conditionally commutative semigroup S, the following conditions are equivalent.*

(i) S is a band of t-archimedean semigroups.

(ii) S is a semilattice of archimedean semigroups.

Proof. (i) implies (ii). Let a, b be arbitrary elements of a conditionally commutative semigroup S with $b \in S^1 a S^1$, that is,

$$b = xay$$

for some $x, y \in S^1$. As S is a band of t-archimedean semigroups, by Theorem 1.7,

$$xay -_t xa^2y$$

and so there is a positive integer n such that

$$xa^2y|_t(xay)^n$$

which implies

$$xa^2y|_l(xay)^n,$$

that is,

$$(xay)^n \in S^1xa^2y \subseteq S^1a^2S^1.$$

Thus S is a Putcha semigroup and so, by Theorem 2.1, S is a semilattice of archimedean semigroups.

(ii) implies (i). Assume that a conditionally commutative semigroup S is a semilattice of (conditionally commutative) archimedean semigroups. Then, by Theorem 2.1, S is a Putcha semigroup. Let $a \in S$, $x, y \in S^1$ be arbitrary elements. As a divides xay ($x, y \in S^1$), it follows that a^2 divides some power of xay, that is,

$$(xay)^n = ua^2v$$

for a positive integer n and some $u, v \in S^1$. In view of Lemma 6.1, we have

$$(xa^2y)^2 = xa^3yxay = xayxa^3y \in S^1xay \cap xayS^1$$

and

$$(xay)^{n+1} = xayua^2v = ua^2vxay,$$

whence

$$(xay)^{n+1} = xa^2yuav = uavxa^2y \in xa^2yS^1 \cap S^1xa^2y$$

and, by Theorem 1.7, S is a band of t-archimedean semigroups. □

Theorem 6.2 *([13]) On a conditionally commutative semigroup S, the following are equivalent.*

(i) S is a rectangular band of t-archimedean semigroups.

(ii) S is archimedean.

Proof. (i) implies (ii). Let S be a rectangular band $B = L \times R$ of t-archimedean semigroups $S_{i,j}$ (L is a left zero semigroup, R is a right zero semigroup, $i \in L$, $j \in R$). Let $a \in S_{i,j}$, $b \in S_{m,n}$ be arbitrary elements of S. Then, for arbitrary $x \in S_{i,m}$ and $y \in S_{n,j}$, we have

$$xby \in S_{i,m}S_{m,n}S_{n,j} \subseteq S_{i,j}.$$

As $S_{i,j}$ is an archimedean semigroup,

$$a^k = uxbyv$$

for some $u, v \in S_{i,j}$ and a positive integer k. Hence

$$a^k \in SbS.$$

We can prove, in a similar way, that

$$b^h \in SaS$$

for some positive integer h. Thus S is archimedean.

(ii) implies (i). Assume that S is a conditionally commutative archimedean semigroup. Then, by the previous theorem, S is a band of t-archimedean semigroups. It is sufficient to show that if τ is the band congruence induced by the decomposition of S then $ab = ba$ implies $a \ \tau \ b$ for every $a, b \in S$. In fact, if $a, b \in S$, there are $x, y \in S^1$ such that

$$a^n = xby.$$

Therefore if $ab = ba$ then it follows that

$$a^{n+2} = axbya = b(xaya) = (axay)b.$$

Hence

$$b|_l a^{n+2} \text{ and } b|_r a^{n+2},$$

that is

$$b|_t a^{n+2}.$$

We can prove, in a similar way, that

$$a|_t b^{m+2}$$

for some positive integer m. Then

$$a -_t b$$

and so, by Theorem 1.7,

$$a\tau b.$$

□

Theorem 6.3 *([13]) For a conditionally commutative semigroup S, the following are equivalent.*

(i) S is a semilattice of t-archimedean semigroups.

(ii) For every $a, b \in S$, there is a positive integer k such that $(ab)^k = (ba)^k$.

Proof. (i) implies (ii). Let $a, b \in S$ be arbitrary elements. Then

$$(ab)^n = xba$$

and

$$(ba)^n = aby$$

for a positive integer n and elements $x, y \in S^1$. Thus

$$(ab)^{2n+1} = a^2bybxba = abybxba^2 = (ba)^{2n+1}.$$

(ii) implies (i). Let $a, b \in S$ be arbitrary elements such that $a = xby$ for some $x, y \in S^1$. Then there are positive integers p and q such that

$$a^p = (xby)^p = (yxb)^p$$

and

$$a^q = (xby)^q = (byx)^q.$$

Hence

$$a^{p+q} = (yxb)^p(byx)^q,$$

that is, b^2 divides a^{p+q}. Thus S is a Putcha semigroup and so S is a semilattice Y of archimedean semigroups S_α, $\alpha \in Y$. Let $c, d \in S_\alpha$, $\alpha \in Y$. Then there is a positive integer r such that

$$c^r = udv$$

for some $u, v \in S_\alpha$. Thus

$$c^{rk} = (udv)^k = (dvu)^k = (vud)^k$$

for some positive integer k. Hence S_α is t-archimedean. □

Theorem 6.4 *([13]) For a conditionally commutative semigroup S, the following are equivalent.*

(i) S is t-archimedean.

(ii) S is archimedean and, for every $a, b \in S$, there is a positive integer m such that $ab^m = b^m a$.

Proof. (i) implies (ii). Let S be a conditionally commutative t-archimedean semigroup. Then, for every $a, b \in S$, there is a positive integer n such that

$$a^n = bx = yb$$

for some $x, y \in S^1$. Therefore

$$a^{2n} = bxby$$

and, by Lemma 6.1, it follows that

$$ba^{2n} = b^2xyb = bxyb^2 = a^{2n}b.$$

Hence S satisfies (ii).

(ii) implies (i). Let S be a conditionally commutative semigroup satisfying (ii). Let $a, b \in S$ be arbitrary elements. Then there are positive integers m, n such that

$$ab^m = b^m a$$

and

$$b^n = xay$$

for some $x, y \in S^1$. From these it follows that

$$b^{n+m} = b^m xay = a(xb^m y)$$

and

$$b^{n+m} = xayb^m = (xb^m y)a.$$

Then S is t-archimedean. □

Corollary 6.2 *If S is a conditionally commutative t-archimedean semigroup then, for every $a, b \in S$, there is a positive integer n such that $a^n = bz = zb$ for some $z \in S$.*

Definition 6.2 *A semigroup S is called a strongly reversible semigroup if, for every $a, b \in S$, there are positive integers h, k, j such that $(ab)^h = a^k b^j = b^j a^k$.*

Theorem 6.5 *([13]) Every conditionally commutative t-archimedean semigroup is strongly reversible.*

Proof. Let S be a conditionally commutative t-archimedean semigroup and $a, b \in S$ be arbitrary elements. By Theorem 6.3, there exists a positive integer k such that

$$(ab)^k = (ba)^k.$$

If $k = 1$ then S is commutative and so

$$(ab)^n = (ba)^n$$

for every positive integer n. Assume $k > 1$. Then

$$(ab)^{k+1} = (ab)(ab)^k = (ab)(ba)^k = ab(ba)^{k-1}ba$$

$$= a(ba)^{k-1}b^2a = (ab)^k ba = (ba)^{k+1}.$$

Hence, by induction,

$$(ab)^n = (ba)^n$$

for every positive integer $n \geq k$. Thus, we can suppose that $k \geq 2$ and $(ab)^n = (ba)^n$ for every positive integer $n \geq k$. By Theorem 6.4, there exists an integer $m > 1$ such that

$$ab^m = b^m a.$$

Then, for every $r \geq 0$, we get

$$ab^{m+r} = b^m ab^r = b^r ab^m = b^{m+r}a.$$

Thus

$$(ab)^{k+2} = a(ba)^{k+1}b = a(ab)^{k+1}b = a^2b(ab)^k b = a^2b^2(ab)^k$$

and

$$(ab)^{k+3} = a(ba)^{k+2}b = a(ab)^{k+2}b = a^3b^2(ab)^k b = a^3b^3(ab)^k,$$

and therefore we get

$$(ab)^{k+m} = a^m b^m (ab)^k.$$

Then, using $ab^{m+r} = b^{m+r}a$ $(r \geq 0)$, we have

$$(ab)^{k+m} = a^m b^m ab(ab)^{k-1} = a^{m+1}b^{m+1}(ab)^{k-1}$$

$$= a^{m+1}b^{m+1}ab(ab)^{k-2} = a^{m+2}b^{m+2}(ab)^{k-2} = \ldots = a^{k+m}b^{k+m} = b^{k+m}a^{k+m}.$$

Hence S is strongly reversible. □

Corollary 6.3 *Every conditionally commutative archimedean semigroup is a disjoint union of power joined semigroups.*

Proof. It follows from Theorem 6.2, Theorem 6.5 and from Lemma 6 of [10]. □

Lemma 6.2 *([13]) In a conditionally commutative weakly separative semigroup S, $a^n b^k = b^k a^n$ implies $ab = ba$ for every $a, b \in S$ and positive integers n, k.*

Proof. Let S be a weakly separative conditionally commutative semigroup and $a, b \in S$ be arbitrary elements such that $a^n b^k = b^k a^n$ for some positive integers n, k. By Lemma 6.1, we have

$$a^{n+1} b^k = a b^k a^n = a^n b^k a = b^k a^{n+1}.$$

Then, by Lemma 1.2, it follows that

$$ab^k = b^k a.$$

Hence, in the same way, it follows that

$$ab = ba.$$

□

Theorem 6.6 *([13]) A weakly separative conditionally commutative semigroup S is a disjoin union of commutative cancellative power joined semigroups P_i, $i \in I$. Moreover, for every $a, a' \in P_i$, $b, b' \in P_j$ $(i, j \in I)$, $ab = ba$ implies $a'b' = b'a'$.*

Proof. Let ρ be the equivalence relation on S defined by

$$\rho = \{(a, b) \in S \times S : \ a^n = b^m \text{ for positive integers } n, m\}.$$

As $a^n = b^m$ implies $a^n b^m = b^m a^n$, by Lemma 6.2,

$$ab = ba.$$

Thus

$$(ab)^m = a^m b^m = a^{n+m}.$$

Hence $(a, b) \in \rho$ implies $(a, ab) \in \rho$ and the ρ-classes P_i $(i \in I)$ are commutative cancellative power joined semigroups (see Prop. 5 of [10]). Next, let $a, a' \in P_i$, $b, b' \in P_j$ $(i, j \in I)$ with $ab = ba$. Then there exist positive integers p, q, r, s such that

$$a^p = (a')^q, \ b^r = (b')^s.$$

Since $ab = ba$ implies $a^p b^r = b^r a^p$ then

$$(a')^q (b')^s = (b')^s (a')^q$$

and so, by Lemma 6.2, $a'b' = b'a'$. □

Theorem 6.7 *([13]) On a t-archimedean semigroup S, the following are equivalent.*

(i) S is conditionally commutative and weakly separative.

(ii) S is commutative and cancellative.

Proof. (i) implies (ii). Let S be a conditionally commutative weakly separative semigroup and $a, b \in S$ be arbitrary elements. By Theorem 6.4,

$$ab^m = b^m a$$

for some positive integer m. Then, by Lemma 1.2,

$$ab = ba,$$

that is, S is commutative. By Theorem 3.3, S is a semilattice of (commutative) cancellative semigroups. As S is t-archimedean, by Theorem 2 of [81], S is cancellative.

(ii) implies (i). It is obvious. □

Theorem 6.8 *(Th. III.4.3 of [75]) The following conditions on a semigroup S are equivalent.*

(i) S is conditionally commutative and weakly cancellative.

(ii) S is a rectangular band of commutative cancellative semigroups.

(iii) S is embeddable into a Rees matrix semigroup over an abelian group.

Theorem 6.9 *(Th. IV.2.1 of [75]) The following conditions on an arbitrary semigroup S are equivalent.*

(i) S is conditionally commutative and the classes of S modulo the least semilattice congruence are weakly cancellative.

(ii) S is a normal band of commutative cancellative semigroups.

Theorem 6.10 *([13]) An archimedean conditionally commutative weakly separative semigroup is weakly cancellative.*

Proof. Let S be an archimedean conditionally commutative weakly separative semigroup. By Theorem 6.2, S is a rectangular band $B = L \times R$ of t-archimedean semigroups $S_{i,j}$ (L is a left zero semigroup, R is a right zero semigroup; $i \in L, j \in R$). By Theorem 6.7, each $S_{i,j}$, $i \in L, j \in R$ is commutative and cancellative. Then, by Theorem 6.8, S is weakly cancellative. □

Theorem 6.11 *(Th. III.5.7.6 of [75]) The following conditions on a semigroup S are equivalent.*

(i) S is conditionally commutative and right cancellative.

(ii) S is right commutative (Definition 10.1) and right cancellative.

Theorem 6.12 *([13]) For a conditionally commutative semigroup S, the following are equivalent.*

(i) S is regular and weakly separative.

(ii) S is right (left) regular.

(iii) S is intra-regular.

(iv) S is a normal band of abelian groups.

Proof. (i) implies (ii). Let S be a conditionally commutative regular weakly separative semigroup. Then, for every $a \in S$, there is an element $x \in S$ such that

$$a^2 = a^2xa^2 = a^2xa^2xa^2 = a^2xa^3xa = (a^2xa)^2.$$

As $a^2 = (a^2xa)a$, we get

$$a = a^2xa,$$

because S is weakly separative. Hence S is right regular. We can prove, in a similar way, that S is left regular.

(ii) implies (iii). If S is right regular then, for every $a \in S$, there is an element $x \in S$ such that

$$a = a^2x$$

and so

$$a = aa^2x^2.$$

Hence S is intra-regular.

(iii) implies (iv). Let S be a conditionally commutative intra-regular semigroup and $a, b \in S$ be arbitrary elements with $b \in S^1aS^1$, that is,

$$b = xay$$

for some $x, y \in S^1$. As S is intra-regular,

$$a = ua^2v$$

for some $u, v \in S$. Then

$$b = xay = xua^2vy \in S^1a^2S^1.$$

Hence S is a Putcha semigroup and so it is a semilattice of archimedean semigroups. Since S is conditionally commutative then, by Theorem 6.1, it is a band of t-archimedean semigroups. Let ρ denote the corresponding band congruence of S. Then we have

$$ua = u^2a^2v \;\rho\; ua^2v = a \;\rho\; a^2,$$

and therefore there exist an element $z \in S$ and a positive integer n such that

$$(ua)^n = a^2z.$$

Then it results

$$a = (ua)av = (ua)^nav^n = a^2zav^n,$$

and S is right regular. We can prove, in a similar way, that S is left regular. Thus S is completely regular (see IV.1.2 of [73]) and so the statement follows from IV.2.7;5 of [75].

(iv) implies (i). It is obvious. □

Theorem 6.13 *([55]) For a semigroup S, the following are equivalent.*

(i) S is a simple conditionally commutative semigroup.

(ii) S is a Rees matrix semigroup over an abelian group.

Proof. (i) implies (ii). Let S be a simple conditionally commutative semigroup. Let a be an arbitrary element of S. Then there are elements $x, y \in S$ such that

$$xa^6y = a.$$

As $a^3a = aa^3$, we get

$$a^3 = axa^3a^3ya = a^3xaaya^3$$

and so xa^2ya^3 is an idempotent element of S. We show that S is completely simple. Assume, in an indirect way, that S is not completely simple. Then, by Theorem 1.22, S has a bicyclic subsemigroup $C(p, q)$ such that $pq = f$, $qp \neq f$, where f is an idempotent element of S. It is evident that $C(p, q)$ is conditionally commutative. So

$$q^2p = q^2p^2q = qp^2q^2 = q$$

which is a contradiction. Consequently S is completely simple, and so S is isomorphic with a Rees matrix semigroup $\mathcal{M}(I, G, J; P)$ over a group G with a $J \times I$ sandwich matrix P. We may assume that P is normalized, that is, there are elements $i_0 \in I$ and $j_0 \in J$ such that

$$p_{j_0,i} = p_{j,i_0} = e$$

for all $i \in I$ and $j \in J$. Here e denotes the identity element of G. Consider elements $(i_0, g, j_0), (i_0, e, j_0)$ and (i_0, a, j_0) of $\mathcal{M}(I, G, J; P)$, where g and a are arbitrary elements of G. As $eg = ge$, we have

$$(i_0, e, j_0)(i_0, g, j_0) = (i_0, g, j_0)(i_0, e, j_0)$$

and so

$$\begin{aligned}(i_0, ag, j_0) = (i_0, eag, j_0) &= (i_0, e, j_0)(i_0, a, j_0)(i_0, g, j_0) \\ &= (i_0, gae, j_0) = (i_0, ga, j_0)\end{aligned}$$

from which we get

$$ag = ga.$$

Thus G is a commutative group and so (ii) is satisfied.

(ii) implies (i). Assume that a semigroup S is isomorphic with a Rees matrix semigroup $\mathcal{M}(I, G, J; P)$ over a commutative group G. We show that S is conditionally commutative. Let (i, g, j) and (k, h, l) be arbitrary elements of S with $(i, g, j)(k, h, l) = (k, h, l)(i, g, j)$. Then

$$(i, gp_{j,k}h, l) = (k, hp_{l,i}g, j),$$

that is, $i = k$ and $j = l$. Let (m, r, n) be arbitrary element of S. Then

$$(i, g, j)(m, r, n)(k, h, l) = (i, gp_{j,m}rp_{n,i}h, j)$$

and

$$(k, h, l)(m, r, n)(i, g, j) = (i, hp_{j,m}rp_{n,i}g, j).$$

As G is commutative,

$$gp_{j,m}rp_{n,i}h = hp_{j,m}rp_{n,i}g$$

and so S is conditionally commutative. □

Corollary 6.4 *([13]) For a semigroup S, the following are equivalent.*

(i) S is right simple and conditionally commutative.

(ii) S is a right abelian group.

Proof. Let S be a right simple conditionally commutative semigroup. Then S is also simple and so it is a Rees matrix semigroup $\mathcal{M}(I, G, J; P)$ over an abelian group G with a sandwich matrix P. We can suppose that P is normalized ($p_{j_0,i} = p_{j,i_0} = e$, the identity of G). Since $(i_0, g, j)S = S$ for every $g \in G$ and $j \in J$ then, for arbitrary $(i, h, k) \in S$, there is an element $(t, x, r) \in S$ such that

$$(i_0, gp_{j,t}x, r) = (i_0, g, j)(t, x, r) = (i, h, k).$$

From this we get that $i = i_0$ and so the elements of P are equal to e. Hence S is the direct product of the right zero semigroup J and the abelian group G. Thus (ii) follows from (i). It is obvious that (ii) implies (i). □

Theorem 6.14 *([13]) For a semigroup S, the following are equivalent.*

(i) S is a conditionally commutative archimedean semigroup containing at least one idempotent element.

(ii) S is a retract extension of a Rees matrix semigroup over an abelian group by a nil semigroup N such that if ϕ is the retract homomorphism of S and $\circ$ denotes the product in N then the relations $a \circ b = b \circ a$, $\phi(a)\phi(b) = \phi(b)\phi(a)$ imply $a \circ x \circ b = b \circ x \circ a$ for every $a, b, x \in N - \{0\}$.

(iii) S is conditionally commutative and a rectangular band of t-archimedean semigroups containing each one idempotent.

Proof. (i) implies (ii). Let S be a conditionally commutative archimedean semigroup containing at least one idempotent element. Then, by Theorem 2.2, S is an ideal extension of a simple semigroup K by a nil semigroup N. Since K is also conditionally commutative then, by Theorem 6.13, it is a Rees matrix semigroup $\mathcal{M}(I, G, J; P)$ over an abelian group G with a sandwich matrix P. Then, by Theorem 1.23, K is the rectangular band $I \times J$ of abelian groups $G_{i,j} = \{(i, g, j) : g \in G\}$. Let $a \in S$ be an arbitrary element. As N is a nil semigroup, there is a positive integer n such that $a^n \in K$. Let

$$a^n = (i, g, j).$$

Then

$$a(i,e,j) = a(i,g^{-1}p_{j,i}^{-1},j)(i,g,j) = a(i,g^{-1}p_{j,i}^{-1},j)a^n$$
$$= a^n(i,g^{-1}p_{j,i}^{-1},j)a = (i,g,j)(i,p_{j,i}^{-1}g^{-1},j)a = (i,e,j)a$$

and so

$$a^{n+1} = aa^n = a(i,g,j) = a(i,e,j)(i,p_{j,i}^{-1}g,j) = (i,e,j)a(i,p_{j,i}^{-1}g,j).$$

As

$$(i,e,j)a(i,p_{j,i}^{-1}g,j) = ((i,e,j)a)(i,p_{j,i}^{-1}g,j) = (k,h,j)$$

and

$$(i,e,j)a(i,p_{j,i}^{-1}g,j) = (i,e,j)(a(i,p_{j,i}^{-1}g,j)) = (i,p,l)$$

for some $k \in I$, $h,p \in G$ and $l \in J$, we get $k = i$, $h = p$ and $l = j$, that is,

$$a^{n+1} = (i,h,j).$$

Consequently, if n is the least positive integer such that $a^n \in G_{i,j}$ then, $a^m \in G_{i,j}$ for every positive integer $m \geq n$. Hence $(i,j) \in I \times J$ is well defined by the element a in the above mentioned sence. Let

$$\phi(a) = ae,$$

where e denotes the identity of $G_{i,j}$. We show that ϕ is a retract homomorphism of S onto K. Let $a,b \in S$ be arbitrary elements. Then there exists a positive integer n such that

$$a^n,\ b^n,\ (ab)^n \in K.$$

Assume $a^n \in G_\alpha$, $b^n \in G_\beta$ $(\alpha,\beta \in I \times J)$. First let us verify that $(ab)^n \in G_{\alpha\beta}$. In fact, calling e the identity of G_α and a^{-n} the inverse of a^n in G_α, it results

$$ae = aa^{-n}a^n = a^na^{-n}a = ea.$$

In the same way, if f is the identity of G_β, it results

$$bf = fb.$$

Thus we have

$$abef = afeb = efab,$$

whence

$$(ab)^nef = ef(ab)^n$$

which implies

$$(ab)^n \in G_{\alpha\beta},$$

because K is a rectangular band of groups $G_{i,j}$. Therefore, calling u the the identity of $G_{\alpha\beta}$, it results

$$abu = uab.$$

As $uf, eu \in G_{\alpha\beta}$, we have

$$(uf)^2 = ufuf = uf$$

and

$$(eu)^2 = eueu = eu.$$

Thus

$$uf = eu = u.$$

That being stated, the function $\phi : a \rightarrow ae$ results to be a retract homomorphism of S onto K. In fact, let $\phi(a) = ae$, $\phi(b) = bf$, $\phi(ab) = abu$. Since $aebf, eb \in G_{\alpha\beta}$, we get

$$abu = uab = euab = aueb = auf(eb) = a(eb)fu = (ae)(bf),$$

that is,

$$\phi(ab) = \phi(a)\phi(b).$$

It remains to verify that N satisfies the condition of the statement. In fact, let $a, b \in N - \{0\}$ with $a \circ b = b \circ a$, $\phi(a)\phi(b) = \phi(b)\phi(a)$. If $a \circ b = b \circ a \neq 0$, we have

$$ab = a \circ b = b \circ a = ba.$$

If $a \circ b = b \circ a = 0$ then it follows

$$ab, ba \in K,$$

and we have

$$ab = \phi(ab) = \phi(a)\phi(b) = \phi(b)\phi(a) = \phi(ba) = ba.$$

Thus in both cases it results

$$ab = ba.$$

As S is conditionally commutative, we get

$$axb = bxa$$

for every $x \in S$. So we can conclude that

$$a \circ x \circ b = b \circ x \circ a$$

for every $x \in N$.

(ii) implies (iii). Let S be a semigroup satisfying (ii). Then S is a retract extension of a rectangural band K of abelian groups G_i $(i \in K)$ and so, by III.2.12;7 of [75], S is a rectangular band of semigroups T_i $(i \in K)$ such that $T_i \cap K = G_i$. For every $a, b \in T_i$, there exist a positive integer n with $a^n, b^n \in G_i$ and an element $x \in G_i$ with

$$a^n = xb^n = b^n x.$$

Thus T_i is t-archimedean with an idempotent. Finally it remains to prove that S is conditionally commutative. Let $a, b, x \in S$ be arbitrary elements with $ab = ba$. If $axb \in S - K$ then

$$a, b, x \in S - K = N - \{0\}$$

and so

$$axb = bxa.$$

In fact $ab = ba$ implies $a \circ b = b \circ a$ and $\phi(a)\phi(b) = \phi(b)\phi(a)$, whence

$$a \circ x \circ \ b = b \circ x \circ a$$

and finally

$$axb = a \circ x \circ \ b = b \circ x \circ a = bxa.$$

Hence it follows also that $axb \in K$ implies $bxa \in K$. In this case, since S is a rectangular band of semigroups T_i, the ralation $ab = ba$ implies that a and b are in the same T_i, whence

$$axb, bxa \in T_i \cap K = G_i.$$

Now it is immediately verifiable that G_i is an ideal of T_i and that

$$ae, ea, be, eb \in G_i$$

(e is the identity of G_i). Then, K being a weakly cancellative conditionally commutative semigroup (see Theorem 6.8), it results

$$eaxbe = eaexbe = beexea = ebexeae = ebxae,$$

whence

$$eaxb = ebxa$$

and

$$axbe = bxae.$$

Hence it follows

$$axb = bxa.$$

(iii) implies (i). It immediately follows from Theorem 6.2. □

Corollary 6.5 *([13]) A conditionally commutative archimedean semigroup with an unique idempotent is t-archimedean.*

Corollary 6.6 *([13]) A semigroup is t-archimedean and conditionally commutative containing at least one idempotent element if and only if it is a retract extension of an abelian group by a conditionally commutative nil semigroup.*

Proof. Suppose that S is a conditionally commutative t-archimedean semigroup with an idempotent. By Theorem 6.14, S is a retract extension of an abelian group G by a nil semigroup $(N, \circ)$. Let $a, b \in N$ be arbitrary elements such that $a \circ b = b \circ a$. If $a \circ b = b \circ a \neq 0$ then

$$ab = a \circ b = b \circ a = ba.$$

If $a \circ b = b \circ a = 0$ then

$$ab, ba \in G$$

and, calling e the identity of G, we find

$$ab = (ab)e = (ae)(be) = (be)(ae) = (ba)e = ba.$$

Thus, in both cases, $a \circ b = b \circ a$ implies $ab = ba$. As S is conditionally commutative, we get

$$axb = bxa$$

for every $x \in S$. Hence, for every $x \in N$,

$$a \circ x \circ b = b \circ x \circ a.$$

The converse easily follows from Theorem 6.14, since a conditionally commutative nil semigroup satisfies the condition contained in the statement of that theorem. □

Chapter 7

$\mathcal{RC}$-commutative semigroups

In this chapter we deal with semigroups which are both $\mathcal{R}$-commutative and conditionally commutative. These semigroups are called $\mathcal{RC}$-commutative semigroups. The $\mathcal{R}$-commutative semigroups and the conditionally commutative semigroups are examined in Chapter 5 and Chapter 6, respectively. From the results of those chapters it follows that every $\mathcal{RC}$-commutative semigroup is a semilattice of conditionally commutative archimedean semigroups. In this chapter, we show that the simple $\mathcal{RC}$-commutative semigroups are exactly the right abelian groups. By the help of this result we show that every $\mathcal{RC}$-commutative archimedean semigroup containing at least one idempotent element is an ideal extension of a right abelian group by a commutative nil semigroup. As a consequence, we prove that every $\mathcal{RC}$-commutative regular semigroup is a spined product of a right normal band and a semilattice of abelian groups. We determine the subdirectly irreducible $\mathcal{RC}$-commutative semigroups with a globally idempotent core. We show that they are those semigroups which are isomorphic to either G or G^0 or F or R or R^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime), F is a two-element semilattice and R is a two-element right zero semigroup. At the end of the chapter we deal with the $\mathcal{RC}$-commutative Δ-semigroups. It is shown that a semigroup S is an $\mathcal{RC}$-commutative Δ-semigroup if and only if it is isomorphic to either G or G^0 or R or R^0 or N or N^1, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime), R is a two-element right zero semigroup and N is a commutative nil semigroup whose ideals form a chain with respect to inclusion.

Definition 7.1 *A semigroup is called an $\mathcal{RC}$-commutative semigroup if it is $\mathcal{R}$-commutative and conditionally commutative.*

Lemma 7.1 *Every $\mathcal{RC}$-commutative semigroup is a semilattice of conditionally commutative archimedean semigroups.*

Proof. Let S be an $\mathcal{RC}$-commutative semigroup. Then, by Theorem 5.3, S is a semilattice of archimedean semigroups. It is clear that the archimedean components of S are conditionally commutative. □

Theorem 7.1 *([55]) A semigroup is simple and $\mathcal{RC}$-commutative if and only if it is a right abelian group.*

Proof. Let S be a simple $\mathcal{RC}$-commutative semigroup. By Theorem 6.13, S is isomorphic with a Rees matrix semigroup $\mathcal{M}(I,G,J;P)$ over an abelian group G. We note that I and J can be considered as a left and a right zero semigroup, respectively. We may assume that the sandwich matrix P is normalized, that is, $p_{j_0,i} = p_{j,i_0} = e$ for some $j_0 \in J$, $i_0 \in I$ and for all $j \in J$, $i \in I$, where e is the identity element of G. Let $a = (i_0, g, j_0)$ and $b = (m, h, j_0)$ be elements of S, where $g, h \in G$ and $m \in I$ are arbitrary. As S is simple, there are elements $x, y \in S$ such that

$$ab = xbay.$$

As S is $\mathcal{R}$-commutative, there is an element z in S^1 such that

$$xba = baxz.$$

Let $xzy = (k, r, l)$. Then

$$(i_0, gh, j_0) = ab = xbay = ba(xzy) = (m, hgr, l).$$

Thus $m = i_0$, for all $m \in I$, that is, $|I| = 1$. Consequently, P has only one column and every element of P is e. This implies that S is a direct product of the commutative group G and the right zero semigroup J, that is, S is a right abelian group.

As a right abelian group is simple and $\mathcal{RC}$-commutative, the theorem is proved. □

Theorem 7.2 *([55]) A semigroup is an $\mathcal{RC}$-commutative archimedean semigroup containing at least one idempotent element if and only if it is a retract extension of a right abelian group by a commutative nil semigroup.*

Proof. Let S be an $\mathcal{RC}$-commutative archimedean semigroup containing at least one idempotent element. Then, by Theorem 6.14, S is a retract extension of a Rees matrix semigroup K over an abelian group by a nil semigroup N. By Lemma 5.2, K is $\mathcal{R}$-commutative. It is clear that K is also conditionally commutative. Thus, by Theorem 7.1, K is a right abelian group. We show that N is $\mathcal{R}$-commutative. Let $a, b \in N$ be arbitrary elements. First, we show that $ab = 0$ in N if and only if $ba = 0$ in N. Assume $ab = 0$ in N. Then $ab \in K$ in S. As S is $\mathcal{R}$-commutative, $ba = abx \in K$ for some $x \in S^1$ and so $ba = 0$ in N. Similarly, $ba = 0$ implies $ab = 0$ in S. Next, assume that $ab \neq 0$ in N. Then $ba \neq 0$ and so $a, b, ab, ba \notin K$ (in S). As S is $\mathcal{R}$-commutative, there is an element $x \in S^1$ such that $ab = bax$. It is clear that $x \notin K$. Hence N is

$\mathcal{R}$-commutative. By Corollary 5.1, N is commutative. Thus the first part of the theorem is proved.

Conversely, assume that a semigroup S is a retract extension of a right abelian group by a commutative nil semigroup. Denote $\circ$ the product in N and ϕ the retract homomorphism of S onto K. Since N is commutative, ϕ satisfies condition (ii) of Theorem 6.14. Thus S is a conditionally commutative archimedean semigroup containing at least one idempotent element. It remains to show that S is also $\mathcal{R}$-commutative. Let $a, b \in S$ be arbitrary elements. As N is commutative, $ab \in K$ if and only if $ba \in K$, and we can suppose $ab, ba \in K$. As K is right simple, $ab = bax$ for some $x \in K$. Thus S is $\mathcal{R}$-commutative. □

Corollary 7.1 *Every regular $\mathcal{RC}$-commutative semigroup is a spined product of a right normal band and a semilattice of abelian groups.*

Proof. Let S be a regular $\mathcal{RC}$-commutative semigroup. By Lemma 7.1, S is a semilattice Y of conditionally commutative archimedean semigroups S_α ($\alpha \in Y$). As S is regular, every S_α is regular and so contains at least one idempotent element. Then, by Theorem 2.2, every S_α is an ideal extension of a simple semigroup containing at least one idempotent element by a nil semigroup. From this we can conclude that every S_α is simple. Let $\alpha \in Y$ be an arbitrary element. Let $R_\alpha = \cup\{S_\beta : \alpha \leq \beta\}$. Let $b, c \in R_\alpha$ be arbitrary elements. Assume $b \in S_\beta$ and $c \in S_\gamma$ for some $\beta, \gamma \geq \alpha$. Then $\alpha \leq \beta\gamma$ and so $bc, cb \in R_\alpha$. Thus R_α is a subsemigroup of S. It is clear that bc and cb are in the same S_δ ($\alpha \geq \delta$). As S is R-commutative, there is an element $x \in S^1$ such that $bc = cbx$. If $x \in S_\xi$ then $\delta = \delta\xi$ and so $\delta \leq \xi$ which implies $\alpha \leq \xi$. Thus $x \in S_\xi \subseteq R_\alpha$. Hence R_α is R-commutative. Since S_α is an ideal of R_α, and S_α is simple then, by Lemma 5.2, it follows that S_α is R-commutative. Then, by Theorem 7.1, S_α is a right abelian group. Thus S is a semilattice of right abelian groups. (If we apply this result for an $\mathcal{RC}$-commutative band B then we get that B is a semilattice of right zero semigroups.) By Theorem 1.27, S is an orthogroup and so E_S is a subsemigroup of S. Let $a, b \in S$ be arbitrary elements with $a^2 = ab = b^2$. Then a and b are in the same semilattice component of S. As the semilattice components of S are right abelian groups, we get $a = b$. Thus S is weakly separative and so, by Theorem 6.12, it is a normal band B of abelian groups. Then S is an orthodox normal band of abelian groups. By Yamada's Theorem S is the spined product of E_S and a semilattice of abelian groups. Moreover, $B \cong E_S$. As a homomorphic image of an $\mathcal{RC}$-commutative semigroup is $\mathcal{RC}$-commutative, we get that B is $\mathcal{RC}$-commutative. By the above remark, B is a semilattice of righ zero semigroups and so it is a right regular band. Let $a, x, y \in B$ be arbitrary element. As B is normal, we get $axya = ayxa$. As B is also right regular, we have $xya = axya = ayxa = yxa$. Thus B and so E_S are right normal. □

Subdirectly irreducible $\mathcal{RC}$-commutative semigroups

Theorem 7.3 *A semigroup S is a subdirectly irreducible $\mathcal{RC}$-commutative semigroup with a globally idempotent core if and only if it satisfies one of the following conditions.*

(i) S is isomorphic to either G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime).

(ii) S is a two-element semilattice.

(iii) S is isomorphic to R or R^0, where R is a two-element right zero semigroup.

Proof. Let S be a subdirectly irreducible $\mathcal{RC}$-commutative semigroup with a globally idempotent core K. First assume that S has no zero element. Then K is simple and, by Lemma 5.2, it is $\mathcal{R}$-commutative. It is clear that K is also conditionally commutative. Then, by Theorem 7.1, K is a right abelian group (that is a direct product of an abelian group G and a right zero semigroup R). By Corollary 1.4, we have either $K = G$ or $K = R$. In the first case S is a homogroup and so, by Theorem 1.47, $S = G$. By Theorem 3.14, S is a non-trivial subgroup of a quasicyclic p-group (p is a prime). Then (i) is satisfied. Assume $K = R$. It is clear that

$$\tau_R = \{(a,b) \in S \times S : \ (\forall r \in R) \ ra = rb\}$$

is a congruence on S and

$$\tau_R|R = id_R,$$

where $\tau_R|R$ denotes the restriction of τ_R to R. As R is a dense ideal of S,

$$\tau_R = id_S.$$

Let $s \in S$ be arbitrary. As S is conditionally commutative and R is right zero,

$$rs = rs^2rs = rsrs^2 = rs^2$$

for every $r \in R$. Hence

$$(s, s^2) \in \tau_R$$

and so

$$s = s^2.$$

Consequently, S is a band. As S is $\mathcal{R}$-commutative, for every $a, b \in S$, there is an element $x \in S^1$ such that

$$ab = bax.$$

Then

$$bab = b^2ax = bax = ab,$$

that is, S is a right regular band. Then, for every $a, b \in S$,

$$b(ab) = ab = (ab)b,$$

that is, b and ab are commute with each other. Then, for every $x \in S$, we have

$$bx(ab) = (ab)xb,$$

because S is conditionally commutative. Let $r \in R$ and $a, b \in S$ be arbitrary elements. Then

$$rab = (ra)b = b(ra)b = br(ab) = (ab)rb = (abr)(rb) = rb,$$

because R is a right zero semigroup and $r \in R$. Thus $(ab, b) \in \tau_R$ and so $ab = b$. Hence S is a right zero semigroup. By Theorem 1.48, $|S| = 2$ and so (iii) is satisfied.

In the second part of the proof, assume that S has a zero element. We show that $S - \{0\}$ is a subsemigroup of S. Assume, in an indirect way, that $ab = 0$ for some $a, b \neq 0$. Then $A_1 = \{x \in S : ax = 0\}$ is a non-trivial right ideal of S. As every right ideal of an $\mathcal{R}$-commutative semigroup is a two-sided ideal (see Lemma 5.1), $K \subseteq A_1$. Thus $aK = \{0\}$. Let $A_2 = \{x \in S : xK = \{0\}\}$. It is easy to see that A_2 is a non-trivial two sided ideal of S and so $K \subseteq A_2$. Hence $K^2 = \{0\}$ which contradicts the fact that K is a globally idempotent core of S. Consequently, $S^* = S - \{0\}$ is a subsemigroup of S. By Lemma 7.1, S is a semilattice of archimedean semigroups. Let η denote the corresponding (least) semilattice congruence of S. It is easy to see that $(a, 0) \notin \eta$ for every $a \in S^*$. If S^* has a zero element 0^* then $K^* = \{0, 0^*\}$ is a non-trivial ideal of S and $\rho_{K^*} \cap \eta = id_S$, where ρ_{K^*} denotes the Rees congruence on S modulo K^*. As S is subdirectly irreducible, we get $\eta = id_S$. Consequently, S is a semilattice and so, by Theorem 3.14, (ii) is satisfied. If S^* does not contain zero elements then it is a subdirectly irreducible RC-commutative semigroup with a globally idempotent core. Consequently, $S \cong G^0$ or $S \cong R^0$, where G is a non-trivial subgroup of a quasicyclic p-group, p is a prime ((i) is satisfied) and R is a two-element right zero semigroup ((iii) is satisfied). As the semigroups listed in the theorem are $\mathcal{RC}$-commutative subdirectly irreducible semigroups with a globally idempotent core, the theorem is proved. □

Theorem 7.4 *An $\mathcal{RC}$-commutative semigroup is subdirectly irreducible with a non-trivial annihilator and a nilpotent core if and only if it has a non-zero disjunctive element.*

Proof. By Theorem 1.49, it is obvious. □

$\mathcal{RC}$-commutative Δ-semigroups

By Remark 5.1, every $\mathcal{RC}$-commutative semigroup is left weakly commutative. Then, by Theorem 4.7,

$$\sigma = \{(a, b) \in S \times S : \ ab^n = b^{n+1},\ ba^n = a^{n+1} \text{ for some positive integer } n\}$$

is a congruence on an $\mathcal{RC}$-commutative semigroup S. In the next, σ will denote this congruence.

Lemma 7.2 *([55]) If S is an $\mathcal{RC}$-commutative Δ-semigroup and S has an (not necessarily proper) ideal which does not contain idempotent elements, then $\sigma = id_S$.*

Proof. Let S be an $\mathcal{RC}$-commutative Δ-semigroup and I be an ideal of S such that I has no idempotent elements. Let ϱ_x denote the Rees congruence on S determined by the ideal S^1xS^1, $x \in S$. As S is a Δ-semigroup, $\sigma \subseteq \varrho_x$ or $\varrho_x \subseteq \sigma$ for every $x \in S$. Assume $\varrho_x \subseteq \sigma$ for some $x \in I$. As $(x, x^2) \in \varrho_x \subseteq \sigma$, we get $x^2x^n = x^{n+1}$. Thus x^{n+1} is an idempotent element of I, contradicting the assumption that I has no idempotent elements. Consequently,

$$\sigma \subseteq \varrho_x$$

for all $x \in I$. We show that $\sigma|I = id_I$. Assume $(a, b) \in \sigma$ for some $a, b \in I$, $a \neq b$. As $\sigma \subseteq \varrho_{a^6}$, we have

$$(a, b) \in \varrho_{a^6},$$

that is,

$$a \in S^1a^6S^1.$$

Then

$$a = xa^6y$$

for some $x, y \in S^1$. As $a^3a = aa^3$, we get

$$a^3 = axa^3a^3ya = a^3xaaya^3$$

and so xa^2ya^3 is an idempotent element of I which is a contradiction. Thus

$$\sigma|I = id_I.$$

As S is a Δ-semigroup, I is a dense ideal of S and so

$$\sigma = id_S.$$

□

Lemma 7.3 *([55]) If S is a conditionally commutative Δ-semigroup and I is an ideal of S such that I is a nil extension of a non-trivial right zero semigroup R then S is a band and $I = R$.*

Proof. Let S be a conditionally commutative Δ-semigroup and I an ideal of S which is a nil extension of a non-trivial right zero semigroup R. Since $R^2 = R$ then, by Theorem 1.14, R is an ideal in S. It is easy to see that

$$\tau_R = \{(a, b) \in S \times S : (\forall r \in R)\ ra = rb\}$$

is a congruence on S such that $\tau_R|R = id_R$. As S is a Δ-semigroup, R is a dense ideal of S. So

$$\tau_R = id_S.$$

As S is conditionally commutative,

$$rara^2 = ra^2ra$$

for all $a \in S$ and $r \in R$. As $ra, ra^2 \in R$, we get

$$ra^2 = ra.$$

Thus

$$(a, a^2) \in \tau_R$$

which implies that

$$a = a^2,$$

that is, S is a band and $I = R$. □

Theorem 7.5 *([55]) S is an archimedean $\mathcal{RC}$-commutative Δ-semigroup if and only if it satisfies one of the following conditions.*

(i) S is a non-trivial subgroup of a quasicyclic p-group, p is a prime.

(ii) S is a two-element right zero semigroup.

(iii) S is a commutative nil semigroup whose ideals form a chain with respect to inclusion.

Proof. Let S be an archimedean $\mathcal{RC}$-commutative Δ-semigropup. If S has a zero element then S is a nil semigroup from which we get that S is a commutative nil semigroup whose ideals form a chain with respect to inclusion (see Corollary 5.1, Theorem 1.56 and Theorem 1.54). Condition (iii) is satisfied.

Next, assume that S does not contain zero element. First, consider the case when S is simple. Then, by Theorem 7.1, S is a direct product of a commutative group G and a right zero semigroup R. As S is a Δ-semigroup, we have either $S = G$ (and so, by Theorem 3.22, (i) is satisfied) or $S = R$ (and so, by Theorem 1.50, (ii) is satisfied).

Consider the case when S has a proper ideal. We show that S has an idempotent element. We may assume that S is not a commutative semigroup. If S is not right cancellative then, by Lemma 5.3, $\sigma \neq id_S$ and so, by Lemma 7.2, S has an idempotent element. Assume that S is right cancellative. As S is not commutative and conditionally commutative, there are elements a, b and x of S such that

$$ab = abx$$

and so

$$xabx = xabx^2 = x^2abx$$

from which we get $x = x^2$. Consequently, S has an idempotent element in both cases. By Theorem 7.2, S is an ideal extension of a right abelian group $K = G \times R$ (G is an abelian group, R is a right zero semigroup). By Theorem 1.52, $|G| = 1$. Thus $K = R$ which contradicts the assumption for S. Thus the first part of the theorem is proved. As the semigroups listed in the theorem are $\mathcal{RC}$-commutative Δ-semigroups, the proof is complete. □

Consider the case when S is a semilattice decomposable $\mathcal{RC}$-commutative Δ-semigroup. Then, by Remark 1.2, S is a semilatice of two semilattice indecomposable semigroups S_0 and S_1 such that $S_0S_1 \subseteq S_0$.

Lemma 7.4 *([55]) If S_1 is an abelian group, then either S_0 is a commutative nil Δ-semigroup and $|S_1| = 1$ or $|S_0| = 1$.*

Proof. By Theorem 3.22, the assertion holds if S is commutative. Assume that S is not commutative. Let e denote the identity element of S_1. As S is an $\mathcal{R}$-commutative semigroup, by Lemma 5.1, eS is a two-sided ideal of S and $eS \cap S_1 \neq \emptyset$, $eS \cap S_0 \neq \emptyset$. As S is a Δ-semigroup, eS must be equal to S. So e is a left identity element of S. We show that S_0 has an idempotent element. Assume, in an indirect way, that S_0 has no idempotent elements. By Lemma 7.2,

$$\sigma = id_S.$$

So, By Lemma 5.3, S_0 is right cancellative. Let a, b be arbitrary elements of S_0. Then

$$aeb = eaeb = eab.$$

As $ae, ea \in S_0$ and S_0 is right cancellative, we get

$$ae = ea.$$

Consequently, e is a (two-sided) identity element of S. As S is conditionally commutative, it follows that it is commutative which is a contradiction. Consequently, S_0 is a conditionally commutative archimedean semigroup with idempotent elements. There are two cases.

Consider the case when S_0 has a zero element. Then S_0 is a nil semigroup. Assume $|S_1| = 1$. We show that e is an identity element of S. Let a be an arbitrary element of S_0. As S is an $\mathcal{R}$-commutative semigroup,

$$a = ea = aex$$

for some $x \in S^1$. We may assume $x \in S_0$. Then

$$a = aex = ax$$

and so

$$a = ax^n$$

for every positive integer n. As S_0 is a nil semigroup, $a = 0$ and so

$$ea = ae = 0.$$

Thus e is a two-sided identity element of S. Hence S is a commutative semigroup which is a contradiction. So we may assume $|S_1| > 1$. By Theorem 1.59, $|S_0| = 1$.

Consider the case when S_0 has no zero element. Let f be an idempotent element of S_0. Then $M = S_0 f S_0$ is the kernel of S_0 and M is simple. It is evident that M is conditionally commutative. Since $M^2 = M$ then, by Theorem 1.14, M is an ideal of S. By Lemma 5.2, M is also $\mathcal{R}$-commutative. Then, by Theorem 7.1, M is a direct product of a commutative group G and a right zero semigroup R. As S is a Δ-semigroup and S_0 is a proper ideal of S, $|G| = 1$ (by Theorem 1.52). Thus $M = R$ and so S_0 is a nil extension of the (non-trivial) right zero semigroup R. By Lemma 7.3, $S_0 = R$. It can be easily verified that

$$\eta = \{(a,b) \in S \times S : (\forall r \in R)\ ra = rb\}$$

is a congruence on S and $\eta|R = id_R$. As S is a Δ-semigroup, R is a dense ideal of S. Thus

$$\eta = id_S.$$

Let g be an arbitrary element of S_1. Then

$$eg = ge$$

(e is the identity element of S_1). As S is conditionally commutative, we get

$$erg = gre$$

for all $r \in R$, from which we get

$$rerg = rgre.$$

As R is right zero and $re, rg \in R$, we get

$$rg = re$$

for all $r \in R$. Thus

$$(g, e) \in \eta,$$

that is,

$$g = e.$$

So $|S_1| = 1$. Let r be an arbitrary element of R. Then

$$re \in R$$

and

$$(re)e = re.$$

So there is an element r_0 of R such that

$$r_0e = r_0 = er_0.$$

As S is conditionally commutative, we have

$$re = r_0re = err_0 = r_0$$

for all $r \in R$. So

$$Re = \{r_0\}.$$

Let α be the equivalence relation on S such that

$$\alpha = \{(a,b) \in S \times S: \ a,b \in \{e,r_0\} \text{ or } a = b\}$$

We show that α is a congruence on S. Let $(a,b) \in \alpha$. We may assume that $a \neq b$. Then, for example, $a = e$ and $b = r_0$. Let r be an arbitrary element of R. Then

$$er = r = r_0r,$$

$$re = r_0 = rr_0,$$

$$e^2 = e \ \alpha \ r_0 = er_0$$

and

$$e^2 = e \ \alpha \ r_0e.$$

So

$$(ax, bx) \in \alpha$$

and

$$(xa, xb) \in \alpha$$

for all $x \in S$. So α is a congruence on S. Let β denote the least semilattice congruence on S. As S is a Δ-semigroup and $e \in \{e, r_0\}$, we have

$$\beta \subseteq \alpha$$

which implies

$$R \subseteq \{e, r_0\}.$$

So $|R| = 1$. Thus the theorem is proved. $\square$

Theorem 7.6 *([55]) S is a semilattice decomposable $\mathcal{RC}$-commutative Δ-semigroup if and only if it satisfies one of the following conditions.*

(i) S is isomorphic to G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime).

(ii) S is isomorphic to R^0, where R is a two-element right zero semigroup.

(iii) S is isomorphic to N^1, where N is a commutative nil semigroup whose ideals form a chain with respect to inclusion.

Proof. Let S be a semilattice decomposable $\mathcal{RC}$-commutative Δ-semigroup. Then, by Remark 1.2, S is a semilattice of two semilattice indecomposable subsemigroups S_1 and S_0 with $S_0S_1 \subseteq S_0$. It is easy to see that S_1 is $\mathcal{R}$-commutative. We can suppose that S is not commutative (the commutative semilattice decomposable Δ-semigroups are exactly semigroups which satisfiy either (i) or (iii)). By Theorem 1.51, S_1^0 is a Δ-semigroup and so, by Remark 1.1, S_1 is a semilattice indecomposable $\mathcal{RC}$-commutative Δ-semigroup. Then S_1 satisfies one of the conditions of Theorem 7.5.

Consider the case when S_1 is a non-trivial subgroup of a quasicyclic p-group (p is a prime). Then, by Lemma 7.4, $|S_0| = 0$. Thus S is isomorphic to G^0 and so (i) is satisfied.

Consider the case when S_1 is a nil semigroup. Then, by Theorem 1.57, $|S_1| = 1$. Assume $S_1 = \{e\}$. As S is $\mathcal{R}$-commutative, eS is a two-sided ideal of S and

$$eS \cap S_1 \neq \emptyset,$$
$$eS \cap S_0 \neq \emptyset.$$

As S is a Δ-semigroup, eS must be equal to S. So e is a left identity element of S.

We show that S_0 has an idempotent element. Assume, in an indirect way, that S_0 has no idempotent elements. By Lemma 7.2, $\sigma = id_S$. So, by Lemma 5.3, S_0 is right cancellative. Let a, b be arbitrary elements of S_0. Then

$$aeb = eaeb = eab.$$

As $ae, ea \in S_0$ and S_0 is right cancellative, we get

$$ae = ea.$$

As $ea = a$, we get that e is a (two-sided) identity element of S. As S is conditionally commutative, it follows that it is commutative which contradicts our assumption for S. Consequently S_0 is a conditionally commutative archimedean semigroup containing idempotent elements. By Theorem 6.14, S_0 is an ideal extension of a Rees matrix semigroup K over an abelian group by a nil semigroup N.

Consider the case when $|K| = 1$, that is, S_0 is a nil semigroup. We show that e is an identity element of S. Let a be an arbitrary element of S_0. As S is an $\mathcal{R}$-commutative semigroup, $a = ea = aex$ for some $x \in S^1$. If $x \notin S_0$ then

$$ae = a.$$

If $x \in S_0$ then

$$a = aex = ax$$

and so

$$a = ax^n$$

for every positive integer n. As S_0 is a nil semigroup,

$$a = 0.$$

But
$$ea = ae = 0.$$
So e is a two-sided identity element of S. Thus S is a commutative semigroup which is a contradiction.

Consider the case when $|K| > 1$, that is, S_0 has no zero element. By Lemma 5.2, K is also $\mathcal{R}$-commutative. Then, by Theorem 7.1, K is a direct product of a commutative group G and a right zero semigroup R. As S is a Δ-semigroup and S_0 is a proper ideal of S, $|G| = 1$ (see Theorem 1.52). Thus $K = R$ and so S_0 is a nil extension of the (non-trivial) right zero semigroup R. By Lemma 7.3, $S_0 = R$. Let r be an arbitrary element of R. Then
$$re \in R$$
and
$$(re)e = re.$$
So there is an element r_0 of R such that
$$r_0 e = r_0 = e r_0.$$
As S is conditionally commutative, we have
$$re = r_0 re = err_0 = r_0$$
for all $r \in R$. So
$$Re = \{r_0\}.$$
Let α be the equivalence relation on S such that
$$\alpha = \{(a, b) \in S \times S : \ a, b \in \{e, r_0\} \ or \ a = b\}$$
We show that α is a congruence on S. Let $(a, b) \in \alpha$. We may assume that $a \neq b$. Then, for example, $a = e$ and $b = r_0$. It is clear that $(ea, eb) \in \alpha$ and $(ae, be) \in \alpha$, because e is a left identity of S and $Re = \{r_0\}$. Let r be an arbitrary element of R. Then
$$er = r = r_0 r,$$
$$re = r_0 = rr_0,$$
$$e^2 = e \ \alpha \ r_0 = er_0$$
and
$$e^2 = e \ \alpha \ r_0 e.$$
So
$$(ax, bx) \in \alpha$$
and
$$(xa, xb) \in \alpha$$

for all $x \in S$. So α is a congruence on S. Let η denote the least semilattice congruence on S (the η-classes are $\{e\}$ and R). As S is a Δ-semigroup and $e \in \{e, r_0\}$, we have

$$\eta \subseteq \alpha$$

which implies

$$R \subseteq \{e, r_0\}.$$

Thus

$$R = \{r_0\}$$

and so S is commutative. But this is a contradiction.

It remains to examine the case when S_1 is a two-element right zero semigroup. Let u and v denote the elements of S_1. As S is a Δ-semigroup and uS is an ideal of S, we have

$$uS = S.$$

Similarly,

$$vS = S.$$

So u and v are left identity elements of S. By Theorem 4.7,

$$\sigma = \{(a, b) \in S \times S : \ ab^n = b^{n+1}, \ ba^n = a^{n+1} \text{ for a positive integer } n\}$$

is a congruence on S and

$$(u, v) \in \sigma.$$

Then the Rees congruence of S modulo S_0 is contained by σ. So

$$(a, b) \in \sigma$$

for all elements $a, b \in S_0$. So

$$(a, a^2) \in \sigma,$$

that is,

$$a^{n+2} = a^{n+1}$$

for some positive integer n ($a \in S_0$). Consequently S_0 has an idempotent element. By Theorem 6.14, S_0 is an ideal extension of a Rees matrix semigroup K over a commutative group by a nil semigroup N. There are two cases.

Consider the case when $|K| = 1$, that is, S is isomorphic to N. Consider the following relation τ_u^* on S:

$$\tau_u^* = \{(a, b) \in S \times S : \ au = bu\}.$$

It is evident that τ_u^* is a left congruence on S. We show that τ_u^* is also right compatible. Assume $(a, b) \in \tau_u^*$ for some $a, b \in S$. Then

$$au = bu.$$

Let x be an arbitrary element of S. Then

$$ax = aux = bux = bx,$$

because u is a left identity element of S. So

$$axu = bxu,$$

that is, τ_u^* is right compatible. Consequently it is a congruence on S. It is evident that

$$(u,v) \in \tau_u^*.$$

As S is a Δ-semigroup, the Rees congruence of S modulo S_0 is contained by τ_u^*. So

$$(a,b) \in \tau_u^*$$

for all $a,b \in S_0$, that is,

$$au = bu$$

for all $a,b \in S_0$. It is evident that the zero of S_0 is a zero of S. So

$$0u = 0.$$

Thus

$$au = 0$$

for all $a \in S_0$. As S is an $\mathcal{R}$-commutative semigroup,

$$ua = aus$$

for some $s \in S^1$. Consequently

$$a = ua = 0.$$

Thus $|S_0| = 1$ and so S is a two-element right zero semigroup with a zero adjoined.

Consider the case when $|K| > 1$, that is, S_0 has no zero element. As K is simple, $K^2 = K$ and so it is an ideal of S. By Lemma 5.2, it is also $\mathcal{R}$-commutative. Then, by Theorem 7.1, K is a right abelian group, that is, K is a direct product of an abelian group G and a right zero semigroup R. By Theorem 1.52, $|G| = 1$. Thus $K = R$. By Lemma 7.3, $S_0 = R$ and so S is a band. By Theorem 1.61, $|R| = 1$ and so S is a two-element right zero semigroup with a zero adjoined. Thus the theorem is proved. □

We summarize our results:

Theorem 7.7 *([55]) A semigroup S is an $\mathcal{RC}$-commutative Δ-semigroup if and only if it satisfies one of the following conditions.*

(i) S is isomorphic to either G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime).

(ii) S is isomorphic to either R or R^0, where R is a two-element right zero semigroup.

(iii) S is isomorphic to either N or N^1, where N is a commutative nil semigroup whose ideals form a chain with respect to inclusion.

We note that our proofs are mainly based on the fact that the conditionally commutative semigroups satisfy the identity $axa^i = a^ixa$ for every positive integer i (≥ 2).

In [79], B. Pondělíček defined the notion of the generalized conditionally commutative (briefly, GC-commutative) semigroup as a semigroup satisfying the identity $axa^2 = a^2xa$. He shoved that every GC-commutative semigroup satisfies the identity $axa^i = a^ixa$ for every integer $i \geq 2$. Using this result, he proved that every GC-commutative Δ-semigroup which is a band of t-archimedean semigroups is weakly exponential. We note that these semigroups are examined in Chapter 14.

Definition 7.2 *For a positive integer n, a semigroup is called generalized conditionally n-commutative (or GC_n-commutative) if it satisfies the identity $a^nxa^i = a^ixa^n$ for every integer $i \geq 2$.*

Definition 7.3 *A semigroup which is $\mathcal{R}$-commutative and GC_n-commutative is called an RGC_n-commutative semigroup.*

Theorem 7.8 *([62]) A semigroup S is an RGC_n-commutative Δ-semigroup if and only if it satisfies one of the following conditions.*

(i) S is isomorphic to either G or G^0 , where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime).

(ii) S is isomorphic to R or R^0, where R is a two-element right zero semigroup.

(iii) S is isomorphic to N or N^1, where N is a commutative nil semigroup whose ideals form a chai with respect to inclusion.

We remark that Theorem 7.7 and Theorem 7.8 show that the subclasses of Δ-semigroups in the class of RGC_n-commutative semigroups and in the class of $\mathcal{RC}$-commutative ones are identical.

Chapter 8

Quasi commutative semigroups

A semigroup S is called left (right) quasi commutative if, for every $a, b \in S$, there is a positive integer r such that $ab = b^r a$ ($ab = ba^r$). A semigroup S is called σ-reflexive if $ab \in H$ implies $ba \in H$ for every $a, b \in S$ and every subsemigroup H of S. In this chapter it is proved that the left quasi commutative semigroups, the right quasi commutative semigroups and the σ-reflexive semigroups are the same. They are called quasi commutative semigroups. As a quasi commutative semigroup is also weakly commutative, they are semilattice of archimedean semigroups. As the commutative archimedean semigroups are describen in Chapter 3, here is considered only the non-commutative case. It is proved that a semigroup is a non-commutative quasi commutative archimedean semigroup containing at least one idempotent element if and only if it is an ideal extension of a hamiltonian group by a commutative nil semigroup. At the end of the chapter, the least weakly separative congruence of a quasi commutative semigroup is constructed. It is shown that, on a quasi commutative semigroup S, σ defined by $a \ \sigma \ b$ ($a, b \in S$) if and only if $a^{n+1} = ba^n$ and $b^{n+1} = ab^n$ for some positive integer n is the least weakly separative congruence.

Definition 8.1 *A semigroup S is said to be a left (right) quasi commutative semigroup if, for any $a, b \in S$, there is a positive integer r such that $ab = b^r a$ ($ab = ba^r$).*

Definition 8.2 *A semigroup is called a σ-reflexive semigroup if any subsemigroup of S is reflexive.*

Lemma 8.1 *([12]) A semigroup S is σ-reflexive if and only if, for every element $a, b \in S$, there is a positive integer m such that $ab = (ba)^m$.*

Proof. As ab is contained by the cyclic subsemigroup of a σ-reflexive semigroup S generated by the element ba, there is a positive integer m such that $ab = (ba)^m$. The converse statement is obvious.

Lemma 8.2 *Every left quasi commutative (right quasi commutative, σ-reflexive) semigroup is weakly commutative.*

Proof. By Definition 8.1 and Lemma 8.1, it is obvious. □

Corollary 8.1 *Every left quasi commutative (right quasi commutative, σ-reflexive) semigroup is a semilattice of left quasi commutative (right quasi commutative, σ-reflexive) archimedean semigroups.*

Proof. By Lemma 8.2 and Theorem 4.3, it is obvious. □

Lemma 8.3 *([12]) If a and b are arbitrary elements of a σ-reflexive semigroup S with $ab \neq ba$ then there is an integer $m > 1$ such that $ab = (ab)^m$.*

Proof. Let a and b be arbitrary elements of a σ-reflexive semigroup S with $ab \neq ba$. Then, By Lemma 8.1, $ab = (ba)^k$ and $ba = (ab)^n$ for some integers $k, n > 1$. Hence $ab = (ab)^{nk}$. □

Lemma 8.4 *([12]) The idempotents of a σ-reflexive semigroup are in the centre.*

Proof. Let $a \in S$, $e \in E_S$ be arbitrary elements of a σ-reflexive semigroup S. If $ae \neq ea$ then, by Lemma 8.3, $(ae)^m = ae$ for a least integer $m > 1$ and so the cyclic subsemigroup $\langle ae \rangle$ of S is a group whose identity element is $(ae)^{m-1}$. Clearly $\langle ea \rangle = \langle ae \rangle$ and $(ea)^m = ea$ from which we get $(ea)^{m-1} = (ae)^{m-1}$. Then

$$ae = (ae)^m = (ae)^{m-1}(ea)e = (ea)(ae)^{m-1}e = (ea)(ae)^{m-1} = ea$$

which is a contradiction. Hence e is in the centre of S □

Theorem 8.1 *([12]) For a group G the following are equivalent.*

(i) G is σ-reflexive.

(ii) G is left quasi commutative.

(iii) G is right quasi commutative.

(iv) Every subgroup of G is normal.

Proof. (i) implies (ii). Let G be a σ-reflexive group. Let H be an arbitrary subgroup of G. Let $g \in G$ and $h \in H$ be arbitrary elements. Then

$$g^{-1}gh = eh = h \in H$$

and so

$$ghg^{-1} \in H.$$

Hence H is a normal subgroup. We can suppose that G is not commutative. Then G is periodic (see p.191 of [30]). Thus, for every element $b \in G$, the subsemigroup $\langle b \rangle$ of G generated by b is a subgroup of G. Then $\langle b \rangle$ is a normal subgroup of G and so, for every $a \in G$, we have

$$a\langle b \rangle = \langle b \rangle a.$$

Thus

$$ab \in \langle b \rangle a$$

which implies that

$$ab = b^r a$$

for a positive integer r. Hence G is left quasi commutative.

(ii) implies (iii). Let G be a left quasi commutative group and $a, b \in G$ be arbitrary elements. Then

$$b^{-1}a = a^r b^{-1}$$

for some positive integer n. From this we get

$$ab = ba^r$$

which means that G is right quasi commutative.

(iii) implies (iv). Let G be a right quasi commutative group and H be an arbitrary subgroup of G. If $g \in G$ and $h \in H$ arbitrary elements then

$$hg = gh^r$$

and

$$hg^{-1} = g^{-1}h^s$$

for some positive integers r and s. From the second equation we get

$$gh = h^s g.$$

Thus

$$Hg \subseteq gH$$

and

$$gH \subseteq Hg.$$

Hence

$$gH = Hg,$$

that is, H is a normal subgroup.

(iv) implies (i). Let G be a group in which every subgroup is normal. We can suppose that G is not commutative (in the commutative case the proof is trivial). Then, by p.191 of [30], G is periodic. Thus the subsemigroup $\langle g \rangle$ of

G generated by an element $g \in G$ is a subgroup of G. Let A be an arbitrary subsemigroup of G. If $ba \in A$ then

$$ab \in \langle ba \rangle \subseteq A,$$

because $\langle ba \rangle$ is reflexive in G. Hence G is σ-reflexive. $\square$

Definition 8.3 *A non-commutative group in called a hamiltonian group if its every subgroup is normal.*

Lemma 8.5 *([12]) If $ab \neq ba$ for some elements a and b of a left quasi commutative (right quasi commutative, σ-reflexive) semigroup S then there is a hamiltonian subgroup of S with identity e which contains ab, ba, ae, be.*

Proof. Let S be a left quasi commutative semigroup and $a, b \in S$ be elements such that $ab \neq ba$. By definition,

$$ab = b^r a$$

and

$$ba = a^s b$$

for some integers $r, s > 1$. Then we have

$$\begin{aligned} ab = b^r a = b^{r-1} ba = b^{r-1} a^s b = b^{r-1} a^{s-1}(ab) \\ = (ab)^h b^{r-2}(ba^{s-1}) = (ab)^h b^{r-2} a^{k(s-1)} b \\ = ab((ab)^{h-1} b^{r-2} a^{k(s-1)-1})ab = ((ab)^{h-1} b^{r-2} a^{k(s-1)-1})^m (ab)^2 \\ = (ab)^{2n}((ab)^{h-1} b^{r-2} a^{k(s-1)-1})^m \end{aligned}$$

for some positive integers h, m, n (here we used the convention $x^0 y = y x^0 = y$, $x, y \in S$). By Lemma 8.2, S is weakly commutative. Then, by Theorem 4.3, it is a semilattice Y of archimedean semigroups S_i $(i \in Y)$. If $a \in S_i$ and $b \in S_j$ then

$$ab \in S_{ij}$$

and, by the previous equation,

$$ab \in (ab)^2 S_{ij} \cap S_{ij}(ab)^2.$$

Then, by Proposition IV.1.2 of [73], ab contained in a subgroup of S_{ij}. As S_{ij} has at most one idempotent, it contains an unique maximal subgroup G. Thus

$$ab, ba \in G.$$

Let e denote the identity of G. As G is an ideal in S_{ij} and $ae, be \in S_{ij}$, we get

$$ae, be \in G.$$

Since G is a left quasi commutative group then, by Theorem 8.1, every subgroup of G is normal. If G was commutative, then we would have

$$ab = (ab)e = (ae)(be) = (be)(ae) = (ba)e = ba$$

which would be a contradiction. Hence G is not commutative and so it is a hamiltonian group. Thus the assertion for left quasi commutative semigroups is proved. The proof is similar for a right quasi commutative semigroup.

Let S be a σ-reflexive semigroup and $ab \neq ba$ for some $a, b \in S$. Then, by Lemma 8.3,

$$ab = (ab)^m$$

and

$$ba = (ba)^n$$

for some integers $m, n > 1$. Thus the cyclic subsemigroups $\langle ab \rangle$ and $\langle ba \rangle$ generated by ab and ba, respectively, are groups. As S is a semilattice of archimedean semigroups, the statement follows as in the preceding case. □

Theorem 8.2 *([12]) For a semigroup S the following are eqiuvalent.*

(i) S is σ-reflexive.

(ii) S is left quasi commutative.

(iii) S is right quasi commutative.

Proof. (i) implies (ii). Let a and b be arbitrary elements of a σ-reflexive semigroup S with $ab \neq ba$. Then, by the previous theorem, there is a hamiltonian subgroup G of S such that $ab, ba, ae, be \in G$, where e denotes the identity of G. Since G is left quasi commutative (see Theorem 8.1), there is a positive integer r such that

$$ab = (ab)e = (ae)(be) = (be)^r(ae) = b^r(ae) = b^{r-1}(ba)e = b^r a$$

and therefore S is left quasi commutative. By a similar process it can be proved that (ii) implies (i) and (i) is equivalent to (iii). □

By the previous theorem we need not distinguish left and right quasi commutative (and σ-reflexive) semigroups.

Definition 8.4 *A semigroup will be called a quasi commutative semigroup if it is left quasi commutative or, equivalently, right quasi commutative or, equivalently, σ-reflexive.*

Theorem 8.3 *([12]) Every quasi commutative semigroup is strongly reversible.*

Proof. Let S be a quasi commutative semigroup and let $a, b \in S$ with $ab \neq ba$. By Theorem 8.5, there is a hamiltonian subgroup G of S such that

$$ab, ba, ae, be \in G,$$

where e is the identity of G. As a hamiltonian group is periodic, there is a positive integer n such that

$$(ab)^n = (ba)^n = (ae)^n = (be)^n = e.$$

Hence

$$(ab)^n = (ae)^n(be)^n = a^n eb^n e = a^n b^n e$$
$$= a^{n-1}abb^{n-1}e = a^{n-1}(ab)eb^{n-1} = a^{n-1}(ab)b^{n-1} = a^n b^n.$$

In the same way it follows that

$$(ab)^n = b^n a^n.$$

Thus S is strongly reversible. □

Theorem 8.4 *([12]) Every quasi commutative nil semigroup is commutative.*

Proof. As the unique maximal subgroup of a nil semigroup N contains only the zero of N, the assertion follows from Lemma 8.5. □

The commutative archimedean semigroups are described in Chapter 3. Next we deal with the non-commutative quasi commutative archimedean semigroups.

Theorem 8.5 *([12]) A semigroup is a non commutative quasi commutative archimedean semigroup if and only if it is an ideal extension of a hamiltonian group by a commutative nil semigroup.*

Proof. Let S be a non-commutative quasi commutative semigroup. Then, by Lemma 8.5, S has an idempotent e. As S is weakly commutative (see Lemma 8.2), S is an ideal extension of a group G by a nil semigroup N (see Theorem 4.5). By Theorem 8.2, S is σ-reflexive. Thus G and N are σ-reflexive. By Theorem 8.1, G is either abelian or hamiltonian. By Theorem 8.4, N is commutative. As an ideal extension of an abelian group by a commutative nil semigroup is commutative, G must be hamiltonian.

Conversely, let S be an ideal extension of a hamiltonian group G by a commutative nil semigroup. By Theorem 2.2, S is archimedean. As G is non-commutative, S is non-commutative. By Lemma 3 of [22], S is σ-reflexive and so, by Theorem 8.2, it is left quasi commutative. □

Corollary 8.2 *A non commutative quasi commutative archimedean semigroup is a periodic power joined semigroup.*

Theorem 8.6 *([12]) A quasi commutative semigroup S is a semilattice of power joined semigroups if and only if every group and group with zero homomorphic image of S is periodic.*

Corollary 8.3 *([12]) A periodic quasi commutative semigroup is a semilattice of power joined semigroups.*

Lemma 8.6 *([12]) Let S be an archimedean semigroup with idempotents. If the idempotents are in the center then S is t-archimedean.*

Theorem 8.7 *([12]) For a semigroup S the following are equivalent.*

(i) S is σ-reflexive (equivalently, quasi commutative).

(ii) The eventual idempotents of S are in the center; the maximal subgroups of S are quasi commutative and for any $a, b \in S$ with $ab \neq ba$, ab belongs to a subgroup of S.

(iii) S is a semilattice Y of quasi commutative archimedean semigroups S_α $(\alpha \in Y)$ and, for every $a, b \in S$ with $ab \neq ba$, ab belongs to a subgroup of S.

Proof. (i) implies (ii). This is obvious if S is commutative, and follows from Lemma 8.4 and Lemma 8.5 if S is not commutative.

(ii) implies (iii). Let $a, b \in S$ be arbitrary elements with $b = xay$ $(x, y \in S^1)$. Then

$$b^2 = xayxay.$$

If $(ay)(xa) = (xa)(ay)$ then

$$b^2 = x^2a^2y^2.$$

If $(ay)(xa) \neq (xa)(ay)$ then

$$(ay)(xa) \in G$$

for a subgroup G of S. Let e denote the identity of G. Then

$$b^2 = x(ayxa)y = xe(ayxa)y = x(ayxa)^{-1}(ayxa)^2y$$

$$= x(ayxa)^{-1}ayxa^2yxay.$$

Thus, whenever a divides b, a^2 divides a power of b. Then S is a Putcha semigroup and so, by Theorem 2.1, S is a semilattice Y of archimedean semigroups S_i $(i \in Y)$. We show that, for each $i \in Y$, S_i is σ-reflexive. Let $a, b \in S_i$ arbitrary elements with $ab \neq ba$. Then, by Lemma 8.5, there are maximal subgroups G_1 and G_2 of S such that

$$ab \in G_1$$

and

$$ba \in G_2.$$

By Corollary 2.1,

$$G_1, G_2 \subseteq S_i.$$

By condition, the idempotents of S are in the center. Thus, by Lemma 8.6, S_i is t-archimedean. Thus

$$G_1 = G_2.$$

Let e denote the idempotent element of G_1. As G_1 is an ideal of S_i, we have

$$ae, be \in G_1.$$

As G_1 is σ-reflexive, there is a positive integer n such that

$$ab = abe = aebe = (beae)^n = (bae)^n = (ba)^n.$$

Thus, by Lemma 8.1, S_i is σ-reflexive.

(iii) implies (i). Let $a, b \in S$ be arbitrary elements with $a \in S_i$ and $b \in S_j$, $ab \neq ba$. Then, there are subgroups G_1, G_2 of S such that

$$ab \in G_1$$

and

$$ba \in G_2.$$

Since $ab, ba \in S_{ij}$, it follows

$$G_1, G_2 \subseteq S_{ij}$$

(see Corollary 2.1). Since S_{ij} has a unique maximal subgroup G, we get

$$ab, ba \in G.$$

Let e denote the identity of G. Then

$$ae, be \in G,$$

because G is an ideal of S_{ij}. As G is σ-reflexive, there is a positive integer n such that

$$ab = abe = aebe = (beae)^n = (ba)^n.$$

Thus, by Lemma 8.1, S is σ-reflexive. □

Theorem 8.8 *([12]) Let S be a quasi commutative semigroup. Then $\sigma = \{(a,b) \in S \times S : a^{n+1} = ba^n,\ b^{n+1} = ab^n$ for some positive integer $n\}$ is the least weakly separative congruence on S.*

Proof. Since a quasi commutative semigroup is left weakly commutative then, by Theorem 4.7, σ is a weakly separative congruence on S. To show that σ is the least weakly separative congruence on S, consider an arbitrary weakly separative congruence ρ of S. It is clear that the factor semigroup $F = S/\rho$ is quasi commutative. By Theorem 8.3, F is strongly reversible. By Proposition 8 of [10], F is left and right separative. Assume $(a, b) \in \sigma$ $(a, b \in S)$. Then, denoting the ρ-class of S containing an element x of S by $[x]$, we have

$$[a]^{n+1} = [b][a]^n$$

and

$$[b]^{n+1} = [a][b]^n.$$

By Lemma II.6.3 of [73],

$$[a] = [b].$$

Hence

$$\sigma \subseteq \rho$$

and so σ is the least weakly separative congruence on S. $\square$

Chapter 9

Medial semigroups

In this chapter we deal with semigroups which satisfy the identity $xaby = xbay$. These semigroups are called medial semigroups. It is shown that every medial semigroup is a semilattice of medial archimedean semigroups. We show that the simple medial semigroups are exactly the rectangular abelian groups, and prove that a semigroup is medial archimedean and contains at least one idempotent element if and only if it is a retract extension of a rectangular abelian group by a medial nil semigroup. It is also shown that every medial archimedean semigroup without idempotent has a non-trivial group homomorphic image. We also deal with the regular medial semigroups. It is shown that they are those semigroups which are orthodox normal bands of abelian groups. We also give other equivalent conditions. It is proved that a medial semigroup is weakly separative, left separartive, right separative, or separative if and only if its archimedean components are weakly cancellative, left cancellative, right cancellative, or cancellative, respectively. It is shown that a medial weakly cancellative semigroup is embeddable into a rectangular abelian group. Moreover, a semigroup can be embedded in a semigroup which is a union of groups if and only if it is weakly separative. The least left separative congruence, the least right separative congruence, the least weakly separative congruence and the least separative congruence of a medial semigroup are also constructed. We deal with the subdirectly irreducible medial semigroups. It is proved that a semigroup is medial and subdirectly irreducible with a globally idempotent core if and only if it is isomorphic to either G or G^0 or F or R or R^0 or L or L^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime), F is a two-element semilattice, R is a two-element right zero semigroup and L is a two-element left zero semigroup. At the end of the chapter we describe the medial Δ-semigroups. It is shown that a semigroup is a medial Δ-semigroup if and only if it is isomorphic to either G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime), or a two-element semilattice, or R or R^0, where R is a two-element right zero semigroup, or L or L^0, where L is a two-element left zero semigroup, or a medial nil semigroup whose principal ideals form a chain with respect to inclusion, or a medial T1 semigroup.

Definition 9.1 *A semigroup is called a medial semigroup if it satisfies the identity $xaby = xbay$.*

Theorem 9.1 *Every finitely generated periodic medial semigroup is finite.*

Proof. By Theorem 1.1, it is obvious. □

Semilattice decomposition of medial semigroups

Theorem 9.2 *Every medial semigroup is a left and right Putcha semigroup.*

Proof. Let S be a medial semigroup and $a, b \in S$ be arbitrary elements with $b \in aS^1$, that is, $b = ax$ for some $x \in S^1$. Then

$$b^2 = (ax)^2 = a^2 x^2,$$

that is,

$$b^2 \in a^2 S^1.$$

Hence S is a left Putcha semigroup. We can prove, in a similar way, that S is a right Putcha semigroup. □

Theorem 9.3 *Every medial semigroup is a semilattice of medial archimedean semigroups.*

Proof. Let S be a medial semigroup. By Lemma 9.2, S is a left and right Putcha semigroup. Then, by Corollary 2.2, S is a semilattice Y of archimedean semigroups S_α $(\alpha \in Y)$. Clearly, the subsemigroups S_α are medial. □

Theorem 9.4 *([66]) Let S be a medial semigroup. S is a semilattice of power joined semigroups if and only if every right and left group and right and left group with zero homomorphic image of S is a periodic group and a periodic group with zero, respectively.*

Theorem 9.5 *([66]) Let S be a medial semigroup. The following are equivalent.*

(i) S is power joined.

(ii) Every subsemigroup of S is t-archimedean.

(iii) Every finitely generated subsemigroup of S is t-archimedean.

Theorem 9.6 *Every medial semigroup is a band of t-archimedean medial semigroups.*

Proof. Since a medial semigroup satisfies the identity $(ab)^3 = a^2b^2(ab) = (ab)a^2b^2$ then, by Theorem 1.8, it is a band of t-archimedean medial semigroups. □

Theorem 9.7 *([66]) Let S be a medial semigroup. S is a band of power joined semigroups if and only if every group and every group with zero homomorphic image of S is a periodic group and a periodic group with zero, respectively.*

Theorem 9.8 *([16]) A semigroup S is medial and simple if and only if it is a rectangular abelian group.*

Proof. Let S be a simple medial semigroup. By Theorem 9.2, S is a left and right Putcha semigroup. Then, by Theorem 2.3, S is completely simple and so S is isomorphic to a Rees matrix semigroup $\mathcal{M}(I, G, J; P)$ over a group G with a normalized sandwich matrix P. Let e denote the identity element of G. As P is normalized, there are elements $i_0 \in I$ and $j_0 \in J$ such that

$$p_{j,i_0} = p_{j_0,i} = e.$$

Then, for every $a, b \in G$,

$$\begin{aligned}(i_0, ab, j_0) &= (i_0, e, j_0)(i_0, a, j_0)(i_0, b, j_0)(i_0, e, j_0)\\ &= ((i_0, e, j_0)(i_0, b, j_0)(i_0, a, j_0)(i_0, e, j_0) = (i_0, ba, j_0).\end{aligned}$$

Hence

$$ab = ba,$$

that is, G is an abelian group. Let $i \in I$, $j \in J$ and $a, b \in G$ be arbitrary elements. Then

$$\begin{aligned}(i, p_{j,i}, j) &= (i, e, j)^2 = ((i, e, j_0)(i_0, e, j))^2 = (i, e, j_0)^2(i_0, e, j)^2\\ &= (i, e, j_0)(i_0, e, j) = (i, e, j)\end{aligned}$$

and so

$$p_{j,i} = e.$$

Hence S is a direct product of the rectangular band $I \times J$ and the abelian group G, that is, S is a rectangular abelian group. As the converse is obvious, the theorem is proved. □

Corollary 9.1 *A semigroup is medial and 0-simple if and only if it is a rectangular abelian group with a zero adjoined.*

Proof. Let S be a medial 0-simple semigroup. By Theorem 9.3, S is a semilattice Y of archimedean semigroups. Let $a, b \in S$ be arbitrary elements with $a, b \neq 0$. Then $S^1aS^1 = S$ and $S^1bS^1 = S$ and so

$$a \in S^1bS^1 \text{ and } b \in S^1aS^1.$$

Thus a and b are in the same η-class A of S, where η denotes the least semilattice congruence on S (see also Theorem 2.1). If 0 was in A then S would a nil semigroup which contradict the assumption that S is 0-simple. Consequently the η-classes of S are A and $\{0\}$. It is clear that A is simple. Hence, by the previous theorem, A is a rectangular abelian group. Thus S is a rectangular abelian group with a zero adjoined. As the converse is trivial, the corollary is proved. □

Theorem 9.9 *([16]) A semigroup S is a medial archimedean semigroup containing at least one idempotent element if and only if it is a retract extension of a rectangular abelian group by a medial nil semigroup.*

Proof. Let S be a medial archimedean semigroup containing at least one idempotent element. Since S is a left and right Putcha semigroup (see Theorem 9.2) then, by Theorem 2.4, it is a retract extension of a completely simple semigroup K by the nil semigroup $N = S/K$. It is clear that K and N are medial. By Theorem 9.8, K is a rectangular abelian group. Hence S is a retract extension of a rectangular abelian group K by the medial nil semigroup N.

Conversely, let the semigroup S be a retract extension of a rectangular abelian group and a medial nil semigroup. By Theorem 2.2, S is an archimedean semigroup containing at least one idempotent. Since a rectangular abelian group is medial and the medial semigroups form a variety then, by Theorem 1.40, S is medial. □

Theorem 9.10 *The following conditions on a semigroup S are equivalent.*

(i) S is medial and regular.

(ii) S is an orthodox normal band of abelian groups.

(iii) S is a strong semilattice of rectangular abelian groups.

(iv) S is a spined pruduct of a normal band and a semilattice of abelian groups.

Proof. (i) implies (ii). Let S be a medial and regular semigroup. Then, by Theorem 9.3, it is a semilattice Y of archimedean medial semigroups S_i, $i \in Y$. As S is regular, each S_i is regular and so has an idempotent element. Then, by Theorem 9.9, each S_i is a retract extension of a rectangular abelian group $K_i = B_i \times G_i$ (B_i is a rectangular band and G_i is an abelian group) by a medial nil semigroup. As K_i contains all idempotent elements of S_i, we can conclude that $S_i = K_i$. Hence S is a semilattice Y of rectangular abelian groups K_i, $i \in Y$. By Theorem 1.27, S is an orthogroup and so the set of all idempotents of S is a subsemigroup. It is clear that each K_i is a union of abelian groups $f \times G_i$, where $f \in B_i$. Hence S is a disjoint union of abelian groups. The idempotent elements of S are $f \times e_i$ ($i \in Y$), where e_i denotes the identity element of the group G_i and $f \in B_i$ be arbitrary. Identify B_i with $B_i \times e_i$. Then

$$B = \cup_{i \in Y} B_i$$

can be considered as the semigroup of all idempotents of S. As S is medial, B is a normal band. To show that S is an orthodox band of (maximal) subgroups $f \times G_i$ ($i \in Y$, $f \in B_i$), by Theorem 1.29, it is sufficient to show that the Green's equivalence $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$ is a congruence on S. Assume $aS^1 = bS^1$ for some $a, b \in S$. Then $a = bx$ and $b = ay$ for some $x, y \in S^1$. We can suppose that $x, y \in S$. Let $s \in S$ be arbitrary. As S is regular, $sts = s$ for some $t \in S$. Thus

$$as = bxsts = bsxts$$

and

$$bs = aysts = asyts,$$

that is,

$$as \in bsS^1$$

and

$$bs \in asS^1.$$

Hence the left congruence $\mathcal{R}$ is a congruence on S. We can prove, in a similar way, that the right congruence $\mathcal{L}$ is a congruence on S. Hence $\mathcal{H}$ is a congruence on S.

(ii) implies (iii) and (iii) implies (iv) by Theorem 1.32. It is obvious that (iv) implies (i). Thus the theorem is proved. □

Theorem 9.11 *Every medial archimedean semigroup without idempotent element has a non-trivial group homomorphic image.*

Proof. Let S be a medial archimedean semigroup without idempotent element. It is clear that S satisfies the identity $(ab)^2 = a^2b^2$. Then, by Theorem 1.42, the principal right congruence $\mathcal{R}_{S_a}$ is a group congruence on S for every $a \in S$. If $S_a \neq S$ then $S/\mathcal{R}_{S_a}$ is a non-trivial group homomorphic image of S. Consider the case $S_a = S$. In this case, for every $x \in S$, there are positive integers i, j, k such that $a^i x a^j = a^k$. Assume that

$$a^p x a^q = a^m$$

also holds for some positive integers p, q, m. Then

$$a^{k+p+q} = a^{i+p} x a^{j+q} = a^{m+i+j}$$

from which we get

$$m - (p+q) = k - (i+j),$$

because S does not contain idempotent element. Thus the integer $k - (i+j)$ is well-determined by the element x. Let φ be the following mapping.

$$\varphi : \ x \in S \ \to k - (i+j),$$

where $k - (i+j)$ is the integer which is determined by x as above. Since $S_a = S$ then φ is defined on S, and it maps S into the additive semigroup of integers. We show that φ is a homomorphism. Let $x, y \in S$ be arbitrary. Assume

$$a^i x a^j = a^k$$

and

$$a^m y a^n = a^h$$

for some positive integers i, j, k, m, n, h. Then

$$a^{k+h} = a^i x a^j a^m y a^n = a^{i+j} x y a^{m+n}$$

and so

$$\varphi(xy) = k + h - (i + j + m + n) = k - (i + j) + h - (m + n) = \varphi(x) + \varphi(y).$$

Hence φ is a homomorphism of S into the additive semigroup of integers. It is clear that $\varphi(a) = 1$. Thus $\varphi(S)$ equals either the additive semigroup of all integers or the additive semigroup of all non-negative integers or the additive semigroup of all positive integers. Since all of these additive semigroups have non-trivial group homomorphic images, the theorem is proved. □

Cancellation and separativity

Let S be a medial left separative semigroup. Assume $a^2 = ab = b^2$ for some $a, b \in S$. Then

$$(ab)(ba) = ab^2a = a^4 = (ab)^2$$

and

$$(ba)(ab) = ba^2b = b^4 = b^2a^2 = (ba)^2.$$

As S is left separative, we get $ab = ba$. Thus $ab = a^2$ and $ba = b^2$. Using again the left separativity of S, we get $a = b$. Hence S is weakly separative. We can prove, in a similar way, that a medial right separative semigroup is weakly separative. We note that a rectangular band $L \times R$ with $|L| \geq 2$ and $|R| \geq 2$ shows that the converse is false.

Lemma 9.1 *([16]) Let S be a weakly separative medial semigroup and x, y be arbitrary elements of S such that $x^{n+1} = x^n y$ ($x^{n+1} = yx^n$) for some positive integer n. Then $xy = x^2$ ($yx = x^2$).*

Proof. Let S be a weakly separative medial semigroup. If, for an integer $n \geq 2$, $x^{n+1} = x^n y$ ($x, y \in S$) then

$$\begin{aligned}(x^n)^2 &= x^{n-1}x^{n+1} = x^{n-1}x^n y = x^n(x^{n-1}y)\\ &= x^{n+1}x^{n-2}y = x^n y x^{n-2} y = (x^{n-1}y)^2,\end{aligned}$$

where $x^{n-2}y = y$ if $n = 2$. Thus

$$x^n = x^{n-1}y,$$

because S is weakly separative. Repeating this process $n - 1$ times, we get

$$x^2 = xy.$$

Similarly, $x^{n+1} = yx^n$ implies $x^2 = yx$ for every positive integer n. □

Lemma 9.2 *If S is a medial archimedean semigroup then, for every $a, x, y \in S$, $ax = ay$ ($xa = ya$) implies $x^{n+1} = x^n y$ and $y^{n+1} = y^n x$ ($x^{n+1} = yx^n$ and $y^{n+1} = xy^n$) for a positive integer n.*

Proof. Let S be a medial archimedean semigroup and $a, x, y \in S$ be arbitrary elements with $ax = ay$. Then there are elements $u, v, z, w \in S$ and a positive integer n such that $x^n = uav$ and $y^n = zaw$. Thus

$$x^{n+1} = uavx = uvax = uvay = uavy = x^n y$$

and

$$y^{n+1} = zawy = zway = zwax = zawx = y^n x.$$

We can prove, in a similar way, that $xa = ya$ implies $x^{n+1} = yx^n$ and $y^{n+1} = xy^n$ for a positive integer n. □

Theorem 9.12 *([16]) If S is a medial semigroup with archimedean components S_α $(\alpha \in Y)$ then*

(i) S is weakly separative if and only if each S_α is weakly cancellative.

(ii) S is left (right) separative if and only if each S_α is left (right) cancellative.

(iii) S is separative if and only if each S_α is cancellative.

Proof. To prove (i), first assume that S is a weakly separative medial semigroup. Let S_α, $\alpha \in Y$ be an arbitrary archimedean component of S and $a, b, x, y \in S_\alpha$ be arbitrary elements with $ax = ay$, $xa = ya$. Then, by Lemma 9.2,

$$x^{m+1} = x^m y$$

and

$$y^{n+1} = xy^n$$

for some positive integers m and n. Then, by Lemma 9.1,

$$x^2 = xy = y^2.$$

As S is weakly separative, we get

$$x = y.$$

Conversely, assume that each S_α is weakly cancellative. Then, by Lemma 1.1, S_α satisfies the condition that, for every $a, b, x, y \in S_\alpha$, $ax = ay$ and $xb = yb$ together imply $x = y$. Assume $x^2 = xy = y^2$ for some $x, y \in S$. Then there is a $\gamma \in Y$ such that

$$x, y, xy \in S_\gamma.$$

Since $xx = xy$ and $xy = yy$ in S_γ, we get $x = y$.

To prove (ii), first assume that S is a left separative medial semigroup. Let S_α, $\alpha \in Y$ be an arbitrary archimedean component of S and $a, b, x, y \in S_\alpha$ be arbitrary elements with $ax = ay$. Then, by Lemma 9.2,

$$x^{n+1} = x^n y$$

and

$$y^{n+1} = y^n x$$

for a positive integer n. By Lemma 9.1,

$$xy = x^2$$

and

$$yx = y^2.$$

As S is left separative, we get $x = y$. Hence S_α is left cancellative.

Conversely, assume that each S_α is left cancellative. Assume $xy = x^2$ and $yx = y^2$ for some $x, y \in S$. Then there is a $\gamma \in Y$ such that $x, y, xy \in S_\gamma$. Since S_γ is left cancellative, we get $x = y$.

We can prove, in a similar way, that S is right separative if and only if each S_α is right cancellative.

The proof of (iii) follows immediately from (ii). □

Theorem 9.13 *([16]) Let S be a medial weakly cancellative semigroup. Then S can be embedded into a rectangular abelian group.*

Proof. Let S be a medial weakly cancellative semigroup. Then, by Lemma 1.1, S satisfies the condition that, for every $a, b, x, y \in S$, $ax = bx$ and $ya = yb$ imply $a = b$. Define a relation ξ on the semigroup $S^* = S \times S \times S$ by

$$(a, b, c)\xi(a', b', c')$$

if and only if

$$cab'c' = c'a'bc.$$

Reflexivity and symmetry of ξ follows immediately. To prove transitivity, let

$$(a_1, b_1, c_1)\xi(a_2, b_2, c_2)$$

and

$$(a_2, b_2, c_2)\xi(a_3, b_3, c_3).$$

By the definition of ξ,

$$c_1 a_1 b_2 c_2 = c_2 a_2 b_1 c_1$$

and

$$c_2 a_2 b_3 c_3 = c_3 a_3 b_2 c_2.$$

Then

$$\begin{aligned} c_2 c_2 a_2 (c_1 a_1 b_3 c_3) &= c_2 c_1 a_1 (c_2 a_2 b_3 c_3) \\ = c_2 c_1 a_1 (c_3 a_3 b_2 c_2) &= c_2 c_3 a_3 (c_1 a_1 b_2 c_2) \\ = c_2 c_3 a_3 (c_2 a_2 b_1 c_1) &= c_2 c_2 a_2 (c_3 a_3 b_1 c_1). \end{aligned}$$

Similarly,

$$(c_1 a_1 b_3 c_3) b_2 c_2 c_2 = (c_3 a_3 b_1 c_1) b_2 c_2 c_2.$$

By our assumption on S,

$$c_1a_1b_3c_3 = c_3a_3b_1c_1,$$

that is,

$$(a_1, b_1, c_1)\xi(a_3, b_3, c_3).$$

The proof that ξ is compatible involves a routine application of mediality. It is clear that the factor semigroup S^*/ξ is medial. Let $[a,b,c]_\xi$ denote the ξ-class of S^* containing the element $(a,b,c) \in S^*$. It is clear that

$$[b,a,g]_\xi[a,b,c]_\xi[e,f,g]_\xi = [e,f,g]_\xi$$

for every $[a,b,c]_\xi, [e,f,g]_\xi \in S^*/\xi$. Hence S^*/ξ is simple. Then, by Theorem 9.8, S^*/ξ is a rectangular abelian group. Let ϕ be a mapping of S to S^*/ξ defined by

$$\phi: \ a \mapsto [a, a^2, a]_\xi.$$

Since

$$\phi(a)\phi(b) = [a,a^2,a]_\xi[b,b^2,b]_\xi$$

$$= [ab, a^2b^2, ab]_\xi = [ab, (ab)^2, ab]_\xi = \phi(ab)$$

then ϕ is a homomorphism. To show that ϕ is an isomorphism, assume

$$\phi(a) = \phi(b)$$

for some $a, b \in S$. Then

$$(a, a^2, a)\xi(b, b^2, b).$$

Thus

$$a^2b^3 = aab^2b = bba^2a = b^2a^3.$$

Consequently,

$$(b^3a^2)a = b(b^2a^3) = b(a^2b^3) = (b^3a^2)b,$$

and

$$a(a^2b^5) = (a^2b^3)(ab^2) = (b^2a^3)(ab^2) = b(b^2a^3)ab$$

$$= b(a^2b^3)ab = b(b^2a^3)b^2 = b(a^2b^3)b^2 = b(a^2b^5).$$

By our assumption on S,

$$a = b.$$

Hence ϕ is an isomorphism. $\square$

Theorem 9.14 *([16]) A medial semigroup can be embedded into a semigroup which is a union of groups if and only if it is weakly separative.*

Proof. It is easy to see that if a semigroup S is embeddable into a semigroup which is a union of groups then S is weakly separative.

Conversely, let S be a weakly separative medial semigroup. By Theorem 9.3, and Theorem 9.12, S is a semilattice Y of weakly cancellative medial archimedean semigroups S_i $(i \in Y)$. Then, by Theorem 9.13, for every $i \in Y$, there is an isomorphism ϕ_i of S_i into the rectangular abelian group

$$R_i = S_i^*/\xi_i,$$

where S_i^* denotes the semigroup $S_i \times S_i \times S_i$ and ξ_i a congruence on S_i^* defined by

$$(a, b, c)\xi_i(a', b', c')$$

if and only if

$$cab'c' = c'a'bc$$

$(a, b, c, a', b', c' \in S_i)$. We can suppose $R_i \cap R_j = \emptyset$ if $i \neq j$. Let

$$R = \cup_{i \in Y} R_i.$$

On R we define a product by

$$[a, b, c]_{\xi_i}[x, y, z]_{\xi_j} = [ax, by, cz]_{\xi_{ij}}.$$

To show that the product is well defined, let

$$(a, b, c)\xi_i(a', b', c')$$

and

$$(x, y, z)\xi_j(x', y', z').$$

Then

$$(ax, by, cz)\xi_{ij}(a'x', b'y', c'z'),$$

because

$$\begin{aligned}(cz)(ax)(b'y')(c'z') &= (cab'c')(zxy'z') \\ &= (c'a'bc)(z'x'yz) = (c'z')(a'x')(by)(cz).\end{aligned}$$

Hence

$$[ax, by, cz]_{\xi_{ij}} = [a'x', b'y', c'z']_{\xi_{ij}}.$$

The operation is obviously associative. Finally, define a mapping $\Phi : S \to R$ by

$$(a)\Phi = \phi_i(a) = [a, a^2, a]_{\xi_i} \in R_i$$

if $a \in S_i$. Since the restriction of Φ to each S_i is injective and since $(S_i)\Phi \cap (S_j)\Phi = \emptyset$ if $i \neq j$, Φ is injective. It is also a homomorphism, because

$$\begin{aligned}(ab)\Phi &= [ab, (ab)^2, ab]_{\xi_{ij}} = [ab, a^2b^2, ab]_{\xi_{ij}} \\ &= [a, a^2, a]_{\xi_i}[b, b^2, b]_{\xi_j} = (a)\Phi(b)\Phi.\end{aligned}$$

Thus S is embedded in the union of groups. □

Next, we give equivalent conditions for a semigroup to be medial and weakly separative.

Theorem 9.15 *(Th. IV.3.5 of [75]) The following conditions on a semigroup S are equivalent.*

(i) S is medial and weakly separative.

(ii) S is a normal band of cancellative semigroups and satisfies the identity $(xy)^2 = x^2y^2$.

(iii) S is embeddable into a strong semilattice of rectangular abelian groups.

(iv) S is a subdirect product of a normal band and a commutative separative semigroup.

Theorem 9.16 *([16]) Let S be a medial semigroup. Then*

$$\tau = \{(a,b) \in S \times S : \ a^{n+1} = a^n b, \ b^{n+1} = b^n a \text{ for some positive integer } n\}$$

is the smallest left separative congruence on S,

$$\sigma = \{(a,b) \in S \times S : \ a^{n+1} = ba^n, \ b^{n+1} = ab^n \text{ for some positive integer } n\}$$

is the smallest right separative congruence on S, $\pi = \tau \cap \sigma$ is the smallest weakly separative congruence on S and

$$\delta = \{(a,b) \in S \times S : \ a^{n+2} = a^n ba, \ b^{n+2} = b^n ab \text{ for some positive integer } n\}$$

is the smallest separative congruence on S.

Proof. By Lemma 4.1, τ is an equivalence on S. We shall show that τ is a congruence on S. Let $a, b \in S$ be arbitrary elements with

$$a \ \tau \ b.$$

Then

$$a^n b = a^{n+1}$$

and

$$b^n a = b^{n+1}$$

for a positive integer n. Let s be an arbitrary element of S. Then

$$(as)^{n+1} bs = as^{n+1} a^n bs = as^{n+1} a^{n+1} s = (as)^{n+2}$$

and

$$(sa)^n (sb) = s^{n+1} a^n b = s^{n+1} a^{n+1} = (sa)^{n+1}.$$

We can prove, in a similar way, that

$$(bs)^{n+1}(as) = (bs)^{n+2}$$

and

$$(sa)^n (sb) = s^{n+1} a^n b = s^{n+1} b^{n+1} = (sb)^{n+1}.$$

Hence τ is a congruence on S. We show that τ is left separative. Assume

$$ab \ \tau \ a^2$$

and

$$ba \ \tau \ b^2$$

for some $a, b \in S$. Then

$$(a^2)^n(ab) = (a^2)^{n+1},$$
$$(b^2)^n(ba) = (b^2)^{n+1},$$

and so

$$a^{2n+3}b = a^{2n+4}$$

and

$$b^{2n+3}a = b^{2n+4}.$$

Hence

$$a \ \tau \ b.$$

It remains to show that τ is the smallest left separative congruence on S. Let α be an arbitrary left separative congruence on S and let

$$a \ \tau \ b$$

for arbitrary $a, b \in S$. Then

$$a^n b = a^{n+1}$$

and

$$b^n a = b^{n+1}$$

for some positive integer n. Thus

$$a^n b \ \tau \ a^{n+1}$$

and

$$b^n a \ \tau \ b^{n+1}.$$

Let $Q = S/\tau$ and let $[a]$ denote the τ-class of S containing the element a of S. Then

$$[a]^n[b] = [a]^{n+1}$$

and

$$[b]^n[a] = [b]^{n+1}.$$

Let η denote the least semilattice congruence on Q. Then

$$[a] \ \eta \ [a]^{n+1}$$
$$= [a]^n[b] \ \eta \ [a][b] \ \eta [b][a] \ \eta \ [b]^n[a]$$
$$= [b]^{n+1} \ \eta \ [b],$$

that is,

$$[a]\eta[b].$$

By Theorem 9.11, the η-classes of Q are left cancellative. Hence $a = b$.

As the proofs are similar in the other cases, the theorem is proved. □

Theorem 9.17 *(Th. III.4.7 of [75]) The following condition on a semigroup S are equivalent.*

(i) S is medial and weakly cancellative.

(ii) S is a rectangular band of cancellative semigroups and satisfies the identity $(ab)^2 = a^2b^2$.

(iii) S is embeddable into a rectangular abelian group.

(iv) S is a subdirect product of a rectangular band and a commutative cancellative semigroup.

Remark 9.1 A medial right (left) cancellative semigroup satisfies the identity $axy = ayx$ $(xya = yxa)$. Semigroups satisfying this identity are examined in the next chapter.

Subdirectly irreducible medial semigroups

Theorem 9.18 *A semigroup S is a subdirectly irreducible medial semigroup with a globally idempotent core if and only if it satisfies one of the following conditions.*

(i) S is isomorphic to either G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime).

(ii) S is a two-element semilattice.

(iii) S is isomorphic to R or R^0, where R is a two-element right zero semigroup.

(iv) S is isomorphic to L or L^0, where L is a two-element left zero semigroup.

Proof. Let S be a subdirectly irreducible medial semigroup with a globally idempotent core K. First, assume that S has no zero element. Then K is simple. As K is also medial, by Theorem 9.8, it is a rectangular abelian group, that is, $K = L \times R \times G$, where L is a left zero semigroup, R is a right zero semigroup and G is an abelian group. By Corollary 1.4, we have either $K = L$ or $K = R$ or $K = G$.

Assume $K = G$. Then S is a homogroup and so, by Theorem 1.47, it is a subdirectly irreducible abelian group. Then, by Theorem 3.14, S is a non-trivial subgroup of a quasicyclic p-group (p is a prime).

Assume $K = L$. It can be easily verified that

$$\delta = \{(a,b) \in S \times S : \ ax = bx \text{ for all } x \in L\}$$

is a congruence on S such that its restriction to L is id_L. As L is a dense ideal of S, we get

$$\delta = id_S.$$

Let $x \in L$ and $s \in S$ be arbitrary elements. Then

$$sx = (sx)^2 = s^2x^2 = s^2x,$$

that is, $(s, s^2) \in \delta$. Hence $s = s^2$. Thus S is a band. Let $x_1, x_2 \in L$ be arbitrary elements. Then, for every $s \in S$,

$$sx_1 = (sx_1)(sx_2x_1) = (sx_2x_1)(sx_1) = sx_2x_1,$$

that is $(s, sx_2) \in \delta$ for every $x_2 \in L$. Thus $s = sx_2 \in L$. So $S = L$, that is, S is a left zero semigroup. As S is subdirectly irreducible, by Theorem 1.48, it has two elements. We can prove, in a similar way, that S is a two-element right zero semigroup if $K = R$. Summarizing our results, S is either a non-trivial subgroup of a quasicyclic p-group (p is a prime) or a two-element left zero semigroup or a two-element right zero semigroup.

Next, assume that S has a zero element 0. As S is a medial semigroup, it is a semilattice of medial archimedean semigroups. Let S_0 denote the archimedean component of S containing 0. Let $a, b \in S$ be arbitrary elements with $a \neq 0$ and $b \neq 0$ and $ab = 0$. Let $B = \{x \in S : ax = 0\}$. It is clear that B is a right ideal of S and $b \in B$. We show that B is also a left ideal. Let $s \in S$, $x \in B$ be arbitrary. Then asx and sax are in the same archimedean component. Hence $asx \in S_0$. If $|S_0| = 1$ then $asx = 0$ and so $sx \in B$. Assume $|S_0| > 1$. Then S_0 is a non-trivial ideal of S and so it contains the core K of S. As S_0 is archimedean and contains the zero of S, it is a nil semigroup. As K is 0-simple and medial, by Corollary 9.1, it is a rectangular abelian group with a zero adjoined. But this contradicts the fact that S_0 is a nil semigroup. Hence B is an ideal of S. As B contains at least two elements, $K \subseteq B$ and so $aK = \{0\}$. Let $A = \{y \in S : yK = \{0\}\}$. It is clear that A is a left ideal of S and $a \in A$. Since $ysK \subseteq yK = \{0\}$ for every $s \in S$ and $y \in A$ then A is also a right ideal of S and so it is an ideal of S. As A has at least two elements, $K \subseteq A$ and so $K^2 = \{0\}$ which contradicts the assumption that the core K of S is globally idempotent. Consequently, the set S' of all non-zero elements of S is a subsemigroup and so S is a semigroup S'^0 with a zero adjoined. If $|S'| = 1$ then S is a two-element semilattice. If $|S'| > 1$ then S' is a subdirectly irreducible medial semigroup without zero. Thus the core S' is globally idempotent. Using also the first part of this proof, we get that S is either G^0 or L^0 or R^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime), L is a two-element left zero semigroup and R is a two-element right zero semigroup. As the semigroups listed in the theorem are subdirectly irreducible medial semigroups, the theorem is proved. □

Theorem 9.19 *A medial semigroup with zero and a non-trivial annihilator is subdirectly irreducible if and only if it has a non-zero disjunctive element.*

Proof. By Theorem 1.49, it is obvious. □

Medial Δ-semigroups

Theorem 9.20 *([23]) A semigroup S is a medial Δ-semigroup if and only if it satisfies one of the following conditions.*

(i) S is isomorphic to either G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime).

(ii) S is a two-element semilattice.

(iii) S is isomorphic to either R or R^0, where R is a two-element right zero semigroup.

(iv) S is isomorphic to either L or L^0, where L is a two-element left zero semigroup.

(v) S is a medial nil semigroup whose principal ideals form a chain with respect to inclusion.

(vi) S is a medial T1 semigroup (if S has an identity then it is commutative).

Proof. Let S be a medial Δ-semigroup. Then, by Theorem 9.3, it is a semilattice of archimedean medial semigroups. By Remark 1.2, S is either archimedean or a disjoint union $S = S_0 \cup S_1$ of an ideal S_0 and a subsemigroup S_1 of S which are archimedean.

Consider the case when S is archimedean. If S has a zero element then it is a nil semigroup whose principal ideals form a chain with respect to inclusion (see also Theorem 1.56). In this case (v) is satisfied.

In the next, we assume that S has no zero element. We have two cases.

First, assume that S is simple. Then, by Theorem 9.8, it is a rectangular abelian group, that is, a direct product of a left zero semigroup L, a right zero semigroup R and an abelian group G. Then, by Corollary 1.3, either $S = G$ or $S = R$ or $S = L$. In the first case (i) is satisfied (see Theorem 3.14). In the second case (iii) is satisfied (see Theorem 1.50). In the third case (iv) is satisfied (see Theorem 1.50).

Consider the case when S is not simple (and S has no zero element). Then, by Theorem 9.11 and Theorem 1.52, S has an idempotent element. By Theorem 9.9, S is a retract extension of a rectangular abelian group K by a medial nil semigroup N. Let λ denote the congruence on S determined by the mentioned retract homomorphism. Then $\lambda \cap \rho_K = id_S$, where ρ_K denotes the Rees congruence of S modulo K. As S is a Δ-semigroup, we have $\lambda = id_S$. Then $S = K$ which contradict the assumption for S.

Next, consider the case when S is a disjoint union $S = S_0 \cup S_1$ of an ideal S_0 and a subsemigroup S_1 of S, where S_0 and S_1 are archimedean. By Theorem 1.51 and Remark 1.1, S_1 is a Δ-semigroup.

If S_1 is a nil semigroup (with zero 0) then, by Theorem 1.57, S_1 has only one element.

Thus S_1 is either a two-element left zero semigroup L or a two-element right zero semigroup R or a subgroup G of a quasicyclic p-group (p is a prime).

If $|S_0| = 1$ then either $S = L^0$ or $S = R^0$ or $S = G^0$. If $|G| = 1$ then S is a two-element semilattice.

Next, assume $|S_0| > 1$. We recall that S_0 is a medial archimedean semigroup. By Theorem 9.11, and Theorem 1.52, S_0 has an idempotent. Then S_0 is a retract extension of a rectangular abelian group K by a medial nil semigroup. By Theorem 1.52, K is a rectangular band, that is $K = R \times L$, where R is a right zero semigroup and L is a left zero semigroup. Since $K^2 = K$ then, by Theorem 1.14, K is an ideal of S.

Assume $|K| = 1$. Then S_0 is a nil semigroup. By Theorem 1.59, we have either $|S_1| = 1$ or S_1 is a two-element right zero semigroup or S_1 is a two-element left zero semigroup. If $|S_1| = 1$ then S is a T1 semigroup and so (vi) is satisfied. Assume that S_1 is a two-element left zero semigroup. Let $S_1 = \{u, v\}$. It is easy to see that

$$\tau_u = \{(a, b) \in S \times S : \ ua = ub\}$$

and

$$\tau_v = \{(a, b) \in S \times S : \ va = vb\}$$

are congruences of S such that $(u, v) \in \tau_u$ and $(u, v) \in \tau_v$. As S is a Δ-semigroup, we have $\rho_{S_0} \in \tau_u$ and $\rho_{S_0} \in \tau_v$, where ρ_{S_0} denotes the Rees congruence of S modulo S_0. Thus $(a, 0) \in \tau_u$ and $(a, 0) \in \tau_v$ for every $a \in S_0$, that is, $ua = va = 0$ for every $a \in S_0$. Let $I = \{a \in S : \ au = av\}$. It is easy to see that I is a left ideal of S. We show that I is also a right ideal of S. Let $a \in I$ and $s \in S$ be arbitrary elements. Then

$$asu = asuu = ausu = avsu = asvu = asv$$

and so $as \in I$. Hence I is an ideal of S. It is clear that $u, v \in I$. As S is a Δ-semigroup, and $u, v \notin S_0$, we have $I = S$. Thus $au = av$ for every $a \in S_0$. Let β be the following equivalence on S.

$$\beta = \{(a, b) \in S \times S : \ a = b \text{ or } a, b \in S_1\}.$$

As $ua = va$ and $au = av$ for every $a \in S_0$, we have that β is a congruence on S. It is clear that $\beta \cap \rho_{S_0} = id_S$, where ρ_{S_0} is the Rees congruence on S determined by the ideal S_0 of S. As S is a Δ-semigroup, either $\beta \subseteq \rho_{S_0}$ or $\rho_{S_0} \subseteq \beta$ and so either $\beta = id_S$ or $\rho_{S_0} = id_S$. As $u \neq v$, we would have only $\rho_{S_0} = id_S$. Hence S_0 has only one element which is a contradiction. If S_1 is a two element right zero semigroup then we get, in a similar way, that S_0 has only one element.

Assume that $|K| > 1$. First consider the case when K is a left zero semigroup. It is easy to see that

$$\alpha = \{(a, b) \in S \times S : \ ax = bx \text{ for all } x \in K\}$$

is a congruence on S such that its restriction to K is the equality relation on K. As K is a dense ideal, it follows that $\alpha = id_S$. Let $x \in k$ and $c \in S$ be arbitrary elements. Then

$$cx = (cx)^2 = c^2x^2 = c^2x$$

which means that $(c, c^2) \in \tau_K$. Thus $c = c^2$. Consequently, S is a band and so $S_0 = L$. By Theorem 1.61, $S = L^1$, where L is a two-element left zero semigroup. As a medial monoid is commutative, $|L| = 1$ which is a contradiction.

We have a contradiction in that case when K is a right zero semigroup. Thus the first part of the theorem is proved. As the semigroups listed in the theorem are medial Δ-semigroups, the theorem is proved. $\square$

Chapter 10

Right commutative semigroups

In this chapter we deal with semigroups which satisfy the identity $axy = ayx$. These semigroups are called right commutative semigroups. It is clear that a right commutative semigroup is medial and so we can use the results of the previous chapter for right commutative semigroups. For example, every right commutative semigroup is a semilattice of right commutative archimedean semigroups and is a band of right commutative t-archimedean semigroups. A semigroup is right commutative and simple if and only if it is a left abelian group. Moreover, a semigroup is right commutative and archimedean containing at least one idempotent element if and only if it is a retract extension of a left abelian group by a right commutative nil semigroup. We characterize the right commutative left cancellative and the right commutative right cancellative semigroups, respectively. Clearly, a semigroup is right commutative and left cancellative if and only if it is a commutative cancellative semigroup. A semigroup is right commutative and right cancellative if and only if it is embeddable into a left abelian group if and only if it is a left zero semigroup of commutative cancellative semigroups. It is shown that a right commutative semigroup is embeddable into a semigroup which is a union of groups if and only if it is right separative. In this chapter we give a complete description of subdirectly irreducible right commutative semigroups. We show that a semigroup is a subdirectly irreducible right commutative semigroup with a globally idempotent core if and only if it is isomorphic to either G or G^0 or F or L or L^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime), F is a two-element semilattice and L is a two-element left zero semigroup. A right commutative semigroup with a zero and a non-trivial annihilator is subdirectly irreducible if and only if it has a non-zero disjunctive element. A right commutative semigroup S with $|A_S^l| = 1$, $|A_S^r| > 1$ is subdirectly irreducible with a nilpotent core if and only if it satisfies all of the following three conditions. (1) S contains a non-zero disjunctive element. (2) S is a disjoint union of a non-trivial ideal R and a subgroup G of S, where R is the set of all right divisors of zero of S and G is a subdirectly irreducible commutative group such that the identity element of G is a right identity element of S. (3) For every $k \in K_0$ and g_1, $g_2 \in G$, $kg_1 = kg_2$ implies $g_1 = g_2$. Finally, we show that a right commutative semigroup S with $\mid A^r_S \mid = \mid A^l_S \mid = 1$ is subdirectly irreducible with a nilpotent core if and only if it is either a commutative subdirectly irreducible semigroup with a nilpotent core and a trivial annihilator or satisfies both of the following two conditions. (1) S is a disjoint union of the set $RI(S)$ of all right identity elements of S and a non-trivial ideal R of all divisors of zero of S, $RI(S)$ has two elements e and f, K_0 has two elements k_1 and k_2 such that both of k_1, k_2 are disjunctive elements of S and $eK_0 = \{k_1\}$, $fK_0 = \{k_2\}$. (2) If $R - K \neq \emptyset$ then $rR \neq \{0\} \neq Rr$ for all $r \in R - K$.

We also determine the right commutative Δ-semigroups. We show that a semigroup S is a right commutative Δ-semigroup if and only if it satisfies one of the following. (1) S is either G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime). (2) S is a two-element semilattice. (3) S is either L or L^0, where L is a two-element left zero semigroup. (4) S is a right commutative nil semigroup whose principal ideals form chain with respect to inclusion. (5) S is a right commutative T1 semigroup. At the end of the chapter the right commutative T1 semigroups are constructed.

Definition 10.1 *A semigroup S is said to be a right commutative semigroup if it satisfies the identity $axy = ayx$.*

Theorem 10.1 *Every finitely generated periodic right commutative semigroup is finite.*

Proof. By Theorem 1.1, it is obvious. □

Lemma 10.1 *On a semigroup S, the following are equivalent.*

(i) *S is right commutative.*

(ii) *$\tau_x = \{(u,v) \in S \times S : xu = xv\}$ is a commutative congruence on S for every $x \in S$.*

(iii) *$\tau = \{(u,v) \in S \times S : xu = xv$ for all $x \in S\}$ is a commutative congruence on S.*

Proof. (i) implies (ii). Let S be a right commutative semigroup and $x \in S$ be arbitrary. It is clear that τ_x is a right congruence on S. As

$$xsu = xus = xvs = xsv$$

for every $s \in S$ and $u, v \in \tau_x$, we get that τ_x is a congruence on S. As $xab = xba$ for every $a, b \in S$, τ_x is commutative.

(ii) implies (iii). It is a consequence of the equation $\tau = \bigcap_{x \in S} \tau_x$.

(iii) implies (i). If τ is a commutative congruence on a semigroup S then, for every $a, b \in S$, we have

$$ab, ba \in \tau,$$

that is,

$$xab = xba$$

for all $x \in S$. Hence S is right commutative. $\square$

Semilattice decomposition of right commutative semigroups

Theorem 10.2 *Every right commutative semigroup is a left and right Putcha semigroup.*

Proof. Let S be a right commutative semigroup and $a, b \in S$ be arbitrary elements with $b \in aS^1$, that is, $b = ax$ for some $x \in S^1$. Then

$$b^2 = a^2x^2 \in a^2S^1.$$

Hence S is a left Putcha semigroup. Assume $b \in S^1a$ for some $a, b \in S$. Then

$$b = ya$$

for some $x \in S^1$. Then

$$b^2 = y^2a^2 \in S^1a^2.$$

Thus S is a right Putcha semigroup. $\square$

Theorem 10.3 *Every right commutative semigroup is a semilattice of right commutative archimedean semigroups.*

Proof. Let S be a right commutative semigroup. By Theorem 10.2 and Corollary 2.2, S is a semilattice Y of archimedean semigroups S_α, $\alpha \in Y$. As the subsemigroups S_α of S are right commutative, the theorem is proved. $\square$

Theorem 10.4 *Every right commutative semigroup is a band of right commutative t-archimedean semigroups.*

Proof. Since a right commutative semigroup satisfies the identity $(ab)^3 = a^2b^2(ab) = (ab)a^2b^2$ then the assertion follows from Theorem 1.8. $\square$

Theorem 10.5 *A semigroup is right commutative and simple if and only if it is a left abelian group.*

Proof. Let S be a right commutative simple semigroup. Then S is medial and so, by Theorem 9.8, it is a rectangular abelian group $S = L \times R \times G$ (L is a left zero semigroup, R is a right zero semigroup and G is an abelian group). It is clear that $|R| = 1$. Hence S is a left abelian group. The converse is obvious. $\square$

Theorem 10.6 *A semigroup is a right commutative archimedean semigroup containing at least one idempotent element if and only if it is a retract extension of a left abelian group by a right commutative nil semigroup.*

Proof. Let S be a right commutative archimedean semigroup containing at least one idempotent element. By Theorem 10.2 and Theorem 2.4, S is a retract extension of a (right commutative) completely simple semigroup K by a (right commutative) nil semigroup $N = S/K$. By Theorem 10.5, K is a left abelian group. Thus the first part of the theorem is proved.

Conversely, assume that the semigroup S is a retract extension of a left abelian group K by a right commutative nil semigroup. Then, by Theorem 2.2, S is archimedean and contains an idempotent. It is clear that K is right commutative. As the right commutative semigroups form a variety, by Theorem 1.40, S is right commutative. □

Corollary 10.1 *The following conditions on a semigroup S are equivalent.*

(i) S is right commutative and regular.

(ii) S is an orthodox left normal band of abelian groups.

(iii) S is a strong semilattice of left abelian groups.

(iv) S is a spined pruduct of a left normal band and a semilattice of abelian groups.

Proof. By the proof of Theorem 9.10, it is obvious. □

Theorem 10.7 *Every right commutative archimedean semigroup without idempotent has a non-trivial group-homomorphic image.*

Proof. As a right commutative semigroup is medial, our assertion follows from Theorem 9.11. □

Cancellation and separativity

Theorem 10.8 *The following conditions on an arbitrary semigroup S are equivalent.*

(i) S is right commutative and left cancellative.

(ii) S is embeddable into a commutative group.

(iii) S is commutative and cancellative.

Proof. It is obvious. □

Theorem 10.9 *The following conditions on an arbitrary semigroup S are equivalent.*

(i) S is right commutative and right cancellative.

(ii) S is embeddable into a left abelian group.

(iii) S is a left zero semigroup of commutative cancellative semigroups.

Proof. (i) implies (ii). Let S be a right commutative right cancellative semigroup. On the semigroup $S \times S$, define a relation ξ as follows.

$$(a,b)\ \xi\ (c,d) \iff ad = cb.$$

It is easy to see that ξ is reflexive and symmetric (for arbitrary semigroup S). To show that ξ is transitive, assume $(a,b)\xi(c,d)$, $(c,d)\xi(e,f)$ for $a,b,c,d,e,f \in S$. Then

$$ad = cb,$$

$$cf = ed$$

and so

$$afd = adf = cbf = cfb = edb = ebd,$$

because S is right commutative. As S is right cancellative, we have

$$af = eb,$$

that is,

$$(a,b)\xi(e,f).$$

Hence ξ is transitive. Using the right commutativity of S, it is easy to see that ξ is a congruence on $S \times S$. Let $[a,b]$ denote the ξ-class of $S \times S$ containing the element (a,b) and let M denote the factor semigroup $(S \times S)/\xi$. It can be easily verified that M is right commutative. As

$$[c,d][a,b][b,a] = [c,d]$$

for every $a,b,c,d \in S$, M is simple. As

$$[a,a]^2 = [a^2,a^2] = [a,a]$$

for every $a \in S$, M contains an idempotent. Let e and f be arbitrary idempotents of M with $e \leq f$, that is, $ef = fe = e$. As M is simple,

$$f = xey$$

for some $x,y \in M$. Then

$$f = f^2 = xeyf = xe^2yf = xeyef = fef = ef = e.$$

Hence every idempotent of M is primitive. Thus M is a completely simple semigroup. As

$$(ef)^2 = efef = e^2f^2 = ef$$

for every idempotents $e, f \in S$, the idempotents of M form a subsemigroup in M. Since M is a completely simple semigroup whose idempotents for a subsemigroup then, by Theorem 1.26, M is a rectangular group. It is easy to see that M is a left abelian group. We show that

$$\phi : a \ \mapsto \ [a^2, a]$$

is an isomorphism of S into M. Assume

$$\phi(a) = \phi(b),$$

that is,

$$[a^2, a] = [b^2, b]$$

for some $a, b \in S$. Then

$$a(ab) = a^2 b = b^2 a = b(ab).$$

As S is right cancellative, we have

$$a = b.$$

Hence ϕ is injective. As

$$\phi(ab) = [(ab)^2, ab] = [a^2 b^2, ab] = [a^2, a][b^2, b] = \phi(a)\phi(b)$$

for every $a, b \in S$, we get that ϕ is a homomorphism. Hence (ii) is satisfied.

(ii) implies (iii). Assume that a semigroup S is embeddable into a left abelian group $B = L \times G$ (L is a left zero semigroup, G is an abelian group). It is clear that B is a left zero semigroup L of commutative groups $f \times G$, $f \in L$. Then S is a left zero semigroup L of subsemigroups S_f ($f \in L$), which are embeddable into commutative groups $f \times G$. Then each S_f is commutative and cancellative. Hence (iii) is satisfied.

(iii) implies (i). Assume that a semigroup S is a left zero semigroup L of commutative cancellative semigroups S_i ($i \in L$). Let $a, b, c \in S$ be arbitrary elements with $a \in S_i$, $b \in S_j$, $c \in S_k$. To show that S is right cancellative, assume

$$ba = ca.$$

As

$$ba \in S_{ji} = S_j$$

and

$$ca \in S_{ki} = S_k,$$

we have

$$j = k$$

and so

$$b, c, baca \in S_j.$$

Then, in S_j,

$$b(ba) = bca = cba = c(ca),$$

because S_j is commutative. Since S_j is cancellative, we get

$$b = c.$$

Hence S is right cancellative. It is clear that, for arbitrary $a, b, c \in S$, we have

$$ab, ac, abc, acb \in S_i.$$

Then

$$(abc)a = a(abc) = a(ab)c = (ab)(ac) = (ac)(ab)$$
$$= (ac)ab = a(ac)b = a(acb) = (acb)a,$$

because S_i is commutative. As S is right cancellative, we get

$$abc = acb.$$

Hence S is right commutative. □

Theorem 10.10 *Let S be a right separative right commutative semigroup and x, y be arbitrary elements of S such that $yx^n = x^{n+1}$ for some positive integer n. Then $yx = x^2$.*

Proof. By Lemma 9.1, it is obvious. □

Theorem 10.11 *If S is a right commutative right separative semigroup then the archimedean components of S are right cancellative.*

Proof. As a right commutative semigroup is medial, our assertion follows from Theorem 9.12. □

Theorem 10.12 *A right commutative semigroup is embeddable in a semigroup which is a union of groups if and only if it is right separative.*

Proof. Assume that the right commutative semigroup is embeddable into a semigroup T which is a union of groups. We can suppose that T is a disjoint union of groups. Let $a, b \in S$ be arbitrary elements with

$$ab = b^2,\ ba = a^2.$$

Then

$$b^4 = (ab)^2 = abab = a^3b = a^2ba = a^4$$

from which we get that

$$a, b \in G$$

for a subgroup G of T. As G is cancellative, we get

$$a = b.$$

Conversely, let S denote a right commutative right separative semigroup. By Theorem 10.3 and Theorem 10.11, S is a semilattice Y of right cancellative right commutative archimedean semigroups S_α, $\alpha \in Y$. By Theorem 10.9, every S_α is embeddable into a left abelian group $M_\alpha = (S_\alpha \times S_\alpha)/\xi_\alpha$. Let ϕ_α denote the corresponding isomorphism. We can suppose that $M_\alpha \cap M_\beta = \emptyset$ if $\alpha \neq \beta$. Let $[a,b]_\alpha$ denote the elements of M_α $(a, b \in S_\alpha)$. On $M = \cup_{\alpha \in Y} M_\alpha$ define the following operation

$$[a,b]_\alpha [c,d]_\beta = [ac, bd]_{\alpha\beta}.$$

We show that the operation is well defined. Assume

$$[a,b]_\alpha = [a',b']_\alpha$$

and

$$[x,y]_\beta = [x',y']_\beta.$$

Then

$$ab' = a'b$$

and

$$xy' = x'y.$$

Thus

$$axb'y' = ab'xy' = a'bx'y = a'x'by,$$

that is,

$$[ax, by]_{\alpha\beta} = [a'x', b'y']_{\alpha\beta}.$$

Hence the operation is well defined. It is obvious that the oparation is associative. Thus M is a semigroup. It is clear that M is a disjoint union of groups. We show that the mapping

$$\Phi : a \mapsto \phi_{\alpha(a)} = [a^2, a]_\alpha, \; a \in S_\alpha$$

is an isomorphism of S into M. Since the restriction of Φ to M_α is ϕ_α and since

$$\Phi(M_\alpha) \cap \Phi(M_\beta) = \emptyset$$

if $\alpha \neq \beta$ then Φ is injective. As

$$\Phi(ab) = [(ab)^2, ab]_{\alpha\beta} = [a^2b^2, ab]_{\alpha\beta}$$

$$= [a^2, a]_\alpha [b^2, b]_\beta = \Phi(a)\Phi(b)$$

for every $a \in S_\alpha$ and $b \in S_\beta$, we have Φ is a homomorphism. Hence Φ is a mapping of S into the semigroup M which is a union of abelian groups. □

Remark 10.1 A right commutative left cancellative semigroup is commutative.

Theorem 10.13 *Every right commutative right cancellative archimedean semigroup with an idempotent element is a left abelian group.*

Proof. Let S be a right commutative right cancellative archimedean semigroup. Assume that S has an idempotent element. Then, by Theorem 10.6, S is a retract extension of a direct product K of an abelian group G and a left zero semigroup L by a nil semigroup N. We show that $|N| = 1$. In the opposite case there is an element a of S such that $a \notin K$. As N is a nil semigroup,

$$a^n \in K$$

for some positive integer n. Let

$$a^n = (g, j)$$

for some $g \in G$ and $j \in L$. Then

$$(e, j)a^n = (e, j)(g, j) = (g, j) = a^n,$$

where e is the identity of G. As S is right cancellative, we get

$$(e, j)a = a$$

and so

$$a \in K,$$

because K is an ideal of S and $(e, j) \in K$. □

Definition 10.2 *A right commutative right cancellative archimedean semigroup without idempotent is called a left $\mathcal{N}$-semigroup.*

Theorem 10.14 *([75]) Let N be the additive semigroup of non-negative integers, L be a left zero semigroup, G be an abelian group, $H = G \times L$, and $I : H \times H \to N$ be a function satisfying:*

(i) $I(\alpha, \beta) + I(\alpha\beta, \gamma) = I(\alpha, \beta\gamma) + I(\beta, \gamma) = I(\gamma, \beta) + I(\alpha, \gamma\beta)$ $(\alpha, \beta, \gamma \in H)$,

(ii) there exists an idempotent ϵ in H such that $I(\epsilon, \epsilon) = 1$,

(iii) for each $\alpha \in H$, there exists a positive integer m such that $I(\alpha^m, \alpha) > 0$.

On the set $S = N \times H$ define a multiplication by

$$(m, \alpha)(n, \beta) = (m + n + I(\alpha, \beta), \alpha\beta).$$

Then S with this multiplication is a left $\mathcal{N}$-semigroup, to be denoted by $(R \times G, I)$. Conversely, every left $\mathcal{N}$-semigroup is isomorphic to some semigroup $(R \times G, I)$.

Subdirectly irreducible right commutative semigroups

Theorem 10.15 *([59]) A semigroup is a subdirectly irreducible right commutative semigroup with a globally idempotent core if and only if it is isomorphic to either G or G^0 or F or L or L^0, where G is a nontrivial subgroup of a quasicyclic p-group (p is a prime), F is a two-element semilattice and L is a two-element left zero semigroup.*

Proof. By Theorem 9.18, it is obvious. □

Theorem 10.16 *([59]) A right commutative semigroup with a zero and a non-trivial annihilator is subdirectly irreducible if and only if it has a non-zero disjunctive element.*

Proof. By Theorem 1.49, it is obvious. □

Next we deal with right commutative semigroups S in which the annihilator A_S is trivial. Recall that $A_S = A^l_S \cap A^r_S$, where A^l_S and A^r_S denotes the left and the right annihilator of S, respectively.

Lemma 10.2 *([59]) Let K be the nilpotent core of a right commutative semigroup S with zero, and let $R = \{r \in S : Kr = \{0\}\}$, $L = \{l \in S : lK = \{0\}\}$. Then the following hold.*

(i) $R \subseteq L$.

(ii) If $L \neq S$ *then* $L = R$.

(iii) If $|A_S| = 1$ *then* $|A^l_S| = 1$.

(iv) $L = S$ *if and only if* $|A^r_S| > 1$.

Proof. To prove (i), let r be an arbitrary element of S with $r \in R$ and $r \notin L$. Then

$$Kr = \{0\}$$

and

$$rK \neq \{0\}.$$

Thus

$$SrK = SKr = \{0\} \subseteq rK$$

and

$$rKS \subseteq rK,$$

that is, rK is a non-trivial ideal of S. So

$$K \subseteq rK$$

which means that

$$rK = K.$$

So

$$K = r^2K = rKr = \{0\}$$

which is a contradiction. Hence

$$R \subseteq L.$$

To prove (ii), assume that there is an element l of L such that $l \notin R$. Then

$$lK = \{0\}$$

and

$$Kl \neq \{0\}.$$

Since $SKl \subseteq Kl$ and $KlS = KSl \subseteq Kl$ then Kl is a non-trivial ideal of S. Then

$$K \subseteq Kl$$

and so

$$K = Kl.$$

If $L \neq S$ then there is an element s of S such that

$$s \notin L.$$

Then

$$\{0\} \neq sK = sKl = slK = \{0\}$$

which is a contradiction. Consequently, $L \neq S$ implies $L \subseteq R$. This and (i) together imply

$$R = L.$$

To prove (iii), let $s \in A_S^l$ be an arbitrary element with $s \neq 0$. Then A_S^l is a non-trivial ideal of S and so

$$K \subseteq A_S^l.$$

Thus

$$KS = \{0\}$$

which means that

$$R = S.$$

By (i),

$$R = L = S$$

and so

$$SK = \{0\}.$$

Thus

$$K \subseteq A_S.$$

Consequently, $|A_S| = 1$ implies $|A_S^l| = 1$.

To prove (iv), assume $L = S$. Then

$$SK = \{0\},$$

that is,

$$K \subseteq A_S^r.$$

So

$$|A_S^r| > 1.$$

Conversely, if $|A_S^r| > 1$ then $K \subseteq A_S^r$ and so $SK = \{0\}$ which means that $L = S$. □

If K is the core of a semigroup S with zero then the set of all non-zero elements of K will be denoted by K_0.

Lemma 10.3 *([59]) If a subdirectly irreducible right commutative semigroup S contains elements k and e such that $k \in K_0$ and $k = ke$ or $k = ek$ then e is a right identity element of S.*

Proof. Let S be a subdirectly irreducible right commutative semigroup. Assume

$$k = ke$$

for some elements $k \in K_0$, $e \in S$. Let

$$Z = \{z \in S : \ z = ze\}.$$

Using the right commutativity of S, it can be easily verified that Z is an ideal of S. As Z is not trivial,

$$K \subseteq Z,$$

that is,

$$k = ke$$

for all $k \in K$. Let

$$\alpha = \{(a, b) \in S \times S : \ ae^n = be^m \text{ for some positive integers } n, m\}.$$

It can be easily verified that α is a congruence on S and

$$\alpha\,|_K = id_K.$$

As S is subdirectly irreducible, K is a dense ideal of S. So $\alpha = id_S$. As

$$ae^2 = (ae)e$$

(that is $(a, ae) \in \alpha$), we get

$$a = ae$$

for all $a \in S$. So e is a right identity element of S.

Assume $k = ek$ for some $k \in K_0$, $e \in S$. Then

$$ke = eke = eek = k.$$

By the first part of the proof, e is a right identity element of S. Thus the lemma is proved. □

In the next, the set of all right identity elements of a semigroup S will be denoted by $RI(S)$.

Theorem 10.17 *([59]) A right commutative semigroup S satisfying $|A^l_S| = 1$, $|A^r_S| > 1$ is subdirectly irreducible with a nilpotent core if and only if all of the following conditions hold.*

(i) S contains a non-zero disjunctive element.

(ii) S is a disjoint union of a non-trivial ideal R and a subgroup G of S, where R is the set of all right divisors of zero of S and G is a subdirectly irreducible commutative group such that the identity element of G is a right identity element of S.

(iii) For every $k \in K_0$ and g_1, $g_2 \in G$, $kg_1 = kg_2$ implies $g_1 = g_2$.

Proof. Let S be a subdirectly irreducible right commutative semigroup such that $| A^l_S |= 1$ and $| A^r_S |> 1$. By Lemma 1.4, S contains a nonzero disjunctive element. So (i) holds.

The proof of (ii): By Lemma 10.2, $S = L$. So $SK = \{0\}$. First we show that S has a right identity element. Let

$$k \in K_0$$

be an arbitrary element. As $A^l_S = \{0\}$, we have

$$k \notin A^l_S$$

and so

$$kS \neq \{0\}.$$

As $SK = \{0\}$, kS is a (non-trivial) ideal of S. So

$$kS = K.$$

Consequently, there is an element e in S such that

$$ke = k.$$

By Lemma 10.3, e is a right identity element of S.

We show that S has only one right identity element. Assume, in an indirect way, that S has a right identity element f with $f \neq e$. Let

$$\alpha = \{(a,b) \in S \times S :\ (\exists x, y \in S^1)\ a, b \in \{xey, xfy\}\}.$$

As $a = ae = af$ for every $a \in S$, α is a reflexive relation of S. It is clear that α is symmetric. Let β denote the transitive closer of α. It is easy to see that β is a congruence. Evidently, $(e,f) \in \beta$. Let $k \in K_0$ be an arbitrary element. Suppose $(k,s) \in \beta$ for some $s \in S$. We show that $k = s$. As $(k,s) \in \beta$, there are elements $x_0, x_1, ..., x_n \in S$ such that $k = x_0$, $s = x_n$, and $(x_i, x_{i+1}) \in \alpha$, for every $i = 0, 1, ..., n-1$. If $x_i = x_{i+1}$ for all $i = 0, 1, ..., n-1$, then $k = s$. So we may assume $x_i \neq x_{i+1}$ for some $i = 0, 1, ..., n-1$. Let

$$j = min\{i : x_i \neq x_{i+1}\}.$$

Then

$$(k, x_{j+1}) \in \alpha$$

and

$$k \neq x_{j+1}.$$

So there are elements $u, v \in S^1$ such that

$$k, x_{j+1} \in \{uev, ufv\}.$$

If $u \neq 1$ then

$$uev = ufv$$

(because e and f are right identity elements of S) contradicting $k \neq x_{j+1}$. Thus $u = 1$ and so $k = ev$ or $k = fv$. Then

$$0 = ek = e(ev) = ev = k$$

or

$$0 = fk = f(fv) = fv = k$$

which contradicts $k \in K_0$. Hence $k = s$. Thus

$$\beta \cap \varrho_K = id_S,$$

where ϱ_K is the Rees congruence on S determined by the ideal K. But this is a contradiction. So $RI(S) = \{e\}$.

Consider the ideal R of S defined in Lemma 10.2. If $R = S$ then $K \subseteq A_S^l$ which contradicts (iii) of Lemma 10.2. So $R \neq S$.

Let $G = S - R$. We show that G is a subsemigroup of S. Let $g, h \in G$ be arbitrary elements. Then $Kg \neq \{0\}$ and $Kh \neq \{0\}$. Using the right commutativity of S, it can be easily verified that Kg and Kh are (non-trivial) ideals of S. So $Kg = Kh = K$. Then $K = Kh = Kgh$ which means that $gh \in G$, that is, G is a subsemigroup.

We show that the elements of G are not right divisors of zero. Let $g \in G$ be arbitrary. Assume $sg = 0$ for some $s \in S$. Let

$$P = \{p \in S : \; pg = 0\}.$$

Using the right commutativity of S, it can be easily verified that P is an ideal of S. As $g \notin R$, $P = \{0\}$. So $s = 0$. Thus g is not a right divisor of zero.

This result and $KR = \{0\}$ together imply that R is the set of all right divisors of zero of S.

We show that G is a subgroup. Evidently, $e \in G$. Let $k \in K_0$ be arbitrary. Then $kS = K$ (see above). As $kR = \{0\}$, we get $kG = K_0$. Let $g \in G$ be arbitrary. Then $kg \in K_0$ which implies $kgG = K_0$. Then there is an element g_1 in G such that $kgg_1 = k$. By Lemma 10.3, $gg_1 = e$. So every element of G has a right inverse in G with respect to the right identity element e. So G is a group. As G is right commutative, it follows that it is commutative.

We prove that G is subdirectly irreducible. Let H_i, $i \in I$ be a family of subgroups of G such that

$$\cap_{i\in I} H_i = \{e\}.$$

Let δ_i, $i \in I$ denote the congruence on G determined by the subgroup H_i. Let

$$\delta_i^* = \{(a,b) \in S \times S : \ aH_i = bH_i\}, \ i \in I.$$

As S is right commutative, δ_i^*, $i \in I$ is a congruence on S. It is evident that

$$\delta_i^* \mid_G = \delta_i.$$

We show that $\cap_{i\in I}\delta_i^* = id_S$. Let

$$(k_1, k_2) \in \cap_{i\in I}\delta_i^*$$

for some $k_1, k_2 \in K_0$. As $k_2G = K_0$, there is an element g in G such that

$$k_2 g = k_1.$$

As

$$(k_1, k_2) \in \delta_i^*, \ i \in I,$$

for every $i \in I$, there is an element g_i in H_i such that

$$k_2 g_i = k_1.$$

So

$$k_2 g = k_2 g_i$$

for all $i \in I$. Let g^{-1} denote the group inverse of g in G. Then

$$k_2 = k_2 e = k_2 g g^{-1} = k_2 g_i g^{-1}.$$

By Lemma 10.3,

$$g_i g^{-1} = e$$

from which it follows that

$$eg = g_i g^{-1} g = g_i e = g_i \in H_i.$$

So

$$eg \in \cap_{i\in I} H_i$$

which means that

$$eg = e.$$

So

$$k_1 = k_2 g = k_2 eg = k_2 e = k_2.$$

As the elements of G are not right divisors of zero,

$$(k,0) \notin \cap_{i \in I} \delta_i^*$$

for all $k \in K_0$. Consequently,

$$\cap_{i \in I} \delta_i^* \mid_K = id_K.$$

As S is subdirectly irreducible, K is a dense ideal of S and so

$$\cap_{i \in I} \delta_i^* = id_S.$$

Using again the condition that S is subdirectly irreducible, there is an element j in I such that

$$\delta_j^* = id_S.$$

So

$$\delta_j = \delta_j^* \mid_G = id_G.$$

Thus G is subdirectly irreducible and so (ii) is proved.

To prove (iii), let $k \in K_0$, $g_1, g_2 \in G$ be arbitrary elements with $kg_1 = kg_2$. Then

$$k = ke = kg_1 g_1^{-1} = kg_2 g_1^{-1}.$$

By Lemma 10.3,

$$g_2 g_1^{-1} = e,$$

that is,

$$g_1 = g_2.$$

Thus (iii) holds, and the first part of the theorem is proved.

To prove the converse, let us suppose that S is a right commutative semigroup such that $\mid A_S^l \mid = 1$, $\mid A_S^r \mid > 1$ and S satisfies all of conditions (i)-(iii) of the theorem. We show that S is a subdirectly irreducible semigroup with a nilpotent core.

By condition (i), S has a core K (see Lemma 1.5). As $\mid A_S^r \mid > 1$, we have $K \subseteq A_S^r$. So $K^2 = \{0\}$, that is, K is nilpotent.

By (ii), S is a disjoint union of a non-trivial ideal R and a subgroup G of S, where R is the set of all right divisors of zero of S and G is a subdirectly irreducible commutative group such that the identity element e of G is a right identity element of S. We note that $RI(S) = \{e\}$.

We show that $KR = \{0\}$. Let $r \in R$ be an arbitrary element. Then there is an element $s \neq 0$ in S such that

$$sr = 0.$$

Let k be an arbitrary element in K. Then $k \in S^1 s S^1$ and so

$$k = xsy$$

for some $x, y \in S^1$. Consequently

$$kr = xsyr = xsry = 0,$$

because S is right commutative. So $KR = \{0\}$, indeed.

By (iv) of Lemma 10.2, $L = S$ and so $SK = \{0\}$. Let $k \in K_0$ be arbitrary. Using $Sk = \{0\}$, it can be easily verified that kS is an ideal of S. As $k \notin A_S^l$, we get $kS \neq \{0\}$. So

$$K = kS = kG \cup \{0\},$$

that is,

$$kG = K_0.$$

Consider the case when the subgroup G has only one element. Then $K_0 = kG = \{k\}$ which means that K has only two elements. With other words, K is primitive. By Lemma 1.6, a semigroup with a primitive core is subdirectly irreducible if and only if its zero is disjunctive. We show that the zero of S is disjunctive. Let a and b be arbitrary elements of S with $a \neq b$. Let k denote the non-zero disjunctive element of S. Evidently,

$$(a, b) \notin C_{\{k\}}.$$

So there are elements $x, y \in S^1$ such that, for example,

$$xay = k$$

and

$$xby \neq k.$$

If $xby = 0$ then $(a, b) \notin C_{\{0\}}$. If $xby \neq 0$ then $xby \notin r\{k\}$ and so there are elements u, v in S^1 such that

$$uxbyv = k.$$

As $xby \neq k$, $u \in S$ or $v \in R$. If $u \in S$ then $ukv = 0$, because $SK = \{0\}$. If $u \notin S$ then $v \in R$. So $kv = 0$, that is, $ukv = 0$. Consequently, $ukv = 0$ in both cases. From this result it follows that

$$uxayv = ukv = 0.$$

This and

$$uxbyv = k$$

together imply that

$$(a, b) \notin C_{\{0\}}.$$

So the zero of S is disjunctive. Thus S is subdirectly irreducible.

Consider the case when the subgroup G has at least two elements. As G is a non-trivial subdirectly irreducible commutative group, it has a least nonunit subgroup H. Let ξ be a congruence on S with $\xi \neq id_S$. Denoting a non-zero disjunctive element of S by k, $\{k\}$ does not form a ξ-class. In the opposite case, $\xi \subseteq C_{\{k\}}$ which contradicts $\xi \neq id_S$. So there exists an element s is S such that $s \neq k$ and $(s,k) \in \xi$.

Consider the case when $s \notin K$. Then $s \in G$ and so

$$S^1 k S^1 = K.$$

Thus there are elements x, y in S^1 such that

$$xsy = k$$

and so

$$(xky, k) \in \xi.$$

As $s \neq k$, we have $x \in S$ or $y \neq e$, where e is the right identity element of S. If $x \in S$ then

$$xky = 0$$

(because $SK = \{0\}$) and so

$$(0, k) \in \xi.$$

If $x \notin S$ and $y \in G - \{e\}$ then

$$sy = k$$

which implies

$$s = se = syy^{-1} = ky^{-1},$$

because $yy^{-1} = e$. So $s \in K$ which is a contradiction. If $x \notin S$ and $y \in R$ then

$$xky = 0$$

and so

$$(k, 0) \in \xi.$$

Consequently,

$$(k, 0) \in \xi$$

in all cases. Thus $(kH, 0) \in \xi$, that is, kH is contained by a ξ-class.

Consider the case when $s \in K_0$. As $kG = K_0$, there is an element a in G such that

$$s = ka = kea.$$

So

$$(kea, k) \in \xi.$$

As $s \neq k$, we have

$$ea \neq e.$$

Let

$$P = \{b \in G : \ (kb, k) \in \xi\}.$$

It can be easily verified that P is a subgroup of G. As $ea \in P$ and $ea \neq e$, P is a non-trivial subgroup of G. Thus $H \subseteq P$ and so

$$(kH, k) \in \xi.$$

Then kH is contained by a ξ-class.

If $s = 0$ then $(k, 0) \in \xi$ and so

$$(kH, 0) \in \xi$$

which means that kH is contained by a ξ-class.

Summarizing our results, kH is in some ξ-class in all cases. By (iii),

$$\mid kH \mid = \mid H \mid > 1.$$

Let ξ_i, $i \in I$ be a family of non-identical congruences on S. Then kH is in some ξ_i-class for all $i \in I$. So

$$\cap_{i \in I} \xi_i \neq id_S$$

which means that S is subdirectly irreducible. Thus the theorem is proved. □

Example: Let S be a semigroup defined by the following Cayley table:

	e	a	u	v	0
e	e	a	0	0	0
a	a	e	0	0	0
u	u	v	0	0	0
v	v	u	0	0	0
0	0	0	0	0	0

It can be easily verified that S is a subdirectly irreducible right commutative semigroup and, using the notations of the previous theorem, $RI(S) = \{e\}$, $G = \{e, a\}$, $A^l_S = \{0\}$, $A^r_S = \{0, u, v\} = K$ and $K^2 = \{0\}$.

Next we describe the subdirectly irreducible right commutative semigroups S with a nilpotent core and the condition $\mid A^r_S \mid = \mid A^l_S \mid = 1$.

Theorem 10.18 *([59]) A right commutative semigroup S satisfying $|A^r_S| = |A^l_S| = 1$ is subdirectly irreducible with a nilpotent core if and only if it is either a commutative subdirectly irreducible semigroup with a nilpotent core and a trivial annihilator or satisfies both of the following conditions.*

(i) S is a disjoint union of $RI(S)$ and a non-trivial ideal R of all divisors of zero of S, $RI(S)$ has two elements e and f, K_0 has two elements k_1 and k_2 such that both of k_1, k_2 are disjunctive elements of S and $eK_0 = \{k_1\}$, $fK_0 = \{k_2\}$.

(ii) If $R - K \neq \emptyset$ then $rR \neq \{0\} \neq Rr$ for all $r \in R - K$.

Proof. Let S be a subdirectly irreducible right commutative semigroup such that $\mid A_S^r \mid = \mid A_S^l \mid = 1$ and the core K of S is nilpotent. We may assume that S is not commutative. (The commutative case has been described in Theorem 3.16.)

By Lemma 1.4, S has a non-zero disjunctive element. Consider the ideals L and R of S defined in Lemma 10.2. As $\mid A_S^r \mid = 1$, we get $R = L \neq S$. Let $G = S - R$. As $R \neq S$, we get $G \neq \emptyset$. We can prove, as in the proof of Theorem 10.17, that G is a subsemigroup.

We show that every element of G is not a divisor of zero. Assume, in an indirect way, that G has an element g which is a divisor of zero. Then there is an element s in S such that $s \neq 0$ and $sg = 0$ or $gs = 0$. Assume $sg = 0$. Let

$$P_g = \{s \in S : \ sg = 0\}.$$

As S is right commutative, P_g is a (non-trivial) ideal of S. So

$$K \subseteq P_g,$$

that is, $Kg = \{0\}$ which means that $g \in R$. But this is a contradiction. Consider the case $gs = 0$. Using again the right commutativity of S, it can be proved that

$$P_g^* = \{s \in S : \ gs = 0\}$$

is a (non-trivial) two-sided ideal of S. So

$$K \subseteq P_g^*$$

which implies $gK = \{0\}$, that is, $g \in L$. But this is also a contradiction. Consequently, every element of G is not a divisor of zero. As $RK = KR = \{0\}$, R is the set of all divisors of zero of S.

Let $k \in K_0$ be arbitrary. As Sk is a non-trivial ideal of S, we have

$$K = Sk = Gk \cup \{0\}$$

and so

$$K_0 = Gk.$$

Thus there is an element e in G such that

$$k = ek.$$

By Lemma 10.3, e is a right identity element of S.

We prove that G is a left group. Let $g \in G$ and $k \in K_0$ be arbitrary elements. Then $gk \in K_0$ and so

$$Ggk = K_0.$$

Thus there is an element g_1 in G such that

$$g_1 gk = k.$$

By Lemma 10.3, $g_1 g \in RI(S)$. Then, for every element b of G,

$$b = b(g_1 g) = (bg_1)g$$

which implies that

$$G = Gg.$$

So G is left simple. Since G has an idempotent element then, by the dual of Theorem 1.27 of [19], G is a left group and so it is a union of its disjoint subgroups, and these subgroups are isomorphic with each other.

We show that the identity elements of the subgroups of G are in $RI(S)$. Let e be the identity element of the subgroup G_e of G. Then e is not a divisor of zero. Thus, for an arbitrary element $k \in K_0$, $ke \in K_0$ and

$$ke = (ke)e.$$

By Lemma 10.3, it follows that $e \in RI(S)$.

It can be easily verified that the Green's relation $\mathcal{H}$ on G is a congruence on G. As the $\mathcal{H}$-classes of G are the maximal subgroups of G, the factor semigroup $G/\mathcal{H}$ is a left zero semigroup.

Let G_e, $e \in RI(S)$ be an arbitrary (maximal) subgroup of G. We show that, for every $g, h \in G_e$ and $k \in K_0$,

$$(*) \qquad gk = hk \text{ or } kg = kh \implies g = h.$$

Assume

$$gk = hk$$

for some $k \in K_0$, $g, h \in G_e$. Then

$$g^{-1}gk = g^{-1}hk$$

from which it follows that

$$ek = g^{-1}hk = g^{-1}hek$$

and

$$ek \in K_0.$$

By Lemma 10.3,

$$g^{-1}h \in RI(S),$$

that is,

$$g^{-1}h = e$$

which implies

$$g = h.$$

Assume

$$kg = kh$$

for some $k \in K_0$, $g, h \in G_e$. Then

$$gk = egk = ekg = ekh = ehk = hk.$$

Using the previous result,

$$g = h.$$

So $(*)$ holds, indeed.

Assume that $ek = k$ for all $e \in RI(S)$ and $k \in K_0$. Let $e \in RI(S)$ be an arbitrary element. As S is right commutative,

$$\tau_e = \{(a, b) \in S \times S : \ ea = eb\}$$

is a congruence on S such that $\tau_e \mid_K = id_K$. As S is subdirectly irreducible, $\tau_e = id_S$. So $ea = e(ea)$ implies $a = ea$ for all $a \in S$. Then e is also a left identity element of S. So S has an identity element, that is, S is commutative which is a contradiction.

Consider the case when there are elements $f \in RI(S)$ and $k_1 \in K_0$ such that $fk_1 \neq k_1$. As $Gk_1 = K_0$, there is an element e in $RI(S)$ such that $ek_1 = k_1$. We note that $e \neq f$. Let $g \in G$ be an arbitrary element. As $k_1 g = k_1 eg$ and $eg \in G_e$, we get $k_1 G = k_1 G_e$. Let

$$\alpha_f = \{(a, b) \in S \times S : \ a = b \text{ or } (\exists g \in G_e) : \ a, b \in \{k_1 g, fk_1 g\}\}.$$

We show that α_f is an equivalence relation on S. Assume $k_1 g = k_1 h$ for some $g, h \in G_e$. Then, by $(*)$, $g = h$. Similarly, $fk_1 g = fk_1 h$ implies $g = h$ for all $g, h \in G_e$. Assume $k_1 g = fk_1 h$, $(g, h \in G_e)$ Then $fk_1 g = fk_1 h$ and so $g = h$. Consequently, the subsets $T_g = \{k_1 g, fk_1 g\}$ have two elements and $T_g \cap T_h = \emptyset$ if $g \neq h$, $g, h \in G_e$. Thus α_f is an equivalence on S and $\alpha_f \neq id_S$.

As $sk_1 g = sfk_1 g$ for all $s \in S$ and $g \in G_e$, it follows that α_f is left compatible. We show that α_f is also right compatible. Let $s \in S$ be arbitrary. If $s \in R$ then

$$(k_1 g)s = 0 = (fk_1 g)s.$$

If $s \in G$ then

$$(k_1 g)s = k_1(gs),$$

$$(fk_1 g)s = fk_1(gs)$$

and $gs \in G_e$. So α_f is right compatible. Consequently, α_f is a congruence on S.

Assume that, for some $t \in RI(S)$, $fk_1 \neq tk_1 \neq k_1$. Then α_t is a congruence on S such that $\alpha_t \neq id_S$.

We show that $\alpha_f \cap \alpha_t = id_S$. If $tk_1 g = fk_1 h$ for some $g, h \in G_e$ then $fk_1 g = fk_1 h$ which implies $g = h$ and so

$$tk_1 = tk_1 e = tk_1 g g^{-1} = fk_1 h g^{-1} = fk_1 h h^{-1} = fk_1 e = fk_1$$

which is a contradiction. So $tk_1 g \neq fk_1 h$ for all $g, h \in G_e$. If $tk_1 g = k_1 h$ then $fk_1 g = fk_1 h$. So $g = h$ and $tk_1 = k_1$ which is a contradiction. So $tk_1 g \neq k_1 h$ for all $g, h \in G_e$. We can prove, in a similar way, that $fk_1g \neq k_1h$ for all $g, h \in G_e$. Thus

$$\{tk_1g, k_1g\} \cap \{k_1h, fk_1h\} = \emptyset \text{ if } g \neq h$$

and

$$\{tk_1g, k_1g\} \cap \{k_1g, fk_1g\} = \{k_1g\}.$$

Consequently, $\alpha_f \cap \alpha_t = id_S$.

As S is subdirectly irreducible, $\alpha_f = id_S$ or $\alpha_t = id_S$ which is a contradiction. So, for all $t \in RI(S)$, $tk_1 = k_1$ or $tk_1 = fk_1$. Thus there are subsets E and F of $RI(S)$ such that $E \cap F = \emptyset$, $E \cup F = RI(S)$ and $Ek_1 = \{k_1\}$, $Fk_1 = \{fk_1\}$.

Let k_2 denote fk_1. We show that $K_0 = \{k_1, k_2\}$. Let $g \in G$ be an arbitrary element. Assume $g \in G_w$ for some $w \in E$. Then

$$gk_1 = wgk_1 = wk_1g = k_1g.$$

If $g \in G_w$, for some $w \in F$, then

$$gk_1 = wgk_1 = wk_1g = k_2g.$$

Thus, for all $g \in G$,

$$gk_1 \in \{k_1g, k_2g\},$$

that is,

$$Gk_1 \subseteq k_1G \cup k_2G.$$

As $Gk_1 = K_0$, we get

$$K_0 \subseteq k_1G \cup k_2G \subseteq K_0.$$

So

$$K_0 = k_1G \cup k_2G = k_1G_e \cup k_2G_e.$$

We note that $k_1G_e \cap k_2G_e = \emptyset$. Let

$$\eta = \{(a, b) \in S \times S : \ a = b \text{ or } a, b \in k_1G_e \text{ or } a, b \in k_2G_e\}.$$

Evidently, η is an equivalence on S. We show that η is a congruence on S. Let $(a, b) \in \eta$, $s \in S$ be arbitrary elements. We may assume $a \neq b$. Consider the case $a = k_1g,\ b = k_1h$ for some $g, h \in G_e,\ g \neq h$.

If $s \in R$ then

$$as = k_1gs = 0 = k_1hs = bs$$

and, similarly,

$$sa = 0 = sb.$$

So

$$(as, bs) \in \eta$$

and

$$(sa, sb) \in \eta.$$

Assume $s \in G_w$, $w \in E$. Then

$$sa = wsa = was = wk_1gs = k_1gs \in k_1G_e,$$

$$sb = wsb = wbs = wk_1hs = k_1hs \in k_1G_e,$$

$$as = k_1gs \in k_1G_e,$$

$$bs = k_1hs \in k_1G_e.$$

So

$$(sa, sb) \in \eta$$

and

$$(as, bs) \in \eta.$$

Assume $s \in G_w$, $w \in F$. Then

$$sa = wsa = was = wk_1gs = k_2gs \in k_2G_e,$$

$$sb = wsb = wbs = wk_1hs = k_2hs \in k_2G_e,$$

$$as = k_1gs \in k_1G_e,$$

$$bs = k_1hs \in k_1G_e.$$

So

$$(sa, sb) \in \eta$$

and

$$(as, bs) \in \eta.$$

Consequently η is a congruence on S. It is evident that

$$\eta \cap \alpha_f = id_S.$$

As S is subdirectly irreducible and $\alpha_f \neq id_S$, we get $\eta = id_S$. So $k_1G_e = \{k_1\}$ and $k_2G_e = \{k_2\}$. Thus

$$K_0 = k_1G_e \cup k_2G_e = \{k_1, k_2\}.$$

We show that k_1 and k_2 are disjunctive elements of S. By Lemma 1.5, one of k_1 and k_2, for example, k_1 is a disjunctive element of S. Assume $(0, k_1) \in C_{\{k_2\}}$. Then, for some $f \in F$, $(0, fk_1) \in C_{\{k_2\}}$ which means that $(0, k_2) \in C_{\{k_2\}}$, because $fk_1 = k_2$. This is a contradiction. So $C_{\{k_2\}} \cap \varrho_K = id_S$ which implies $C_{\{k_2\}} = id_S$, that is, k_2 is a disjunctive element of S.

We show that $\mid E \mid = \mid F \mid = 1$. Let e_1, e_2 be arbitrary elements of E. Let ξ denote the congruence on S generated by $\{e_1, e_2\}$. Assume $(k, s) \in \xi$ for some $k \in K$ and $s \in S$, $s \neq k$. Then there are elements $x, y \in S^1$ such that $k, s \in \{xe_1y, xe_2y\}$. As $s \neq k$ and e_1, e_2 are right identity of S, we have that $x \neq 1$. Let, for example, $k = e_1y$, $s = e_2y$. From this equations it follows that

$$k = e_1y = e_1e_2y = e_1s$$

and

$$s = e_2y = e_2e_1y = e_2k,$$

that is, k and c generate the same ideal of S. We note that $s \neq 0$. As $k \in K_0$, we get $s \in K_0$. As $K_0 = \{k_1, k_2\}$, we get, for example, $k = k_1$ and $s = k_2$. Then

$$e_2k_2 = e_2e_2y = e_2y = k_2$$

which contradicts $EK_0 = \{k_1\}$. So

$$\xi \cap \varrho_K = id_S,$$

where ϱ_K denotes the Rees congruence on S generated by K. As S is subdirectly irreducible, $\xi = id_S$, that is, $e_1 = e_2$. So $| E |= 1$. We can prove, in a similar way, that $| F |= 1$.

Let $E = \{e\}$ and $F = \{f\}$. Consider two elements g, h in G_e. As $k_1G_e = \{k_1\}$ (see above), $k_1g = k_2h$ which implies that $g = h$ (see $(*)$). So $| G_e |= 1$. As G_e and G_f are isomorphic, $G = \{e, f\} = RI(S)$. As $G \cup R = S$, (i) holds.

The proof of (ii): Assume $R - K \neq \emptyset$. We show that, for every $r \in R - K$, $rR \neq \{0\} \neq Rr$. Assume, in an indirect way, that $rR = \{0\}$ or $Rr = \{0\}$. If $rR = \{0\}$ then $erR = \{0\}$ and so $eRr = \{0\}$. As e is not a divisor of zero, $Rr = \{0\}$. Similarly, $Rr = \{0\}$ implies $rR = \{0\}$. So $rR = Rr = \{0\}$. As k_1 is a disjunctive element, $r \notin r\{k_1\}$ and so there are elements $x, y \in S^1$ such that $xry = k_1$. Evidently, $x, y \notin R$, because $Rr = rR = \{0\}$. So $x, y \in G$ and $xr = k_1$. From this it follows that

$$k_1 = ek_1 = exr = er.$$

Then

$$er = eer = ek_1 = k_1$$

and

$$fr = fer = fk_1 = k_2.$$

Consider the relation β on S defined by

$$\beta = \{(a, b) \in S \times S : \ a = b \textit{ or } a, b \in \{r, k_1\}\}.$$

Evidently, β is an equivalence relation on S. Let $s \in S$ be an arbitrary element. If $s \in R$ then

$$sr = sk_1 = rs = k_1s = 0.$$

So $(sr, sk_1) \in \beta$ and $(rs, k_1s) \in \beta$. If $s = e$ then $sr = sk_1 = k_1$, $rs = r$ and $k_1s = k_1$. So $(sr, sk_1) \in \beta$ and $(rs, k_1s) \in \beta$. If $s = f$ then $sr = sk_1 = k_2$, $rs = r$ and $k_1s = k_1$. So $(sr, sk_1) \in \beta$ and $(rs, k_1s) \in \beta$. Consequently β is a congruence on S.

We can prove, in a similar way, that

$$\delta = \{(a, b) \in S \times S : \ a = b \textit{ or } a, b \in \{r, k_2\}\}$$

is a congruence on S. It is evident that $\beta \cap \delta = id_S$. As S is subdirectly irreducible, $\beta = id_S$ or $\delta = id_S$ which is a contradiction. So $rR \neq \{0\} \neq Rr$ as it was asserted. So (ii) holds. Thus the first part of the theorem is proved.

To prove the converse, we may assume that S is a right commutative semigroup sucht that $\mid A_S^r \mid = \mid A_S^l \mid = 1$ and S satisfies both of conditions (i)-(ii) of this theorem (the commutative subdirectly irreducible semigroups with a nilpotent core and a trivial annihilator has been described in Theorem 3.16). As S has a non-zero disjunctive element, S has a core K and every disjunctive element of S is contained by K (see Lemma 1.5). We prove that $RK = KR = \{0\}$. Let $r \in R$ be an arbitrary element. Then there is an element s in S such that $s \neq 0$ and $sr = 0$ or $rs = 0$. Let $g \in RI(S)$ be an arbitrary element. If $sr = 0$ then $gsr = 0$ and so $grs = 0$. As g is not a divisor of zero, $rs = 0$. Similarly, $rs = 0$ implies $sr = 0$. So $rs = sr = 0$. Let $k \in K$ be an arbitrary element. Then $k \in S^1sS^1$, that is, $k = xsy$ for some $x, y \in S^1$. Thus

$$rk = rxsy = rsxy = 0$$

and

$$kr = xsyr = xsry = 0$$

which implies that $RK = KR = \{0\}$, indeed. As $K \subseteq R$, we get $K^2 = \{0\}$, that is, the core of S is nilpotent.

We show that S is subdirectly irreducible. Let α be a congruence on S such that $\alpha \neq id_S$. Assume that k_1 is a non-zero disjunctive element of S. As k_1 is a disjunctive element, there is an element s is S such that $(s, k_1) \in \alpha$ and $s \neq k_1$. Let k_2 denote the element of K_0 which differs from k_1. We show that $(k_1, k_2) \in \alpha$. Consider the case when $s = e$. Then $k_1 s = k_1$ and $k_1^2 = 0$ imply that $(k_1, 0) \in \alpha$ and so $(k_2, 0) \in \alpha$, because $fk_1 = k_2$. So $(k_1, k_2) \in \alpha$. Consider the case when $s = f$. Then $s^2 = s$ and $sk_1 = k_2$ imply $(s, k_2) \in \alpha$ and so $(k_2, 0) \in \alpha$, because $sk_2 = k_2$ and $k_2^2 = 0$. Thus $(k_1, 0) \in \alpha$, because $k_1 = ek_1$ and $e0 = 0$. So $(k_1, k_2) \in \alpha$. Consider the case when $s = 0$. Then $0 = f0$, $fk_1 = k_2$ imply $(k_2, 0) \in \alpha$, that is, $(k_1, k_2) \in \alpha$. If $s \in K_0$ then $s = k_2$ and so $(k_1, k_2) \in \alpha$. Assume $s \in R - K$. From $(s, k_1) \in \alpha$, we get $(rs, rk_1) \in \alpha$ and $(sr, k_1 r) \in \alpha$ for all $r \in R$. As $rk_1 = k_1 r = 0$, we get $(rs, 0) \in \alpha$ and $(sr, 0) \in \alpha$ for all $r \in R$. Let

$$Q = \{q \in R : \ (q, 0) \in \alpha\}.$$

Then Q is an ideal of S and so $K \subseteq Q$ or $Q = \{0\}$. If $K \subseteq Q$ then $(k_1, k_2) \in \alpha$. If $Q = \{0\}$ then $sr = rs = 0$ for all $r \in R$, that is, $Rs = sR = \{0\}$ which contradicts (ii). Consequently $(k_1, k_2) \in \alpha$ in all cases. Let α_i, $i \in I$ be an arbitrary family of non-identical congruences on S. As $(k_1, k_2) \in \alpha_i$, for all $i \in I$, that is, $\cap_{i \in I} \alpha_i \neq id_S$, we get that S is subdirectly irreducible (see Corollary 1.1.). Thus the theorem is proved. □

Example. Let S be a semigroup defined by the following Cayley table:

	e	f	k_1	k_2	0
e	e	e	k_1	k_1	0
f	f	f	k_2	k_2	0
k_1	k_1	k_1	0	0	0
k_2	k_2	k_2	0	0	0
0	0	0	0	0	0

S is a right commutative subdirectly irreducible semigroup such that $|A_S^r| = |A_S^l| = 1$ and the core of S is nilpotent. We note that $RI(S) = \{e, f\}$, $K_0 = \{k_1, k_2\}$, and $R - K = \emptyset$.

Example. Let S be a semigroup defined by the following Cayley table:

	e	f	r	t	k_1	k_2	0
e	e	e	r	r	k_1	k_1	0
f	f	f	t	t	k_2	k_2	0
r	r	r	k_1	k_1	0	0	0
t	t	t	k_2	k_2	0	0	0
k_1	k_1	k_1	0	0	0	0	0
k_2	k_2	k_2	0	0	0	0	0
0	0	0	0	0	0	0	0

It can be easily verified that S is a subdirectly irreducible right commutative semigroup with a nilpotent core such that $| A_S^r |=| A_S^l |= 1$ and S satisfies both of the conditions (i)-(ii) of Theorem 10.8. We note that $K_0 = \{k_1, k_2\}$, $RI(S) = \{e, f\}$ and $R = \{r, t, k_1, k_2, 0\}$. So $R - K \neq \emptyset$.

Right commutative Δ-semigroups

Lemma 10.4 *([63]) If a right commutative semigroup S is a disjoint union $S = N \cup L$ of an ideal N of S and a subsemigroup L of S which is a left zero semigroup then the τ_u-class $[u]_{\tau_u}$ of S containing u equals L for every $u \in L$ (τ_u is defined in Lemma 10.1).*

Proof. Let $u \in L$ be arbitrary. By Lemma 10.1, τ_u is a congruence on S. As $u^2 = uv$ for every $v \in L$, we have

$$L \subseteq [u]_{\tau_u}.$$

For every $a \in N$, $u^2 \notin N$ and $ua \in N$ imply $(u, a) \notin \tau_u$. Thus

$$L = [u]_{\tau_u}.$$

□

Lemma 10.5 *([63]) If a right commutative Δ-semigroup S is a disjoint union $S = N \cup L$ of an ideal N of S and a subsemigroup L of S which is a left zero semigroup with $|L| \geq 2$ then $|N| = 1$.*

Proof. Let ρ_N denotes the Rees congruence on S modulo N. By Lemma 10.1 and Lemma 10.4, τ_u is a congruence on S and

$$L = [u]_{\tau_u}$$

for every $u \in L$. As S is a Δ-semigroup,

$$\rho_N \subseteq \tau_u$$

for every $u \in L$. Thus

$$(a, 0) \in \tau_u$$

and so

$$ua = 0$$

for every $a \in N$ and $u \in L$. Let

$$\gamma = \{(x, y) \in S \times S : \ x = y \text{ or } x, y \in L\}.$$

Clearly, γ is an equivalence relation on S. As

$$su = suv = svu = sv$$

for every $s \in S$ and $u, v \in L$, γ is left compatible. To show that γ is also right compatible, let $(x, y) \in \gamma$, $x \neq y$ for some $x, y \in S$. Then $x, y \in L$. If $s \in N$ then

$$xs = ys = 0$$

by the above. If $s \in L$ then

$$xs, ys \in L.$$

Consequently,

$$(xs, ys) \in \gamma.$$

Thus γ is right compatible, that is, it is a congruence on S. As $|L| \geq 2$, we have

$$\gamma \neq id_S.$$

As

$$\gamma \cap \rho_N = id_S$$

and S is a Δ-semigroup, we get

$$\rho_N \subseteq \gamma$$

and so

$$\rho_N = id_S.$$

Hence

$$|N| = 1.$$

□

It is clear that a right commutative right zero semigroup is trivial (it has only one element). This and Lemma 10.5 together imply the following.

Lemma 10.6 *([63]) There is no right commutative T2R or T2L semigroup.*

Theorem 10.19 *A semigroup S is a right commutative Δ-semigroup if and only if it satisfies one of the following conditions.*

(i) S is isomorphic to G or G^0, where G is a non-trivial subgroup of a quasi-cyclic p-group (p is a prime).

(ii) S is a two-element semilattice.

(iii) S is isomorphic to L or L^0, where L is a two-element left zero semigroup.

(iv) S is a right commutative nil semigroup whose principal ideals form a chain with respect to inclusion.

(v) S is a right commutative T1 semigroup.

Proof. By Theorem 9.20 and Lemma 10.6, it is obvious. □

We note that if S is a right commutative T1 semigroup ($S = N \cup \{e\}$) such that e is the identity of S then S is commutative and $S = N^1$.

Newt, we give a construction for right commutative T1 semigroups.

Lemma 10.7 *([63]) If S is a right commutative semigroup such that $S^1eS^1 = S$ for some idempotent element e of S then e is a right identity element of S.*

Proof. Let a be an arbitrary element of S. Then

$$a = xey$$

for some $x, y \in S^1$. As S is right commutative, we get

$$ae = xeye = xe^2y = xey = a.$$

□

In our investigation we need some notions from the theory of automata.

Let S be a semigroup. By a *right S-act* (briefly, an *S-act* or an *act*) we mean a triplet (A, S, δ) such that A is an arbitrary set, $\delta : A \times S \to A$ is a mapping such that $\delta(a, st) = \delta(\delta(a, s), t)$ for every $a \in A$ and $s, t \in S$. If S has an identity element e then we suppose that $\delta(a, e) = a$ for every $a \in A$. Sometimes $\delta(a, s)$ will be denoted by as and the act (A, S, δ) is denoted by $\mathbf{A}$.

An equivalence relation α of A is called a congruence of the S-act $\mathbf{A}$ if $(a,b) \in \alpha$ implies $(as, bs) \in \alpha$ for every $a, b \in A$ and $s \in S$.

An S-act $\mathbf{A}$ is called a *Δ-act* if the congruences of $\mathbf{A}$ form a chain with respect to inclusion.

If $\mathbf{B}$ is a subact of the act $\mathbf{A}$ then $\beta = \{(x,y) \in A \times A : \ x = y \text{ or } x, y \in B\}$ is a congruence of $\mathbf{A}$. This congruence is called the Rees congruence of $\mathbf{A}$ modulo $\mathbf{B}$.

If the subact $\mathbf{B}$ has only one element, denoted by b, then b is called a *trap* of $\mathbf{A}$. If I is an ideal of S then $\mathbf{AI}$ is a subact of $\mathbf{A}$, where $AI = \{ai;\ a \in A,\ i \in I\}$. The S-act $\mathbf{A}$ is called a *full act* if $\mathbf{AI} = \mathbf{A}$ for every non-zero ideal I of S.

Construction 10.1 *Let A be a nonempty set and $(B, \diamond)$ be an arbitrary semigroup with zero 0_B. Suppose that $\mathbf{A} = (A, B, \delta)$ is a (right) B-act with a trap 0_A such that $a0_B = 0_A$ for every $a \in A$. We note that A can be considered as a null semigroup with zero 0_A. Assume $A \cap B = \emptyset$ or $A \cap B = \{0_A\} = \{0_B\}$. Let $B^* = B - \{0_B\}$ and $S = A \cup B^*$. On S we define an operation $\circ_\delta$ as follows:*

$$a \circ_\delta b = \begin{cases} a \diamond b, & \text{if } a, b \in B^*\ , a \diamond b \neq 0_B; \\ 0_A, & \text{if } a, b \in B^*,\ a \diamond b = 0_B \text{ or } b \in A; \\ \delta(a,b), & \text{if } a \in A,\ b \in B^*. \end{cases}$$

It is easy to see that S is a semigroup under the operation $\circ_\delta$ in which 0_A is a null element, A (as a null semigroup with a zero 0_A) is an ideal of S, and S is an ideal extension of A by B. This semigroup will be denoted by $[A, B, \circ_\delta]$.

Definition 10.3 *A semigroup which is isomorphic to the semigroup $[A, B, \circ_\delta]$ defined in Construction 10.1 will be called an overact of the null semigroup A by the semigroup B.*

We note that an overact S of a null semigroup A by a commutative semigroup B with an identity element is commutative if and only if $|A| = 1$. In this case S is isomorphic to B. This will be used in Theorem 10.20.

Definition 10.4 *An overact of a null semigroup A by a semigroup B will be called a Δ-overact if $\mathbf{A}$ (as a B-act) is a Δ-act.*

Definition 10.5 *An overact of a null semigroup A by a semigroup B is called a full overact if $\mathbf{A}$ is a full B-act.*

Theorem 10.20 *([63]) A semigroup S is a right commutative T1 semigroup if and only if it is a full Δ-overact $[A, B^1, \circ_\delta]$ of a null semigroup A by a commutative nil Δ-semigroup B^1 with an identity adjoined. In case $|A| = 1$, B has at least two elements and S is isomorphic to B^1.*

Proof. Assume that S is a right commutative T1 semigroup. Then S is a disjoint union $S = N \cup \{e\}$ of an ideal N of S which is a non-trivial right commutative nil semigroup and a one-element subsemigroup $\{e\}$ of S. By Theorem 1.58, N is a Δ-semigroup and $S^1eS^1 = S$. By Lemma 10.7, e is a right identity element of S. By Theorem 1.17, S is $\mathcal{J}$-trivial and so, Theorem 1.18, every non-identity congruence on S is a Rees congruence. Let

$$B' = \{b \in S : \ eb = b\}.$$

It is clear that B' is a subsemigroup of S and

$$e, 0 \in B'.$$

Moreover, e is an identity element of B'. As $a, b \in B'$ implies $ab = eab = eba = ba$, B' is commutative and $B = B' - \{e\} = B' \cap N$ is a subsemigroup of S. Let

$$A = N - B \cup \{0\}.$$

If $a \in N - B$ then

$$ea \neq a.$$

It is clear that

$$(ea, a) \in \tau,$$

where τ is defined in Lemma 10.1. Hence

$$\tau \neq id_S.$$

By Lemma 10.1, τ is a congruence on S. As every non-identity congruence of S is a Rees congruence,

$$(a, 0) \in \tau$$

which implies that

$$xa = x0 = 0$$

for every $x \in S$. Consequently,

$$A = \{a \in S : \ Sa = \{0\}\}.$$

Clearly, A is a null semigroup and an ideal of S, and the Rees factor semigroup S/A of S modulo A is isomorphic to B^1. We define a mapping $\delta : \ A \times B^1 \to A$. For arbitrary $a \in A$ and $b \in B^1$, let $\delta(a, b) = ab$ in S. Then $\mathbf{A} = (A, B^1, \delta)$ is a B^1-act. It is clear that $ab = a \circ_\delta b$ for every $a, b \in S$, where $\circ_\delta$ is defined in Construction 10.1. Consequently, S is an overact of A by $B^1 = B \cup \{e\}$. By Theorem 1.51 and Remark 1.1, B^1 (and so B) is a Δ-semigroup. It remains to show that this overact is full and a Δ-overact. Let I be an arbitrary non-zero ideal of B^1. Then $\mathbf{A}I$ is a subact in the B^1-act $\mathbf{A}$. It is clear that $J = I^* \cup AI$ is an ideal of S, where $I^* = I - \{0\}$. The ideals A and J of S are comparable only that case when $A \subseteq J$, that is, $AI = A$. Consequently, the overact is full. To show that the overact is a Δ-overact, we must show that $\mathbf{A} = (A, B^1, \delta)$ is a

Δ-act. Let α_i, $i = 1,2$ be arbitrary congruences of the B^1-act $\mathbf{A}$. Consider the following relations

$$\alpha_i^* = \{(x,y) \in S \times S : \ x,y \in A \text{ and } (x,y) \in \alpha_i \text{ or } x = y\},$$

$i = 1,2$. Clearly, α_i^*, $i = 1,2$ is an equivalence relation on S. We show that α_i^*, $i = 1,2$ is right and left compatible on S. Assume $(x,y) \in \alpha_i^*$, $x \neq y$. Then $x,y \in A$ and $(x,y) \in \alpha_i$. As α_i is a congruence of the B^1-act $\mathbf{A}$, we get $xs, ys \in A$ and $(xs,ys) \in \alpha_i$ for every $s \in B^1$. Moreover, $xs = ys = 0$ for every $s \in A$. Thus $(xs,ys) \in A$ and $(xs,ys) \in \alpha$ for every $s \in S$. Thus α_i^* is right compatible on S. As $sx = sy = 0$ for every $s \in S$ and $x,y \in A$, we get that α_i^* is also left compatible on S. Thus α_i^*, $i = 1,2$ is a congruence on S. As S is a Δ-semigroup, $\alpha_1^* \subseteq \alpha_2^*$ or $\alpha_2^* \subseteq \alpha_1^*$. As the restriction of α_i^* to A equals α_i, $i = 1,2$, we get $\alpha_1 \subseteq \alpha_2$ or $\alpha_2 \subseteq \alpha_1$. Thus $\mathbf{A}$ is a Δ-act (as a B^1-act). Consequently, S is a full Δ-overact of A by B^1. It is clear that if $|A| = 1$ then $B = N$ and S is isomorphic to B^1. Thus the first part of the theorem is proved.

Conversely, let $S = S[A, B^1, \circ_\delta]$ be a full Δ-overact of a null semigroup A by a commutative nil Δ-semigroup $B^1 = B \cup \{e\}$ with an identity e adjoined. If $|A| = 1$ then S is isomorphic to B^1. Next we suppose $|A| \geq 2$. Then $N = A \cup B^*$ is a non-trivial nil semigroup and S is a disjoint union of the ideal N and the one-element subsemigroup $\{e\}$ of S. Clearly, $S^1 \circ_\delta e \circ_\delta S^1 = S$ and e is a right identity element of S. Moreover, $e \circ_\delta s = 0_A$ if and only if $s \in A$. Thus, for every $a,b \in S$, $a \circ_\delta b \in A$ if and only if $b \circ_\delta a \in A$ and, in this case,

$$e \circ_\delta (a \circ_\delta b) = 0_A = e \circ_\delta (b \circ_\delta a).$$

If $a \circ_\delta b \notin A$ then $a \circ_\delta b = b \circ_\delta a$ and so

$$e \circ_\delta (a \circ_\delta b) = a \circ_\delta b = b \circ_\delta a = e \circ_\delta (b \circ_\delta a).$$

Thus, for every $a,b \in S$,

$$e \circ_\delta (a \circ_\delta b) = e \circ_\delta (b \circ_\delta a).$$

Consequently, for arbitrary $a,b,c \in S$, we get

$$\begin{aligned} c \circ_\delta (a \circ_\delta b) &= c \circ_\delta (e \circ_\delta (a \circ_\delta b)) \\ &= c \circ_\delta (e \circ_\delta (b \circ_\delta a)) = c \circ_\delta (b \circ_\delta a). \end{aligned}$$

Thus S is right commutative. By Theorem 1.58 and Theorem 1.56, it remains to prove that the ideals of N are chain-ordered by inclusion. Let I and J be arbitrary non-zero ideals of N. Then $I \cap A$ and $J \cap A$ are ideals of S. If $I, J \subseteq A$ then $\mathbf{I}$ and $\mathbf{J}$ are subacts of the B^1 act $\mathbf{A}$. The subacts of $\mathbf{A}$ form a chain with respect to inclusion. Thus $I \subseteq J$ or $J \subseteq I$. Next, assume $I \not\subseteq A$. Then $I^* = I - A \cup \{0_B\}$ is a non-zero ideal of B^1. As the overact is full and $a \circ 0_B = 0_A$ for every $a \in A$, we get $A = A \circ I \subseteq I \cap A$ and so $A = I \cap A$, that is, $A \subseteq I$. Thus we can suppose that $J \not\subseteq A$ and so $A \subseteq J$. As the ideals of B are chain ordered with respect to inclusion, we get $J^* \subseteq I^*$ or $I^* \subseteq J^*$. Then $J \subseteq I$ or $I \subseteq J$. Thus N is a Δ-semigroup. $\square$

Corollary 10.2 *([63]) A full overact of a null semigroup A by a commutative nil Δ-semigroup B^1 with an identity adjoined is a Δ-overact if and only if the subacts of the B^1-act $\mathbf{A}$ form a chain with respect to inclusion.*

Proof. Let $\mathbf{A} = (A, B^1, \delta)$ be an act such that $S = [A, B^1, \circ_\delta]$ is a full overact. If this overact is a Δ-overact then $\mathbf{A}$ is a Δ-act which implies that the Rees congruences and so the subacts of $\mathbf{A}$ form a chain with respect to inclusion. Conversely, if the subacts of the act $\mathbf{A}$ form a chain with respect to inclusion then, by the second part of the proof of Theorem 10.20, the ideals of N form a chain with respect to inclusion. Then, by Theorem 1.58 and Theorem 1.56, S is a right commutative T1 semigroup. From the first part of the proof of Theorem 10.20, it follows that $\mathbf{A}$ is a Δ-act. □

If $\mathbf{A} = (A, S, \delta)$ is an S-act and $a \in A$ then let $\mathbf{R}(a)$ denote the subact of $\mathbf{A}$ generated by the element a. $\mathbf{R}(a)$ is called the principal subact of the act $\mathbf{A}$ generated by a. Clearly, $R(a) = \{\delta(a, s);\ s \in S^1\}$.

Lemma 10.8 *([63]) In an S-act $\mathbf{A}$ the subacts form a chain with respect to inclusion if and only if the principal subacts do it.*

Proof. Assume that the principal subacts of $\mathbf{A}$ form a chain with respect to inclusion. Let $\mathbf{I}$ and $\mathbf{J}$ be two arbitrary subacts of $\mathbf{A}$ with $I \neq I \cap J \neq J$. Then there are elements $x \in I$ and $y \in J$ such that $x \notin J$ and $y \notin I$. Clearly, $R(x) = \{x\} \cup xS \subseteq I$ and $R(y) \in J$. By the assumption, $R(x) \subseteq R(y)$ or $R(y) \subseteq R(x)$. Then $x \in J$ or $y \in I$ which is a contradiction. Consequently, $I \subseteq J$ or $J \subseteq I$. □

A construction of right commutative T1 semigroups

In the next eight lemmas, B denotes a commutative nil Δ-semigroup and $\mathbf{A} = (A, B^1, \delta)$ is a B^1-act such that $[A, B^1, \circ_\delta]$ is a full Δ-overact. 0_A denotes the trap of $\mathbf{A}$ and the zero of B^1 is denoted by 0_B.

Lemma 10.9 *([63]) $\mathbf{A}$ is an $\mathcal{R}$-trivial B^1-act (that is $R(a_1) = R(a_2)$ if and only if $a_1 = a_2$ for every $a_1, a_2 \in A$).*

Proof. Assume $R(a_1) = R(a_2)$ for some $a_1, a_2 \in A$ with $a_1 \neq a_2$. Then there are elements $x, y \in B$ such that $a_1 = a_2 \circ_\delta x$, $a_2 = a_1 \circ_\delta y$. Then $a_1 = (a_1 \circ_\delta y) \circ_\delta x = a_1 \circ_\delta (y \circ_\delta x)^n$ for every positive integer n. As B is a nil semigroup and $y \circ_\delta x \in B$, we get $a_1 = 0_A = a_2$ which is a contradiction. □

Lemma 10.10 *([63]) A is totally ordered by $\leq_A$ defined by $a_1 \leq_A a_2$ if and only if $R(a_1) \subseteq R(a_2)$, $a_1, a_2 \in A$.*

Proof. It is evident that $\leq_A$ is reflexive and transitive. By Lemma 10.9, $\mathbf{A}$ is $\mathcal{R}$-trivial. Thus $\leq_A$ is antisymmetric. As $\mathbf{A}$ is a Δ-act, Corollary 10.2 shows that $a_1 \leq_A a_2$ or $a_2 \leq_A a_1$ for every $a_1, a_2 \in A$. Thus A is totally ordered by $\leq_A$. □

Lemma 10.11 *([63]) B^1 is totally ordered by $\leq_B$ defined by $b_1 \leq_B b_2$ if and inly if $J(b_1) \subseteq J(b_2)$, where $J(b)$ denotes the two-sided ideal of B^1 generated by $b \in B^1$.*

Proof. By Theorem 1.17, B^1 is $\mathcal{J}$-trivial. Thus B^1 is totally ordered by $\leq_B$. □

Lemma 10.12 *([63]) For every $a \in A$ and $b \in B^1$, $R(a) = \{a' \in A : a' \leq_A a\}$ and $J(b) = \{b' \in B^1 : b' \leq_B b\}$.*

Proof. By the definition of $\leq_A$ and $\leq_B$, the proof is trivial. □

Lemma 10.13 *([63]) For every $a \in A$, $I_a = \{b \in B^1 : a \circ_\delta b = 0_A\}$ is an ideal of B^1. $I_a = B^1$ if and only if $a = 0_A$.*

Proof. It is obvious. □

Lemma 10.14 *([63]) $a_1 \leq_A a_2$ implies $I_{a_2} \subseteq I_{a_1}$ for every $a_1, a_2 \in A$.*

Proof. Let $a_1 \leq_A a_2$ for some $a_1, a_2 \in A$. Then $R(a_1) \subseteq R(a_2)$ and so $a_1 = a_2 \circ_\delta x$ for some $x \in B^1$. Let $b \in I_{a_2}$ be arbitrary. Then $a_1 \circ_\delta b = (a_2 \circ_\delta x) \circ_\delta b = a_2 \circ_\delta (x \circ_\delta b) = a_2 \circ_\delta (b \circ_\delta x) = (a_2 \circ_\delta b) \circ_\delta x = 0_B \circ_\delta x = 0_A$ which means that $b \in I_{a_1}$. Hence $I_{a_2} \subseteq I_{a_1}$. □

Lemma 10.15 *([63]) $b_1 <_B b_2$ implies $a \circ_\delta b_1 <_A a \circ_\delta b_2$ for every $0_A \neq a \in A$ and $b_1, b_2 \in B^1 - I_a$.*

Proof. Assume that $b_1 <_B b_2$ for some $b_1, b_2 \in B^1 - I_a$. Then $b_1 = b_2 \circ_\delta x$ for some $x \in B$. Thus $a \circ_\delta b_1 = a \circ_\delta b_2 \circ_\delta x$ and so $R(a \circ_\delta b_1) \subseteq R(a \circ_\delta b_2)$ which means that $a \circ_\delta b_1 \leq_A a \circ_\delta b_2$. Assume $a \circ_\delta b_1 = a \circ_\delta b_2$. Then $a \circ_\delta b_2 \circ_\delta x = a \circ_\delta b_2$ and $a \circ_\delta b_2 \circ_\delta x^n = a \circ_\delta b_2$ for every positive integer n. As B is a nil semigroup, $a \circ_\delta b_2 = 0_A$ and so $b_2 \in I_a$ contradicting $b_2 \notin I_a$. Hence $a \circ_\delta b_1 <_A a \circ_\delta b_2$. □

Lemma 10.16 *([63]) For every $a \in A$, φ_a defined by $\varphi_a(b) = a \circ_\delta b$, is a mapping of B^1 onto $[0_A, a] = \{a' \in A;\ a' \leq_A a\}$ such that $\varphi_a(b) = 0_A$ if and only if $b \in I_a$, and, for every $0_A \neq a \in A$, φ_a is an order-preserving bijection of $B^1 - I_a$ onto $(0_A, a]$.*

Proof. By Lemma 10.12, Lemma 10.13 and Lemma 10.15, it is obvious. □

Construction 10.2 *Let B^1 be a commutative nil Δ-semigroup with an identity adjoined. Let A be a totally ordered set by an ordering $\leq_A$. Assume that A has a least element 0_A. Let Φ be a mapping of A into the set of all ideals of B^1 with the following properties:*

(i) Φ is monotone decreasing;

(ii) $\Phi(a) = B^1$ if and only if $a = 0_A$;

(iii) *For every* $a \in A$, *there is a mapping* φ_a *of* B^1 *onto* $[0_A, a] = \{a' \in A : a' \leq_A a\}$ *such that* $\varphi_a(\Phi(a)) = 0_A$ *and if* $a \neq 0_A$ *then* φ_a *is an order-preserving bijection of* $B^1 - \Phi(a)$ *onto* $(0_A, a]$;

(iv) *For every* $a \in A$ *and* $b_1, b_2 \in B^1$, $\varphi_a(b_1 b_2) = \varphi_{\varphi_a(b_1)}(b_2)$;

(v) *For every* $a \in A$ *and every non-zero ideal* I *of* B^1, *there is an element* $a' \in A$ *such that* $a \leq_A a'$ *and* $a \in \varphi_{a'}(I)$.

Let $\gamma_\varphi : A \times B^1 \to A$ *be a mapping defined by* $\gamma_\varphi(a, b) = \varphi_a(b)$. *Clearly,* γ_φ *is well-defined and* $\gamma_\varphi(a, 0_B) = 0_A$, $\gamma_\varphi(0_A, b) = 0_A$, $\gamma_\varphi(a, e) = a$ *for every* $a \in A$, $b \in B^1$ *and the identiy element* e *of* B^1. *If* $b_1, b_2 \in B^1$ *are arbitrary elements then, by property (iv),* $\gamma_\varphi(a, b_1 b_2) = \varphi_a(b_1 b_2) = \varphi_{\varphi_a(b_1)}(b_2) = \gamma_\varphi(\gamma_\varphi(a, b_1), b_2)$. *Thus* $\mathbf{A} = (A, B^1, \gamma_\varphi)$ *is a* B^1*-act. Let* $[(A, \leq_A); B^1; \Phi, \{\varphi_a,\ a \in A\}, \circ_{\gamma_\varphi}]$ *denote the overact of* A *by* B^1, *given by Construction 10.1.*

Example. Let $B = \{0_B\}$ be the trivial semigroup. Then B^1 is a commutative nil Δ-semigroup with an identity adjoined. Let $A = \{a, 0_A\}$. Define an ordering $\leq_A$ on A such that $0_A <_A a$. Let $\Phi(0_A) = B^1$, $\Phi(a) = B$ and $\varphi_{0_A}(B^1) = \{0_A\}$, $\varphi_a(B) = \{0_A\}$, $\varphi_a(e) = a$, where e denotes the identity element of B^1. It can be checked that $\Phi, \varphi_a, \varphi_{0_A}$ satisfy the conditions (i)-(v) of Construction 10.2. Let $\gamma_\varphi(a, b) = \varphi_a(b)$ for every $a \in A$ and $b \in B^1$. Then (A, B^1, γ_φ) is an act. It can be verified that this act is a full Δ-act. The Cayley multiplication table of the semigroup $S = [(A, \leq_A); B^1; \Phi, \{\varphi_a,\ a \in A\}, \circ_{\gamma_\varphi}]$ defined as in Construction 10.2 is the following:

	a	0_A	e
a	0_A	0_A	a
0_A	0_A	0_A	0_A
e	0_A	0_A	e

It can be directly verified that S is a right commutative T1 semigroup which is a disjoint union of the ideal $N = \{a, 0_A\}$ and the one-element subsemigroup $\{e\}$ of S such that N is a nil semigroup.

Theorem 10.21 *([63]) The act* $\mathbf{A} = (A, B^1, \gamma_\varphi)$ *defined in Construction 10.2 is a full* Δ*-act. Conversely, every full* Δ*-act* (A, B^1, δ) *defined by a null semigroup* A *and a commutative nil* Δ*-semigroup* B^1 *with an identity adjoined is isomorphic to an act* (A, B^1, γ_φ) *given in Construction 10.2.*

Proof. To show that $\mathbf{A} = (A, B^1, \gamma_\varphi)$ given in Construction 10.2 is full, let a be an arbitrary element of A and $I \neq \{0_B\}$ be an ideal of B^1. By property (v), there is an element $a' \in A$ with $a \leq_A a'$ such that

$$a \in \varphi_{a'}(I) = \gamma_\varphi(a', I).$$

Hence

$$\gamma_\varphi(A, I) = A,$$

that is, $\mathbf{A} = (A, B^1, \gamma_\varphi)$ is a full B^1-act. To show that $\mathbf{A}$ is a Δ-act, by Corollary 10.2 and Lemma 10.8, it is sufficient to show that the principal subacts of $\mathbf{A}$ form a chain with respect to inclusion. Let a be an arbitrary element of A. By the definition of γ_φ, we have

$$\gamma_\varphi(a, b) \leq_A a$$

for every $b \in B^1$. Thus

$$R(a) \subseteq [0_A, a].$$

We show that $[0_A, a] \subseteq R(a)$. Let $a' \leq_A a$ be arbitrary. We can suppose that $a \neq 0_A$. Then, by condition (iii) of Construction 10.2, there is an element $b' \in B^1 - \Phi(a)$ such that

$$a' = \varphi_a(b') = \gamma_\varphi(a, b') \in R(a).$$

Thus

$$[0_A, a] \subseteq R(a).$$

Consequently,

$$R(a) = [0_A, a]$$

for every $a \in A$. Hence the principal subacts of $\mathbf{A}$ form a chain with respect to inclusion.

Conversely, let $\mathbf{A} = (A, B^1, \delta)$ be a full Δ-act, where A is a null semigroup and B^1 is a commutative nil Δ-semigroup with an identity adjoined. Then the semigroup $[A, B^1, \circ_\delta]$ defined in Construction 10.1 is a full Δ-overact of A by B^1. Then, by Lemma 10.10, A is totally ordered by $\leq_A$. A has a least element 0_A which is the trap of the B^1-act $\mathbf{A}$. By Lemma 10.13, for every $a \in A$, I_a is an ideal of B^1. Let $\Phi(a) = I_a$. Property (i) is satisfied by Lemma 10.14. Condition (ii) follows from Lemma 10.13. By Lemma 10.16, for every $a \in A$, φ_a defined by $\varphi_a(b) = a \circ_\delta b$ is a mapping of B^1 onto $[0_A, a]$ such that $\varphi_a(b) = 0_A$ if and only if $b \in I_a$, and, for every $0_A \neq a \in A$, φ_a is an order-preserving bijection of $B^1 - I_a$ onto $(0_A, a]$. Thus conditions (iii) and (iv) are satisfied. As S is a full overact of A by B^1, $\delta(A, I) = A$ for every non-zero ideal of B^1. This means that, for every element $a \in A$, there are elements $a' \in A$ and $b \in B^1$ such that

$$a = a' \circ_\delta b = \varphi_{a'}(b) \in \varphi_{a'}(I).$$

Thus condition (v) is satisfied. Let γ_φ be the mapping defined as in Construction 10.2. Then

$$\delta(a, b) = a \circ_\delta b = \varphi_a(b) = \gamma_\varphi(a, b).$$

Thus the acts (A, B^1, δ) and (A, B^1, γ_φ) are isomorphic. $\square$

Theorem 10.22 *([63]) The semigroup* $[(A, \leq_A); B^1; \Phi, \{\varphi_a,\ a \in A\}, \circ_{\gamma_\varphi}]$ *defined in Construction 10.2 is a right commutative T1 semigroup, and every right commutative T1 semigroup is isomorphic to a semigroup*

$$[(A, \leq_A); B^1; \Phi, \{\varphi_a,\ a \in A\}, \circ_{\gamma_\varphi}]$$

defined in Construction 10.2.

Proof. By Theorem 10.21, the B^1-act $\mathbf{A} = (A, B^1, \gamma_\varphi)$ defined in Construction 10.2 is a full Δ-act. Thus $[(A, \leq_A); B^1; \Phi, \{\varphi_a,\ a \in A\}, \circ_{\gamma_\varphi}]$ (defined in Construction 10.2) is a full Δ-overact of A by B^1 and so, by Theorem 10.20, it is a right commutative T1 semigroup.

Conversely, let S be a right commutative T1 semigroup. Then, by Theorem 10.20, there is a null semigroup A and a commutative nil Δ-semigroup B^1 with an identity adjoined such that the act (A, B^1, δ) is a full Δ-act and S is isomorphic to the overact $[A, B^1, \circ_\delta]$ of A by B^1 defined in Construction 10.1. By Theorem 10.21, (A, B^1, δ) is isomorphic to an act (A, B^1, γ_φ) defined in Construction 10.2. Thus S is isomorphic to the semigroup $[(A, \leq_A); B^1; \Phi, \{\varphi_a,\ a \in A\}, \circ_{\gamma_\varphi}]$ defined in Construction 10.2. □

Chapter 11

Externally commutative semigroups

In this chapter we deal with semigroups satisfying the identity $axb = bxa$. These semigroups are called externally commutative semigroups. It is clear that an externally commutative semigroup is medial. Thus the externally commutative semigroups are semilattice of externally commutative archimedean semigroups. A semigroup is externally commutative and 0-simple if and only if it is a commutative group with a zero adjoined. A semigroup is externally commutative and archimedean containing at least one idempotent element if and only if it is an ideal extension of a commutative group by an externally commutative nil semigroup. Moreover, every externally commutative archimedean semigroup without idempotent has a non-trivial group homomorphic image. We show that an externally commutative semigroup is regular if and only if it is a semilattice of commutative groups. We construct the least separative, left separative, right separative and weakly separative congruence on an externally commutative semigroup, respectively. We determine the subdirectly irreducible externally commutative semigroups. We prove that a semigroup is subdirectly irreducible and externally commutative with a globally idempotent core if and only if it is isomorphic to either G or G^0 or F, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime) and F is a two-element semilattice. An externally commutative semigroup with a zero and a non-trivial annihilator is subdirectly irreducible if and only if it has a non-zero disjunctive element. If S is a subdirectly irreducible externally commutative semigroup with zero such that $|A_S| = 1$ and the core of S is nilpotent then S is commutative (and so is describen in Chapter 3). At the end of the chapter we determine the externally commutative Δ-semigroups. We prove that a semigroup S is an externally commutative Δ-semigroup if and only if it satisfies one of the following conditions. (1) S is isomorphic to G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime). (2) S is a two-element semilattice. (3) S is an externally commutative nil semigroup whose principal ideals form a chain with respect to inclusion. (4) S is isomorphic to N^1 where N is a non-trivial commutative nil semigroup whose principal ideals form a chain with respect to inclusion.

Definition 11.1 *A semigroup is called an externally commutative semigroup if it satisfies the identity $axb = bxa$.*

We note that, in [48] and [57], the externally commutative semigroups are called completely symmetrical.

It is clear that an externally commutative semigroup is conditionally commutative and weakly commutative.

Lemma 11.1 *Every externally commutative semigroup is medial.*

Proof. Let S be an externally commutative semigroup and $a, b, x, y \in S$ be arbitrary elements. Then

$$axyb = (axy)b = (yxa)b = y(xa)b = b(xa)y = b(xay) = b(yax)$$

$$= b(ya)x = x(ya)b = (xya)b = (ayx)b = ayxb.$$

□

Theorem 11.1 *Every finitely generated periodic externally commutative semigroup is finite.*

Proof. By Theorem 1.1, it is obvious. □

Semilattice decomposition of externally commutative semigroup

Theorem 11.2 *([57]) Every externally commutative semigroup is decomposable into a semilattice of externally commutative archimedean semigroups.*

Proof. By Theorem 9.3 and Lemma 11.1, it is obvious. □

Theorem 11.3 *([57]) A semigroup is externally commutative and 0-simple if and only if it is a commutative group with a zero adjoined.*

Proof. Let S be an externally commutative 0-simple semigroup. Since S is medial then, by Corollary 9.1, it is a rectangular abelian group with a zero adjoined. As an externally commutative left (right) zero semigroup has only one element, S is an abelian group with a zero adjoined. The converse statement is obvious. □

Lemma 11.2 *([57]) A semigroup is externally commutative and archimedean containing at least one idempotent element if and only if it is an ideal extension of a commutative group by an externally commutative nil semigroup.*

Proof. Let S be an externally commutative archimedean semigroup containing at least one idempotent element. As S is also a medial semigroup, it is an ideal extension of a rectangular Abelian group K by a medial nil semigroup Q (see Theorem 9.9). Since K is simple and externally commutative then, by Theorem 11.3, it is a commutative group. It is evident that Q is also externally commutative.

Conversely, assume that the semigroup S is an ideal extension of a commutative group G by an externally commutative nil semigroup Q. Then, by Theorem 2.2, S is an archimedean semigroup with an idempotent element. It is easy to see that

$$\phi : s \mapsto es$$

is a retract homomorphism of S onto G, where e denotes the identity of G. As the externally commutative semigroups form a variety, S is externally commutative (see Theorem 1.40). □

Theorem 11.4 *On an externally commutative semigroup S, the following conditions are equivalent.*

(i) S is regular.

(ii) S is left regular.

(iii) S is right regular.

(iv) S is intra-regular.

(v) S is a semilattice of commutative groups

(vi) S is an inverse semigroup.

Proof. Let a, x and y be arbitrary elements of an externally commutative semigroup S. Then $axa = a$ implies

$$a = a(xa)xa = xaxaa = xa^2,$$

$xa^2 = a$ implies

$$a = xaa = aax = a^2x,$$

$a = a^2x$ implies

$$a = a^2x = a^3x^2 = a(a^2)x^2$$

and $a = xa^2y$ implies

$$a = xa^2y = (xa)x(a^2y^2) = (a^2y^2)x(xa).$$

Hence (i), (ii), (iii) and (iv) are equivalent. As

$$ef = eef = fee = fe$$

for every idempotent elements e and f of S, (i) and (vi) are equivalent. By Theorem 11.2, S is a semilattice Y of externally commutative archimedean semigroups S_α $(\alpha \in Y)$. If S is regular then each S_α is regular and so has an idempotent element. Thus, by Lemma 11.2, each S_α is an ideal extension of a commutative group by an externally commutative nil semigroup. We can conclude that each S_α is a commutative group. Hence (i) implies (v). As (v) implies (i), the theorem is proved. □

Theorem 11.5 *Every externally commutative archimedean semigroup without idempotent has a non-trivial group homomorphic image.*

Proof. By Lemma 11.1 and Theorem 9.11, it is obvious. □

Theorem 11.6 *Let S be an externally commutative semigroup and*

$$\tau = \{(a,b) \in S \times S : \ a^{n+1} = a^n b,\ b^{n+1} = b^n a \text{ for some positive integer } n\},$$

$$\sigma = \{(a,b) \in S \times S : \ a^{n+1} = ba^n,\ b^{n+1} = ab^n \text{ for some positive integer } n\}.$$

Then $\tau = \sigma$ and it is the least left, right and weakly separative congruence on S. Moreover,

$$\delta = \{(a,b) \in S \times S : \ a^{n+2} = a^n ba,\ b^{n+2} = b^n ab \text{ for some positive integer } n\}$$

is the least separative congruence on S.

Proof. By Lemma 11.1 and Theorem 9.16, it is obvious, because $ab^n = b^n a$ for every integer $n \geq 2$ and every elements a and b of an externally commutative semigroup. □

Subdirectly irreducible externally commutative semigroups

Theorem 11.7 *([57]) A semigroup is subdirectly irreducible externally commutative with a globally idempotent core if and only if it is isomorphic to either G or G^0 or F, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime) and F is a two-element semilattice.*

Proof. As an externally commutative left (right) zero semigroup has only one element, our statement follows from Theorem 9.18. □

Theorem 11.8 *([57]) An externally commutative semigroup with a zero and a non-trivial annihilator is subdirectly irreducible if and only if it has a non-zero disjunctive element.*

Proof By Theorem 1.49, it is obvious. □

Theorem 11.9 *([57]) If S is a subdirectly irreducible externally commutative semigroup with zero such that $|A_S| = 1$ and the core of S is nilpotent then S is commutative.*

Proof. Let S be an externally commutative subdirectly irreducible semigroup such that $|A_S| = 1$ and the core K of S is nilpotent. Consider subsets F_1 and F_2 of S defined as follows:

$$F_1 = \{f \in S : \ fK = \{0\}\}$$

and

$$F_2 = \{f \in S : \ Kf = \{0\}\}$$

We show that $F_1 = F_2$. Let f be an arbitrary element of F_1. Assume, in an indirect way, that $f \notin F_2$. Then $Kf = Kf \cup fK \cup KfK$ and so $K = Kf$. This implies $Kf = Kff = ffK = \{0\}$, because S is externally commutative and $f \in F_1$. But this is a contradiction. Consequently, $f \in F_2$ and so $F_1 \subseteq F_2$. Similarly, $F_2 \subseteq F_1$. So $F_1 = F_2$.

Let $F = F_1 (= F_2)$ and $B = S - F$. If B were the empty set then S would be equal to F from which it would follow that $SK = KS = \{0\}$, that is, $K \subseteq A_S$ which is a contradiction.

We show that B is a subsemigroup of S. Let $a, b \in B$ be arbitrary elements. Then

$$aK \cup Ka \cup KaK$$

and

$$bK \cup Kb \cup KbK$$

are non-trivial ideals of S. As K is nilpotent,

$$KaK = \{0\}$$

and

$$KbK = \{0\}.$$

So

$$aK \cup Ka = K = bK \cup Kb.$$

Assume that $ab \notin B$, that is, $abK = Kab = \{0\}$. Then

$$K = aK \cup Ka = a(bK \cup Kb) \cup (bK \cup Kb)a$$

$$= abK \cup aKb \cup bKa \cup Kba = aKb,$$

because $abK = \{0\}$, $Kba = abK = \{0\}$ and $aKb = bKa$.

Since

$$aKb = a(bK \cup Kb)b = abK \cup a(Kbb) = abK \cup abbK \subseteq abK = \{0\},$$

we get

$$K = aKb = \{0\}$$

which is a contradiction. So

$$ab \in B,$$

that is, B is a subsemigroup of S.
Let $k \neq 0$ be an arbitrary element of K. As $k \notin A_S$, $Sk \cup kS \cup SkS$ is a non-trivial ideal of S. So

$$Sk \cup kS \cup SkS = K.$$

As $Fk = kF = \{0\}$, we have

$$Bk \cup kB \cup BkB \cup \{0\} = K.$$

So $k = e_1k$ or $k = ke_2$ or $k = e_3ke_4$ for some $e_1, e_2, e_3, e_4 \in B$.
We show that the third case implies the first two cases. Assume that

$$k = e_3ke_4$$

for some $e_3, e_4 \in B$. Then

$$e_3k = e_3e_3ke_4 = e_4ke_3e_3 = e_3ke_4e_3 = ke_3$$

and, similarly,

$$e_4k = ke_4.$$

So

$$k = e_3ke_4 = e_3e_4k,$$

where $e_3e_4 \in B$. Consequently, we may consider only the first two cases.
Assume

$$k = e_1k$$

for some $e_1 \in B$. We show that $Z = \{s \in S : \ s = e_1s\}$ is an ideal of S. As $k = e_1k$, Z contains at least two elements of S. It is evident that Z is a right ideal. Let $z \in Z$ and $s \in S$ be arbitrary elements. Then

$$e_1z = z.$$

As

$$e_1sz = e_1se_1z = e_1zse_1 = sze_1e_1 = se_1e_1z = sz,$$

we get

$$sz \in Z.$$

So Z is also a left ideal of S. Thus Z is a non-trivial ideal of S.
Then

$$K \subseteq Z,$$

that is,

$$e_1k = k$$

for all $k \in K$.
Let α denote the relation on S defined as follows:

$$\alpha = \{(a, b) \in S \times S : \ e_1^n a = e_1^m b \text{ for some positive integers } n, m\}.$$

It is clear that α is a right congruence on S. We show that α is also left compatible. Let $(a,b) \in \alpha$ and $x \in S$ be arbitrary elements $(a,b \in S)$. Then

$$e_1^n a = e_1^m b$$

for some positive integers n and m. So

$$e_1^{n+1} xa = e_1^n e_1 xa = e_1^n axe_1 = e_1^m bxe_1 = e_1^{m+1} xb,$$

that is,

$$(xa, xb) \in \alpha.$$

Consequently, α is left compatible and so it is a congruence on S.
Since $e_1 k = k$ for all $k \in K$ (see above),

$$\alpha|K = id_K.$$

As S is subdirectly irreducible, K is a dense ideal of S. Thus

$$\alpha = id_S.$$

So $e_1 e_1^2 = e_1^2 e_1$ (that is, $(e_1^2, e_1) \in \alpha$) implies $e_1^2 = e_1$. Consequently, for all $s \in S$,

$$e_1 s = e_1 e_1 s = s e_1 e_1 = s e_1.$$

As $e_1 s = e_1 e_1 s$ (that is, $(s, e_1 s) \in \alpha$) for all $s \in S$, we get

$$e_1 s = s$$

for all $s \in S$. Consequently, for all $s \in S$,

$$e_1 s = s e_1 = s$$

which means that e_1 is a two-sided identity element of S. So, for all $s, t \in S$,

$$st = se_1 t = te_1 s = ts,$$

because S is externally commutative. Thus S is commutative.
In case $k = ke_2$, $e_2 \in B$, we can prove the commutativity of S in a similar way. Thus the theorem is proved. □

We note that the commutative subdirectly irreducible semigroups with a nilpotent core and a trivial annihilator have been described in Chapter 3.

Externally commutative Δ-semigroups

Theorem 11.10 *A semigroup S is an externally commutative Δ-semigroup if and only if it satisfies one of the following conditions.*

(i) S is isomorphic to G or G^0, where G is a non-trivial subgroup of a quasi-cyclic p-group (p is a prime).

(ii) *S is a two-element semilattice.*

(iii) *S is an externally commutative nil semigroup whose principal ideals form a chain with respect to inclusion.*

(iv) *S is isomorphic to N^1 where N is a non-trivial commutative nil semigroup whose principal ideals form a chain with respect to inclusion.*

Proof. Let S be an externally commutative Δ-semigroup. By Lemma 11.1 and the fact that there is no non-trivial externally commutative left (right) zero semigroup, S is satisfies either (i) or (ii) or (v) or (vi) of Theorem 9.20. In case (v), S is also externally commutative. Assume that S is an externally commutative T1 semigroup. Then S is a disjoint union $S = P \cup N$ of a one-element subsemigroup $P = \{e\}$ of S and an ideal N of S which is a (non-trivial) nil semigroup. Since SeS is an ideal of S and $e \in SeS$, $N \cap SeS \neq \emptyset$, we get

$$SeS = S.$$

Let $a \in S$ be arbitrary. Then

$$a = xey$$

for some $x, y \in S$. Then

$$a = xey = (xe)ey = ye(xe) = yexe$$

and

$$a = xey = xe(ey) = (ey)ex = eyex.$$

Thus

$$ae = (yexe)e = yexe = a$$

and

$$ea = e(eyex) = eyex = a,$$

that is, e is an identity element of S. As an externally commutative monoid is commutative, S is N^1, where N is a non-trivial commutative nil semigroup. By Remark 1.1, N is a Δ semigroup and so, by Theorem 1.56, the principal ideals of N form a chain with respect to inclusion.

As the semigroups listed in the theorem are externally commutative Δ-semigroups, the theorem is proved. $\square$

Chapter 12

E-m semigroups, exponential semigroups

In this chapter we deal wih the E-m semigroups and the exponential semigroups. A semigroup is called an E-m semigroup (m is an integer with $m \geq 2$) if it satisfies the identity $(ab)^m = a^m b^m$. A semigroup which is an E-m semigroup for every integer $m \geq 2$ is called an exponential semigroup. We show that a semigroup is an exponential semigroup if and only if it is an E-2 and E-3 semigroup. It is proved that every E-m semigroup (exponential semigroup) is a semilattice of archimedean E-m semigroups (exponential semigroups). It is also shown that every exponential semigroup is a band of t-archimedean semigroups. We show that a semigroup is a 0-simple E-m semigroup if and only if it is a completely simple E-m semigroup with a zero adjoined. We characterize the completely simple E-m semigroups and show that a semigroup is an archimedean E-m semigroup containing at least one idempotent element if and only if it is a retract extension of a completely simple E-m semigroup by a nil E-m semigroup. It is proved that every archimedean E-2 semigroup without idempotent has a non-trivial group homomorphic image. We show that a regular E-m semigroup is a semilattice of completely simple E-m semigroups. Moreover, a semigroup is an inverse E-m semigroup if and only if it is a semilattice of E-m groups. We deal with the regular E-2 semigroups. We show that a semigroup is a regular E-2 semigroup if and only if it is a spined product of some band and a semilattice of abelian groups and so it is a regular exponential semigroup. At the end of the chapter we describe the translational hull of a regular E-2 semigroup.

For an arbitrary semigroup S, let $E(S)$ denote the set of all positive integers m for which S satisfies the identity $(xy)^m = x^m y^m$. It is clear that $1 \in E(S)$ and $E(S)$ is a subsemigroup of the multiplicative semigroup of all positive integers. $E(S)$ is called the *exponent semigroup* of S.

The structure of $E(S)$ seems to be complicated in general and has been known only in some special cases.

Example 12.1. Let $X = \{a, b\}$ and X^+ be the set of all finite sequences of elements of X. Define an operation $*$ on X^+. If $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_m)$ are elements of X^+ then let $(x_1, x_2, \ldots, x_n) * (y_1, y_2, \ldots, y_m)$ be equal to the sequence consisting of the last two elements of the sequence $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ in the original order. More precisely, the result of the product $*$ is (x_n, y_1) if $m = 1$ and (y_{m-1}, y_m) if $m > 1$. It is easy to see that $S = (X^+, *)$ is a semigroup and $E(S)$ contains only 1.

Example 12.2. Let k be an integer with $k \geq 2$ and denote S the Rees factor semigroup of a free semigroup $\mathcal{F}_X$ over the set $X = \{a, b\}$ modulo the ideal I of $\mathcal{F}_X$ containing all elements of $\mathcal{F}_X$ whose length greater then or equl to $2k$. It is easy to see that $E(S) = \{m \in N^+;\ m \geq k\}$.

Example 12.3. Let $X = \{a, b\}$ and m be a fixed integer with $m \geq 2$. Define the following relations on the free semigroup $\mathcal{F}_X$:

$$\alpha = \{(u, v) :\ u = v \text{ or } (\exists x, y \in \mathcal{F}_X)\ u = (xy)^m,\ v = x^m y^m\},$$

$$\alpha_s = \{(u, v) :\ (u, v) \in \alpha \text{ or } (v, u) \in \alpha\},$$

$$\beta = \{(u, v) :\ (\exists e, f \in \mathcal{F}_X^1,\ u', v' \in \mathcal{F}_X)\ u = eu'f,\ v = ev'f \text{ and } (u', v') \in \alpha_s\}.$$

Let γ be the transitive closer of β. It is a matter of checking to see that γ is a congruence on $\mathcal{F}_X$. We note that if two distinct elements of $\mathcal{F}_X$ are in relation modulo γ then their length must be at least $2m$. Let S be the factor semigroup of $\mathcal{F}_X$ modulo γ. It is clear that $((xy)^m, x^m y^m) \in \gamma$ for all $x, y \in \mathcal{F}_X$, that is, $m \in E(S)$.

Clearly, $((ab)^{m-k}, a^{m-k} b^{m-k}) \notin \gamma$ and so $m - k \notin E(S)$, if $1 \leq k \leq m - 2$.

It is clear that an element $w \in \mathcal{F}_X$ is in relation β with an element of the set $A = \{(ab)^{m+1}, aba^m b^m, ab^m a^m b, a^m b^m ab\}$ if and only if $w \in A$. Hence $((ab)^{m+1}, a^{m+1} b^{m+1}) \notin \gamma$, and so $m + 1 \notin E(S)$. Hence $E(S) \subseteq \{m\} \cup \{m + 2, m + 3, \ldots\}$.

Definition 12.1 *For a fixed integer $m \geq 2$, a semigroup S is called an E-m semigroup if $m \in E(S)$.*

Definition 12.2 *A semigroup S is called an exponential semigroup if $m \in E(S)$ for every integer $m \geq 2$. With other words, a semigroup is called an exponential semigroup if it satisfies the identity $(ab)^m = a^m b^m$ for every integer $m \geq 2$.*

Theorem 12.1 *([17]) If a semigroup satisfies the identity $(xy)^2 = x^2 y^2$ then it satisfies the identity $(xy)^n = x^n y^n$ for all positive integers $n \geq 4$.*

Proof. Since $2 \in E(S)$ implies $4 \in E(S)$, it is sufficient to verify the following: if $n > 2$ and $2, n \in E(S)$ then $n + 1 \in E(S)$. Assume $2, n \in E(S)$ for an integer $n > 2$. First, assume that n is odd. Then there is a positive integer k such that $n - 1 = 2k$ and, for arbitrary $x, y \in S$,

$$x^{n+1} y^{n+1} = x(x^n y^n) y = x(xy)^n y = x^2 (yx)^{n-1} y^2$$

$$= x^2((yx)^k)^2y^2 = (x(yx)^k)^2y^2 = ((xy)^kx)^2y^2$$
$$(xy)^{2k}x^2y^2 = (xy)^{n-1}(xy)^2 = (xy)^{n+1}.$$

Next, we suppose that n is even. Then there is a positive integer k such that $n - 2 = 2k$ and, for arbitrary $x, y \in S$,

$$x^{n+1}y^{n+1} = x(x^ny^n)y = x(xy)^ny = x^2(yx)^{n-1}y^2$$
$$= x^2(yx)^{n-3}(yx)^2y^2 = x^2(yx)^{n-3}(yxy)^2$$
$$= x^2(yx)^{n-2}y^2xy = x^2((yx)^k)^2y^2xy$$
$$= (x(yx)^k)^2y^2xy = ((xy)^kx)^2y^2xy = (xy)^{2k}x^2y^2xy$$
$$(xy)^{n-2}(xy)^2(xy) = (xy)^{n+1}.$$

□

Corollary 12.1 *A semigroup S is exponential if and only if it satisfies the identities $(xy)^2 = x^2y^2$ and $(xy)^3 = x^3y^3$.*

Corollary 12.2 *If S is an E-2 semigroup then either $E(S) = N^+$ (and so S is an exponential semigroup) or $E(S) = N^+ - \{3\}$, where N^+ denotes the set of all positive integers.*

Proof. By Theorem 12.1, it is obvious. □

We remark that if we apply Example 12.3 for $m = 2$ then the semigroup $S = \mathcal{F}_X/\gamma$ is an E-2 semigroup and $E(S) = N^+ - \{3\}$.

The exponent semigroup $E(S)$ of an E-m semigroup is fairly simple in case $m = 2$, but the situation is much more complicated if $m > 2$. We have a very usefull information of $E(S)$ for arbitrary E-m semigroup S.

Theorem 12.2 *([15]) If S is an E-m semigroup for some integer $m \geq 2$ then, for every $h \in E(S)$, there exists a positive integer λ_0 such that $h + \lambda m(m-1) \in E(S)$ for every $\lambda \geq \lambda_0$.*

In the literature, $\overline{E}(S) = \{k \in \mathbf{Z}_{m(m-1)} : (\exists \lambda_k \geq 0)\ k + \lambda_k m(m-1) \in E(S)\}$ is described instead of $E(S)$, where S is an E-m semigroup and $\mathbf{Z}_{m(m-1)}$ denotes the multiplicative semigroup of integers modulo $m(m-1)$.

If $k_1, k_2 \in \overline{E}(S)$ (S is an E-m semigroup) then $k_1 + \lambda_{k_1}m(m-1), k_2 + \lambda_{k_2}m(m-1) \in E(S)$ for some integers $\lambda_{k_1}, \lambda_{k_2} \geq 0$. Let $k \cong k_1k_2$ modulo $m(m-1)$ such that $1 \geq k \geq m(m-1)$. Then there is a positive integer h such that $(k_1 + \lambda_{k_1}m(m-1))(k_2 + \lambda_{k_2}m(m-1)) = k + hm(m-1)$. As $E(S)$ is closed under the multiplication, $k \in \overline{E}(S)$. Consequently, $\overline{E}(S)$ is a subsemigroup of the multiplicative semigroup $\mathbf{Z}_{m(m-1)}$.

Definition 12.3 *For an E-m semigroup S, the multiplicative subsemigoup*

$$\overline{E}(S) = \{0 \leq n < m(m-1);\ (\exists\lambda_0)(\forall\lambda \geq \lambda_0)\ n + \lambda m(m-1) \in E(S)\}$$

of $\mathbf{Z}_{m(m-1)}$ is called the exponent semigroup modulo $m(m-1)$ of S.

For brevity, we shall write $\overline{E}(S) = \{\lambda k + 1\}$ if there are integers $k > 0$ and $\lambda \geq 0$ such that $\overline{E}(S) = \{n | 0 \leq n < m(m-1),\ n \cong \lambda k + 1 \bmod m(m-1)\}$ and similarly in the analogous cases.

In [15] and [34], $\overline{E}(S)$ is described when S is an E-m semigroup, $m = 3, \ldots, 9$. Next, we list these results without proof.

Theorem 12.3 *([34]) Let S be an E-3 semigroup. Then $\overline{E}(S)$ is one of the following four subsemigroups of $\mathbf{Z}_6$:*

$$\{1,3\},\ \{1,3,5\},\ \{0,1,3,4\},\ \{0,1,2,3,4,5\}.$$

Theorem 12.4 *([15]) Let S be an E-4 semigroup. Then $\overline{E}(S)$ belongs to one of the following types ($\lambda \geq 0$):*

$$\{\lambda\};\ \{3\lambda, 3\lambda+1\};\ \{4\lambda, 4\lambda+1\};$$

$$\{3\lambda+1\};\ \{1,4,9,0\};\ \{1,4,\}.$$

Conversely, each of these subsemigroups is an exponent semigroup mod 12 of some E-4 semigroup.

Theorem 12.5 *([15]) Let S be an E-5 semigroup. Then $\overline{E}(S)$ belongs to one of the following types ($\lambda \geq 0$):*

$$\{\lambda\};\ \{2\lambda+1\};\ \{4\lambda, 4\lambda+1\};$$

$$\{5\lambda, 5\lambda+1\};\ \{4\lambda+1\};\ \{1,5,11,15\};$$

$$\{1,5,16,0\};\ \{1,5\}.$$

Conversely, each of these subsemigroups is an exponent semigroup mod 20 of some E-5 semigroup.

Theorem 12.6 *([15]) Let S be an E-6 semigroup. Then $\overline{E}(S)$ belongs to one of the following types (where $\lambda \geq 0$):*

$$\{\lambda\},\ \{3\lambda, 3\lambda+1\},\ \{5\lambda, 5\lambda+1\},$$

$$\{6\lambda, 6\lambda+1\},\ \{5\lambda+1\},\ \{1,6,16,21\},$$

$$\{1,6,10,15,16,21,25,0\},\ \{1,6,25,0\},\ \{1,6\}.$$

Conversely, each of these subsemigroups is an exponent semigroup mod 30 of some E-6 semigroup.

Theorem 12.7 *([15]) Let S be an E-7 semigroup. Then $\overline{E}(S)$ belongs to one of the following types (where $\lambda \geq 0$):*

$$\{\lambda\},\ \{2\lambda+1\},\ \{7\lambda, 7\lambda+1\},$$

$$\{3\lambda, 3\lambda+1\},\ \{3\lambda+1\},\ \{6\lambda, 6\lambda+1\},$$

$$\{6\lambda+1\},\ \{1,7,15,21,22,28,36,0\},$$

$$\{6\lambda+1, 6\lambda+3\},\ \{1,7,15,21,29,35\},$$

$$\{1,7,15,21\},\ \{1,7,22,28\}, \{1,7,36,0\},\ \{1,7\}.$$

Conversely, each of these subsemigroups is an exponent semigroup mod 42 of some E-7 semigroup.

Theorem 12.8 *([15]) Let S be an E-8 semigroup. Then $\overline{E}(S)$ belongs to one of the following types (where $\lambda \geq 0$):*

$$\{\lambda\},\ \{4\lambda, 4\lambda+1\},\ \{7\lambda, 7\lambda+1\},$$

$$\{8\lambda, 8\lambda+1\},\ \{7\lambda+1\},$$

$$\{1,8,21,28,29,36,49,0\},$$

$$\{1,8,29,36\},\ \{1,8,49,0\},\ \{1,8\}.$$

Conversely, each of these subsemigroups is an exponent semigroup mod 56 of some E-8 semigroup.

Theorem 12.9 *([15]) Let S be an E-9 semigroup. Then $\overline{E}(S)$ belongs to one of the following types (where $\lambda \geq 0$):*

$$\{\lambda\},\ \{2\lambda+1\},\ \{4\lambda, 4\lambda+1\},$$

$$\{3\lambda, 3\lambda+1\},\ \{9\lambda, 9\lambda+1\},\ \{4\lambda+1\}$$

$$\{8\lambda, 8\lambda+1\},\ \{6\lambda+1, 6\lambda+3\},$$

$$\{8\lambda+1\},\ \{12\lambda, 12\lambda+1, 12\lambda+4, 12\lambda+9\},$$

$$\{12\lambda+1, 12\lambda+9\},\ \{18\lambda+1, 18\lambda+9\}$$

$$\{1,9,16,24,25,33,40,48,49,57,64,0\},$$

$$\{1,9,28,36,37,45,64,0\},$$

$$\{1,9,25,33,49,57\},\ \{1,9,37,45\},$$

$$\{1,9,64,0\},\ \{1,9\}.$$

Conversely, each of these subsemigroups is an exponent semigroup mod 72 of some E-9 semigroup.

Semilattice decomposition of E-m semigroups

Theorem 12.10 *Every E-m semigroup is a right and left Putcha semigroup.*

Proof. Let S be an E-m semigroup for some m. Let $x, y \in S$ be arbitrary elements with $y \in xS^1$, that is, $y = xu$ for some $u \in S^1$. Then $y^m = (xu)^m = x^m y^m \in x^m S^1$. Therefore S is a left Putcha semigroup. Similarly, S is a right Putcha semigroup. □

Corollary 12.3 *Every exponential semigroup is a right and left Putcha semigroup.*

Proof. By Theorem 12.10, it is obvious. □

Theorem 12.11 *([65]) Every E-m semigroup is a semilattice of archimedean E-m semigroups.*

Proof. By Theorem 12.10 and Corollary 2.2, it is obvious. □

Corollary 12.4 *([101]) Every exponential semigroup is a semilattice of exponential archimedean semigroups.*

Proof. By Theorem 12.11, it is obvious. □

Theorem 12.12 *([65]) A strong semilattice of E-m semigroups is also an E-m semigroup.*

Proof. It is obvious. □

Theorem 12.13 *Every E-2 semigroup is a band of t-archimedean semigroups.*

Proof. As an E-2 semigroup satisfies the identity $(ab)^3 = a^2b^2(ab) = (ab)a^2b^2$, the assertion follows from Theorem 1.8. □

Corollary 12.5 *Every exponential semigroup is a band of t-archimedean semigroups.*

Proof. By Theorem 12.13, it is obvious. □

Theorem 12.14 *([51],[65]) A semigroup is a 0-simple E-m semigroup if an only if it is a completely simple E-m semigroup with a zero adjoined.*

Proof. Let S be a 0-simple E-m semigroup. It follows immediately that the semilattice decomposition of S has exactly two arhimedean components S_0 and S_1 such that $S_0 = \{0\}$ and S_1 is a simple semigroup, that is, S is a simple E-m semigroup with a zero adjoined. As S_1 is an E-m semigroup, by Theorem 12.10, it is a left and right Putcha semigroup. Then, by Theorem 2.3, it is completely simple. As the converse statement is trivial, the theorem is proved. □

Theorem 12.15 *([65]) Let $S = \mathcal{M}(I,G,J;P)$ be a completely simple E-m semigroup expressed as a Rees matrix semigroup over a group G with a sandwich matrix P normalized by $p_{j_0,i} = p_{j,i_0} = e$, the identity of G, for all $i \in I$ and $j \in J$. Then S is an E-m semigroup if and only if G is an E-m group and $p_{j,i}^{m-1} = e$ for all $i \in I$ and $j \in J$.*

Proof. Let $(i,a,j) \in S$, a completely simple E-m semigroup as above.

$$(i, a(p_{j,i}a)^{m-1}, j) = (i,a,j)^m = (i,a,j_0)^m(i_0,e,j)^m$$
$$= (i,a^m,j_0)(i_0,e,j) = (i,a^m,j)$$

and so

$$a^{m-1} = (p_{j,i}a)^{m-1}$$

for all $a \in G, i \in I, j \in J$. It follows, letting $a = e$, that

$$p_{j,i}^{m-1} = e$$

for all $i \in I$ and $j \in J$. As

$$(i_0, a^m b^m, j_0) = (i_0,a,j_0)^m(i_0,a,j_0)^m = ((i_0,a,j_0)(i_0,b,j_0))^m$$
$$= (i_0,ab,j_0)^m = (i_0,(ab)^m,j_0),$$

we get

$$(ab)^m = a^m b^m.$$

Hence G is an E-m group.

Conversely, suppose that G is an E-m group and $p_{j,i}^{m-1} = e$ for all $i \in I$ and $j \in J$. Let $(i,a,j),(k,b,n) \in S$ be arbitrary elements. Then

$$(ap_{j,i})^m = a^m p_{j,i}^m = a^m p_{j,i} = a^{m-1}(ap_{j,i})$$

and so

$$(ap_{j,i})^{m-1} = a^{m-1},$$

and dually

$$(p_{j,i}a)^{m-1} = a^{m-1}.$$

Thus

$$(i,a,j)^m(k,b,n)^m = (i, a(p_{j,i}a)^{m-1}p_{j,k}b(p_{n,k}b)^{m-1}, n)$$
$$= (i, aa^{m-1}p_{j,k}bb^{m-1}, n) = (i, a^m p_{j,k} b^m, n).$$

Also we have

$$((i,a,j)(k,b,n))^m = (i, ap_{j,k}b, n)^m = (i, (ap_{j,k}b)(p_{n,i}ap_{j,k}b)^{m-1}, n)$$
$$= (i, (ap_{j,k}b)(ap_{j,k}b)^{m-1}, n) = (i, (ap_{j,k}b)^m, n)$$
$$= (i, a^m p_{j,k}^m b^m, n) = (i, a^m p_{j,k} b^m, n)$$

and so S is an E-m semigroup. $\square$

Corollary 12.6 *A semigroup is an 0-simple E-2 (exponential) semigroup if and only if it is a rectangular abelian group with a zero adjoined.*

Proof. By Theorem 12.14 and Theorem 12.15, it is obvious. □

Theorem 12.16 *([65]) A semigroup is an E-m archimedean semigroup containing at least one idempotent element if and only if it is a retract extension of a completely simple E-m semigroup by an E-m nil semigroup.*

Proof. Let S be an E-m archimedean semigroup containing at least one idempotent element. Since S is a right and left Putcha semigroup (see Theorem 12.10) then, by Theorem 2.4, it is a retract extension of a completely simple semigroup K by a nil semigroup N. It is clear that K and N are E-m semigroups.

Conversely, assume that a semigroup S is a retract extension of a completely simple E-m semigroup K by an E-m nil semigroup N. By Theorem 1.40, S is an E-m semigroup. By Theorem 2.2, S is archimedean and contains at least one idempotent element. □

Corollary 12.7 *S is an E-2 (exponential) archimedean semigroup containing at least one idempotent element if and only if S is a retract extension of a rectangular abelian group by an E-2 (exponential) nil semigroup.*

Proof. By Theorem 12.16 and Corollary 12.6, it is obvious. □

Theorem 12.17 *([65]) Every E-2 (exponential) archimedean semigroup without idempotent element has a non-trivial group homomorphic image.*

Proof. Let S be an E-2 archimedean semigroup without idempotent element. Then, from Theorem 1.42, it follows that, for every $a \in S$,

$$S_a = \{x \in S;\ (\exists i, j, k \in N^+)\ a^i = a^j x a^k\}$$

is the least reflexive unitary subsemigroup of S that contains a. As S is archimedean, the principal right congruence $\mathcal{R}_{S_a}$ is a group congruence on S. If $S_a \neq S$ then $S/\mathcal{R}_{S_a}$ is a non-trivial group homomorphic image of S. Assume $S_a = S$. If s is an arbitrary element of S then

$$a^i = a^j s a^k$$

for some positive integers i, j, k. If

$$a^m = a^n s a^t$$

also holds for some positive integers m, n, t then

$$a^{n+i+t} = a^n a^j s a^k a^t = a^j a^n s a^t a^k = a^{j+m+k}$$

and so

$$n + i + t = j + m + k,$$

that is,

$$i - (j + k) = m - (n + t),$$

because S does not contain idempotent elements. Thus $s' = i-(j+k)$ is defined for each $s \in S$. It is clear that $\varphi\ s \mapsto s'$ is a well-defined mapping of S into the additive semigroup of all integers. We show that φ is a homomorphism. Assume $a^i = a^j u a^k$ and $a^m = a^n v a^t$ for some positive integers i, j, k, m, n, t. As S is an E-2 semigroup,

$$a^{2(i+m)} = (a^j u a^k a^n v a^t)^2 = a^{2j}(ua^{k+n}v)^2 a^{2t}$$

$$= a^j a^j u a^k a^n v u a^{k+n} v a^t a^t = a^{j+i+n} v u a^{k+m+t}$$

and so

$$\varphi(vu) = (vu)' = 2i + 2m - (j + i + n + k + m + t)$$

$$= i + m - (j + n + k + t) = m - (n + t) + i - (j + k) = v' + u' = \varphi(v) + \varphi(u).$$

As $\varphi(a) = 1$, $\varphi(S)$ is isomorphic to the additive semigroup of either the integers or the non-negative integers or the positive integers. These semigroups have non-trivial group homomorphic images.

As an exponential semigroup is an E-2 semigroup, the theorem is proved in both E-2 and exponential semigroups. □

Theorem 12.18 *([108]) On an E-m semigroup S, the following are equivalent.*

(i) S is regular.

(ii) S is right regular.

(iii) S is left regular.

(iv) S is a union of disjoint groups.

Proof. (i) implies (ii). If S is regular then, for any $a \in S$, $a = axa$ for some $x \in S$. Since ax is an idempotent element, we have

$$a = (ax)^m a$$

and so

$$a = a^2(a^{m-2}x^m)a,$$

where $a^{m-2}x^m a = x^2 a$ if $m = 2$. Hence S is right regular

In the same way we may prove that (i) implies (iii).

(ii) implies (iv). If S is right regular then, by Theorem 4.2 of [19], it is a union of disjoint right simple semigroup S_i, $i \in I$. By Theorem 12.14, each S_i has an idempotent element and so, by Theorem 1.27 of [19] each S_i is a right group. As each S_i is a union of disjoint subgroups (see Theorem 1.27 of [19]), S is a union of disjoint subgroups.

In the same way we may prove that (iii) implies (iv). As (iv) implies (i) in an obvious way, the theorem is proved. □

Theorem 12.19 *([65]) A regular E-m semigroup is a semilattice of completely simple E-m semigroups.*

Proof. This follows from Theorem 12.11 and Theorem 12.16. □

Corollary 12.8 *A regular E-2 (exponential) semigroup is a semilattice of rectangular abelian groups.*

Proof. By Theoerem 12.19 and Corollary 12.6, it is obvious. □

Theorem 12.20 *([65]) A semigroup is an inverse E-m semigroup if and only if it is a semilattice of E-m groups.*

Proof. Let S be an inverse E-m semigroup. By Theorem 12.19, S is a semilattice of completely simple E-m semigroups. It is easy to see that the semilattice components are inverse semigroups. As a completely simple inverse semigroup is a group, we have that S is a semilattice of E-m groups. As a semilattice of groups is a strong semilattice of groups, the converse follows from Theorem 1.21 and Theorem 12.12. □

Corollary 12.9 *An E-2 (exponential) semigroup is an inverse semigroup if and only if it is a semilattice of commutative groups.*

Proof. By Theorem 12.20, it is obvious. □

Theorem 12.21 *([65]) The following conditions on an arbitrary semigroup S are equivalent.*

(i) S is a regular E-2 semigroup.

(ii) S is a regular exponential semigroup.

(iii) S is an orthodox band of abelian groups.

(iv) S is a spined product of some band and a semilattice of abelian groups.

Proof. (i) implies (ii): Let S be a regular E-2 semigroup. To show that S is an exponential semigroup, by Corollary 12.1, it is sufficient to show that S is also an E-3 semigroup. By Theorem 12.18, S is a union of disjoint subgroups G_i of S $(i \in I)$. Let $x, y \in S$ be arbitrary elements. Assume $x \in G_i$ and $y \in G_j$ for some $i, j \in I$. Let x^{-1} and y^{-1} denote the inverse of x and y in G_i and in G_j, respectively. Then

$$x^3y^3 = xx^2y^2y = x(x^{-1}x)^2x^2y^2(yy^{-1})^2y$$
$$= x(x^{-1})^2x^4y^4(y^{-1})^2y = x(x^{-1})^2(x^2y^2)^2(y^{-1})^2y$$
$$= x(x^{-1})^2x^2y^2x^2y^2(y^{-1})^2y = x(x^{-1}x)^2y^2x^2(yy^{-1})^2y$$
$$= xy^2x^2y = x(yx)^2y = (xy)^3.$$

(ii) implies (iii). Let S be a regular exponential semigroup. Then, by Corollary 12.8, S is a semilattice of rectangular abelian groups. Then, by Theorem 1.27, S is an orthogroup. By Theorem 1.29, an orthogroup is an orthodox band of its maximal subgroups if and only if the Green's equivalence $\mathcal{H}$ is a congruence on S. The $\mathcal{H}$-classes of our semigroup are the abelian groups which appears in the semilattice decomposition mentioned above. We must show that if $e, f \in E_S$, and if $a \in H_e$, $b \in H_f$ then $ab \in H_{ef}$, where H_e denotes the $\mathcal{H}$-class of S containing the idempotent e. Let a^{-1} and b^{-1} denote the inverse of a and b in H_e and H_f, respectively. Then

$$abb^{-1}a^{-1}ab = afeb = aefefb = a(ef)^2b = aefb = ab,$$

because $ef \in E_S$. So, by the foregoing,

$$g = ab(b^{-1}a^{-1})^2ab \in E_S,$$

and

$$ab \in H_g.$$

As

$$\begin{aligned} g &= ab(b^{-1})^2(a^{-1})^2ab = afb^{-1}a^{-1}eb = ab^{-1}a^{-1}b \\ &= aeb^{-1}a^{-1}fb = a^2a^{-1}b^{-1}a^{-1}b^{-1}b^2 = a^2(a^{-1})^2(b^{-1})^2b^2 \\ &= (aa^{-1}b^{-1}b)^2 = (ef)^2 = ef, \end{aligned}$$

we have

$$ab \in H_{ef}.$$

Hence S is an orthodox band of abelian groups.

(iii) implies (iv) by Theorem 1.30. As the spined product of a band and a semigroup which is a semilattice of abelian group is clearly a regular and $E-2$ semigroup, that is, (iv) implies (i), the theorem is proved. □

Let S be a regular E-2 semigroup. Then, by Corollary 12.8, S is a semilattice Y of rectangular abelian groups $S_\alpha = I_\alpha \times G_\alpha \times M_\alpha$ $(\alpha \in Y)$. By Theorem 1.27, S is an orthogroup and so, by Theorem 1.28, the product in S is determined by left representations $t_{\alpha,\beta}(\ )$ of S_α by transformations of I_β, right representations $(\ )\tau_{\alpha,\beta}$ of S_α by transformations of M_β and homomorphisms $(\ )\psi_{\alpha,\beta}$ of G_α into G_β $(\alpha, \beta \in Y$with $\alpha \geq \beta)$. If $A = (i_\alpha, g_\alpha, m_\alpha) \in S_\alpha$ and $B = (i_\beta, g_\beta, m_\beta) \in S_\beta$ are arbitrary elements and $\alpha \geq \beta$, then

$$AB = ((t_{\alpha,\beta}A)i_\beta, (g_\alpha\psi_{\alpha,\beta})g_\beta, m_\beta)$$

and

$$BA = (i_\beta, g_\beta(g_\alpha\psi_{\alpha,\beta}), m_\beta(A\tau_{\alpha,\beta})).$$

Since

$$\begin{aligned} (AB)^2 &= ((t_{\alpha,\beta}A)i_\beta, (g_\alpha\psi_{\alpha,\beta})g_\beta, m_\beta)^2 \\ &= ((t_{\alpha,\beta}A)i_\beta, ((g_\alpha\psi_{\alpha,\beta})g_\beta)^2, m_\beta) \end{aligned}$$

$$= ((t_{\alpha,\beta}A)i_\beta, (g_\alpha\psi_{\alpha,\beta})^2 g_\beta^2, m_\beta)$$
$$= ((t_{\alpha,\beta}A)i_\beta, (g_\alpha^2\psi_{\alpha,\beta})g_\beta^2, m_\beta)$$

and

$$A^2B^2 = (i_\alpha, (g_\alpha)^2, m_\alpha)(i_\beta, (g_\beta)^2, m_\beta)$$
$$= ((t_{\alpha,\beta}A^2)i_\beta, (g_\alpha^2\psi_{\alpha,\beta})g_\beta^2, m_\beta),$$

we get

$$t_{\alpha,\beta}A = t_{\alpha,\beta}A^2.$$

Since $t_{\alpha,\beta}$ is a homomorphism then

$$t_{\alpha,\beta}(i_\alpha, e_\alpha, m_\alpha)$$
$$= t_{\alpha,\beta}(i_\alpha, g_\alpha g_\alpha^{-1}, m_\alpha)$$
$$= t_{\alpha,\beta}(i_\alpha, g_\alpha, m_\alpha)t_{\alpha,\beta}(i_\alpha, g_\alpha^{-1}, m_\alpha)$$
$$= t_{\alpha,\beta}(i_\alpha, g_\alpha^2, m_\alpha)t_{\alpha,\beta}(i_\alpha, g_\alpha^{-1}, m_\alpha)$$
$$= t_{\alpha,\beta}(i_\alpha, g_\alpha, m_\alpha).$$

Thus $t_{\alpha,\beta}(i_\alpha, g_\alpha, m_\alpha)$ does not depend on g_α and so it can be considered as a homomorphism of $I_\alpha \times M_\alpha$ into $\mathcal{T}_{I_\beta}$, the semigroup of all transformations of I_β acting on the left. More precisely, if $(i_\alpha, m_\alpha) \in I_\alpha \times M_\alpha$ then $t_{\alpha,\beta}(i_\alpha, m_\alpha) = t_{\alpha,\beta}(i_\alpha, a_\alpha, m_\alpha)$ for some $a_\alpha \in G_\alpha$. Similarly, $\tau_{\alpha,\beta}(i_\alpha, g_\alpha, m_\alpha)$ does not depend on g_α and so $\tau_{\alpha,\beta}$ can be considered as a homomorphism of $I_\alpha \times M_\alpha$ into $\mathcal{T}_{M_\beta}$, the semigroup of all transformations of M_β acting on the right.

Let (λ, ρ) be an arbitrary bitranslation of S, that is, $(\lambda, \rho) \in \Omega(S)$. Consider an element $(i_\alpha, e_\alpha, m_\alpha)$ of S_α (e_α is the identity of G_α). Assume $\lambda(i_\alpha, e_\alpha, m_\alpha) \in S_\beta$ and $(i_\alpha, e_\alpha, m_\alpha)\rho \in S_\delta$. As

$$\lambda(i_\alpha, e_\alpha, m_\alpha) = \lambda((i_\alpha, e_\alpha, m_\alpha)(i_\alpha, e_\alpha, m_\alpha))$$
$$= (\lambda(i_\alpha, e_\alpha, m_\alpha))(i_\alpha, e_\alpha, m_\alpha),$$

we have $\beta = \beta\alpha$. Similarly, $\delta = \delta\alpha$. Moreover,

$$(\lambda(i_\alpha, e_\alpha, m_\alpha))\rho = (\lambda(i_\alpha, e_\alpha, m_\alpha)(i_\alpha, e_\alpha, m_\alpha))\rho$$
$$= (\lambda(i_\alpha, e_\alpha, m_\alpha))((i_\alpha, e_\alpha, m_\alpha)\rho) \in S_{\beta\delta}$$

and, similarly,

$$\lambda((i_\alpha, e_\alpha, m_\alpha)\rho) = (\lambda(i_\alpha, e_\alpha, m_\alpha))((i_\alpha, e_\alpha, m_\alpha)\rho) \in S_{\beta\delta}.$$

As

$$(\lambda(i_\alpha, e_\alpha, m_\alpha))^2$$
$$= (\lambda(i_\alpha, e_\alpha, m_\alpha))(\lambda(i_\alpha, e_\alpha, m_\alpha))$$
$$= ((\lambda(i_\alpha, e_\alpha, m_\alpha))\rho)(i_\alpha, e_\alpha, m_\alpha),$$

we have $\beta = \beta\delta\alpha = \beta\delta$. As

$$((i_\alpha, e_\alpha, m_\alpha)\rho)^2$$
$$= ((i_\alpha, e_\alpha, m_\alpha)\rho)((i_\alpha, e_\alpha, m_\alpha)\rho)$$
$$= (i_\alpha, e_\alpha, m_\alpha)(\lambda((i_\alpha, e_\alpha, m_\alpha)\rho)),$$

we have $\delta = \alpha\beta\delta = \beta\delta$. Consequently $\beta = \delta$. Hence for every $g^*_\alpha \in G_\alpha$ and $m^*_\alpha \in M_\alpha$, we have $\lambda(i_\alpha, g^*_\alpha, m^*_\alpha) = (\lambda(i_\alpha, e_\alpha, m_\alpha))(i_\alpha, g^*_\alpha, m^*_\alpha) \in S_{\beta\alpha} = S_\beta$. Similarly, $(i^*_\alpha, g^*_\alpha, m_\alpha)\rho \in S_\beta$ for every $g^*_\alpha \in G$ and $i^*_\alpha \in I_\alpha$. Let $(i^*_\alpha, g^*_\alpha, m^*_\alpha) \in S_\alpha$ be an arbitrary element. Then $(i^*_\alpha, g^*_\alpha, m_\alpha)\rho \in S_\beta$ and so $\lambda(i^*_\alpha, g^*_\alpha, m_\alpha) \in S_\beta$ from which we can conclude that $(\lambda(i^*_\alpha, g^*_\alpha, m^*_\alpha) \in S_\beta$. Consequently, $\lambda(S_\alpha) \subseteq S_\beta$. Similarly, $(S_\alpha)\rho \subseteq S_\beta$. Let θ denote the canonical homomorphism of S onto Y. Then λ' and ρ' defined by

$$\lambda'(\theta(y)) = \theta(\lambda(y))$$

and

$$(\theta(y))\rho' = \theta((y)\rho)$$

$(y \in S)$ are well defined mappings of Y into itself such that $\lambda' = \rho'$. As

$$(\lambda')^2(\theta(y)) = \lambda'(\lambda'(\theta(y))) = \lambda'(\lambda'(\theta(y))\theta(y))$$
$$= \lambda'(\theta(y)\lambda'(\theta(y))) = \lambda'(\theta(u))\lambda'(\theta(y)) = \lambda'(\theta(y))$$

for every $y \in S$, we get $(\lambda')^2 = \lambda'$. Moreover, for every $a, b \in S$,

$$\lambda'(\theta(a)\theta(b)) = \lambda'(\theta(ab)) = \theta(\lambda(ab)) = \theta(\lambda(a)b) = \theta(\lambda(a))\theta(b)$$
$$= \lambda'(\theta(a))\theta(b) = (\lambda')^2(\theta(a))\theta(b) = \rho'(\lambda'(\theta(a)))\theta(b)$$
$$= ((\lambda'(\theta(a)))\rho'(\theta(b)) = \lambda'(\theta(a))\lambda'(\theta(b)).$$

Consequently, $\lambda' = \delta'$ is an idempotent homomorphism. Let $A = \lambda'(Y)$ and denote λ' by Γ_A. Then A is a retract ideal of Y. By [73], the set R_Y of retract ideals of Y forms a semilattice under intersection and that associated with each $A \in R_Y$ is a unique retract homomorphism Γ_A.

Theorem 12.22 *([65]) Let S be an E-2 regular semigroup such that S is a semilattice Y of rectangular abelian groups $I_\alpha \times G_\alpha \times M_\alpha$, $\alpha \in Y$. Let R_Y denote the set of retract ideals of Y and let $\Omega(S)$ be the translational hull of S. Then*

$$\Omega(S) \cong \cup_{A \in R_Y} \{([k_\alpha(\ ), a_\alpha, (\ )l_\alpha])_{\alpha \in A} \in \prod_{\alpha \in A} (\mathcal{T}_{I_\alpha} \times G_\alpha \times \mathcal{T}_{M_\alpha}) :$$

$$(\forall \alpha, \beta \in A \textit{ with } \alpha \geq \beta)\ (a_\alpha)\psi_{\alpha,\beta} = a_\beta, \textit{ and}$$
$$(\forall (i_\alpha, m_\alpha) \in I_\alpha \times M_\alpha)\ k_\beta \circ t_{\alpha,\beta}(i_\alpha, m_\alpha) = t_{\alpha,\beta}(k_\alpha(i_\alpha), m_\alpha),$$
$$t_{\alpha,\beta}(i_\alpha, m_\alpha) \circ k_\beta = t_{\alpha,\beta}(i_\alpha, (m_\alpha)l_\alpha),$$

$$l_\beta \circ \tau_{\alpha,\beta}(i_\alpha, m_\alpha) = \tau_{\alpha,\beta}(k_\alpha(i_\alpha), m_\alpha),$$

$$\tau_{\alpha,\beta}(i_\alpha, m_\alpha) \circ l_\beta = \tau_{\alpha,\beta}(i_\alpha, (m_\alpha)l_\alpha),$$

$$(\forall \alpha \in Y - A) \text{ and } \beta = \Gamma_A(\alpha))(\forall (i_\alpha, m_\alpha) \in I_\alpha \times M_\alpha)$$

$$k_\beta \circ t_{\alpha,\beta}(i_\alpha, m_\alpha),\ t_{\alpha,\beta}(i_\alpha, m_\alpha) \circ k_\beta,$$

$$l_\beta \circ \tau_{\alpha,\beta}(i_\alpha, m_\alpha),\ \tau_{\alpha,\beta}(i_\alpha, m_\alpha) \circ l_\beta \text{ are all constant functions.}\}$$

The product in $\Omega(S)$ *is given by*

$$([k_\alpha(\ ), a_\alpha, (\ )l_\alpha])_{\alpha\in A}([j_\beta(\ ), b_\beta, (\ )p_\beta])_{\beta\in B}$$

$$= ([k_\gamma \circ j_\gamma(\ ), a_\gamma b_\gamma, (\ )l_\gamma p_\gamma])_{\gamma\in A\cap B}.$$

Proof. Let S be an E-2 regular semigroup such that S is a semilattice Y of rectangular abelian groups $S_\alpha = I_\alpha \times G_\alpha \times M_\alpha$, $\alpha \in Y$. Let $(\lambda, \rho) \in \Omega(S)$ be arbitrary. Then, by the remark befor the theorem, λ' and ρ' defined by $\lambda'(\theta(y)) = \theta(\lambda(y))$ and $(\theta(y))\rho' = \theta((y)\rho)$ $(y \in S)$ are idempotent homomorphisms of Y into itself such that $\lambda' = \rho'$. Moreover, $A = \lambda'(Y)$ is a retract ideal of Y. Denoting λ' by Γ_A, the set R_Y of retract ideals of Y forms a semilattice under intersection and that associated with each $A \in R_Y$ is a unique retract homomorphism Γ_A. For all $\beta \in A$, $(\lambda, \rho)|S_\beta \in \Omega(S_\beta)$, where $\Omega(S_\beta) \cong \mathcal{T}_{I_\alpha} \times G_\alpha \times \mathcal{T}_{M_\alpha}$ by Theorem 1.34. Suppose $\alpha \in Y - A$ and let $\beta = \Gamma_A(\alpha)$. Let $(i_\alpha, g_\alpha, m_\alpha) \in S_\alpha$ and $(i_\beta, g_\beta, m_\beta) \in S_\beta$ arbitrary elements. Since $\Gamma_A(\alpha) = \beta$, we have $\lambda(i_\alpha, g_\alpha, m_\alpha) = (i^*_\beta, g^*_\beta, m^*_\beta) \in S_\beta$. Then there is an element a_β of G_β such that

$$a_\beta(g_\alpha\psi_{\alpha,\beta}) = g^*_\beta$$

and so

$$\lambda(i_\alpha, g_\alpha, m_\alpha) = (i^*_\beta, a_\beta(g_\alpha\psi_{\alpha,\beta}), m^*_\beta).$$

Similarly, we may assume that

$$(i_\alpha, g_\alpha, m_\alpha)\rho = (i'_\beta, (g_\alpha\psi_{\alpha,\beta})a'_\beta, m'_\beta).$$

Then we have

$$(\lambda(i_\alpha, g_\alpha, m_\alpha))(i_\beta, g_\beta, m_\beta) = (i^*_\beta, a_\beta(g_\alpha\psi_{\alpha,\beta}), m^*_\beta)(i_\beta, g_\beta, m_\beta)$$

$$= (i^*_\beta, a_\beta(g_\alpha\psi_{\alpha,\beta}), m_\beta)$$

and

$$\lambda((i_\alpha, g_\alpha, m_\alpha)(i_\beta, g_\beta, m_\beta)) = \lambda(t_{\alpha,\beta}(i_\alpha, m_\alpha))i_\beta, (g_\alpha\psi_{\alpha,\beta})g_\beta, m_\beta)$$

$$= ((k_\beta \circ t_{\alpha,\beta}(i_\alpha, m_\alpha))i_\beta, a_\beta(g_\alpha\psi_{\alpha,\beta})g_\beta, m_\beta).$$

So

$$(k_\beta \circ t_{\alpha,\beta}(i_\alpha, m_\alpha))I_\beta = i^*_\beta.$$

We can prove, in a similar way, that

$$M_\beta((i_\alpha, m_\alpha)\tau_{\alpha,\beta} \circ l_\beta) = m'_\beta.$$

As λ and ρ are linked, we get from

$$\begin{aligned}((i_\alpha, g_\alpha, m_\alpha)\rho)(i_\beta, g_\beta, m_\beta) &= (i'_\beta, (g_\alpha\psi_{\alpha,\beta})a'_\beta, m'_\beta)(i_\beta, g_\beta, m_\beta)\\ &= (i'_\beta, (g_\alpha\psi_{\alpha,\beta})a'_\beta g_\beta, m_\beta)\end{aligned}$$

and

$$\begin{aligned}(i_\alpha, g_\alpha, m_\alpha)(\lambda(i_\beta, g_\beta, m_\beta)) &= (i_\alpha, g_\alpha, m_\alpha)(k_\beta(i_\beta), a_\beta g_\beta, m_\beta)\\ &= ((t_{\alpha,\beta}(i_\alpha, m_\alpha) \circ k_\beta)i_\beta, (g_\alpha\psi_{\alpha,\beta})a_\beta g_\beta, m_\beta)\end{aligned}$$

that

$$(t_{\alpha,\beta}(i_\alpha, m_\alpha) \circ k_\beta)I_\beta = i'_\beta.$$

We can prove, in a similar way, that

$$M_\beta(l_\beta \circ (i_\alpha, m_\alpha)\tau_{\alpha,\beta}) = m^*_\beta.$$

Hence $k_\beta \circ t_{\alpha,\beta}(i_\alpha, m_\alpha)$, $t_{\alpha,\beta}(i_\alpha, m_\alpha) \circ k_\beta$, $l_\beta \circ \tau_{\alpha,\beta}(i_\alpha, m_\alpha)$ and $\tau_{\alpha,\beta}(i_\alpha, m_\alpha) \circ l_\beta$ are all constant functions. Moreover, it is clear that if we know how (λ, ρ) acts on $\cup_{\beta\in A} S_\beta$ then we know how (λ, ρ) acts on S. Thus with (λ, ρ) we can associate

$$([k_\beta(\ ), a_\beta, (\ )l_\beta])_{\beta\in A} \in \prod_{\beta\in A}(\mathcal{T}_{I_\beta} \times G_\beta \times \mathcal{T}_{M_\beta}).$$

Let $\alpha, \beta \in A$ with $\alpha \geq \beta$ and $(i_\alpha, g_\alpha, m_\alpha) \in S_\alpha$, $(i_\beta, g_\beta, m_\beta) \in S_\beta$. Then

$$\begin{aligned}&(\lambda(i_\alpha, g_\alpha, m_\alpha))(i_\beta, g_\beta, m_\beta)\\ &= (k_\alpha(i_\alpha), a_\alpha g_\alpha, m_\alpha)(i_\beta, g_\beta, m_\beta) = (t_{\alpha,\beta}(k_\alpha(i_\alpha), m_\alpha))i_\beta, ((a_\alpha g_\alpha)\psi_{\alpha,\beta})g_\beta, m_\beta)\\ &= \lambda((i_\alpha, g_\alpha, m_\alpha)(i_\beta, g_\beta, m_\beta)) = \lambda(t_{\alpha,\beta}(i_\alpha, m_\alpha))i_\beta, (g_\alpha\psi_{\alpha,\beta})g_\beta, m_\beta)\\ &= k_\beta \circ t_{\alpha,\beta}(i_\alpha, m_\alpha))i_\beta, a_\beta(g_\alpha\psi_{\alpha,\beta})g_\beta, m_\beta).\end{aligned}$$

Thus

$$a_\alpha\psi_{\alpha,\beta} = a_\beta$$

and

$$k_\beta \circ t_{\alpha,\beta}(i_\alpha, m_\alpha) = t_{\alpha,\beta}(k_\alpha(i_\alpha), m_\alpha).$$

We can prove, in a similar way, that

$$\tau_{\alpha,\beta}(i_\alpha, m_\alpha) \circ l_\beta = \tau_{\alpha,\beta}(i_\alpha, (m_\alpha)l_\alpha).$$

As λ and ρ are linked, we find that

$$t_{\alpha,\beta}(i_\alpha, m_\alpha) \circ k_\beta = t_{\alpha,\beta}(i_\alpha, (m_\alpha)l_\alpha)$$

and

$$l_\beta \circ \tau_{\alpha,\beta}(i_\alpha, m_\alpha) = \tau_{\alpha,\beta}(k_\alpha(i_\alpha), m_\alpha).$$

Thus the conditions of the theorem are necessary. It is a matter of checking to see that they are sufficient. $\square$

Chapter 13

WE-m semigroups

In this chapter we deal with semigroups in which, for every elements a and b, there is a non-negative integer k such that $(ab)^{m+k} = a^m b^m (ab)^k = (ab)^k a^m b^m$, where m is a fixed integer $m \geq 2$. These semigroups are called WE-m semigroups. It is clear that every E-m semigroup is a WE-m semigroup. The examination of WE-m semigroups need some results about E-m semigroups. Thus the E-m semigroups were examined in the previous chapter. As a WE-m semigroup is a left and right Putcha semigroup, it is a semilattice of WE-m archimedean semigroups. We show that the 0-simple WE-m semigroups are the completely simple E-m semigroups with a zero adjoined. A semigroup is a WE-m archimedean semigroup containing at least one idempotent element if and only if it is a retract extension of a completely simple E-m semigroup by a nil semigroup. We also prove that every WE-2 archimedean semigroup without idempotent element has a non-trivial group homomorphic image. We deal with the regular WE-m semigroups. We show that the regular WE-m semigroups are exactly the regular exponential semigroups. Moreover, we show that a semigroup which is an ideal extension of a regular semigroup K by a nil semigroup N is a WE-2 semigroup if and only if K is an E-2 semigroup and the extension is retract. We deal with the subdirectly irreducible WE-2 semigroups. It is shown that a semigroup is a subdirectly irreducible WE-2 semigroup with a globally idempotent core if and only if it is isomorphic to either G or G^0 or B, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime) and B is a non-trivial subdirectly irreducible band.

For an arbitrary semigroup S, let $WE(S)$ denote the set of all positive integers m which satisfies the condition that, for every couple $(a,b) \in S \times S$, there is a non-negative integer k such that $(xy)^{m+k} = x^m y^m (xy)^k = (xy)^k x^m y^m$.

We note that if the equation $(xy)^{m+k} = x^m y^m (xy)^k = (xy)^k x^m y^m$ holds for some non-negative integer k then it holds for all integers $t \geq k$.

Theorem 13.1 *([51]) For every semigroup S, $WE(S)$ is a subsemigroup of the multiplicative semigroup of all positive integers.*

Proof. Let S be an arbitrary semigroup. $WE(S)$ is not empty, because 1 is in $WE(S)$. Consider two elements n and m in $WE(S)$. Let x and y be arbitrary elements of S. Then there are positive integers k and t such that

$$(xy)^{n+k} = x^n y^n (xy)^k = (xy)^k x^n y^n$$

and

$$(x^n y^n)^{m+t} = x^{nm} y^{nm} (x^n y^n)^t = (x^n y^n)^t x^{nm} y^{nm}.$$

Then

$$\begin{aligned}(xy)^{nm+nt+k} &= (xy)^{n(m+t)+k} = (xy)^{n+n(m+t-1)+k}\\ &= x^n y^n (xy)^{n(m+t-1)+k} = x^n y^n (xy)^{n+n(m+t-2)+k}\\ &= (x^n y^n)^2 (xy)^{n(m+t-2)+k} = \ldots = (x^n y^n)^{m+t} (xy)^k\\ &= x^{nm} y^{nm} (x^n y^n)^t (xy)^k = x^{nm} y^{nm} (x^n y^n)^{t-1} x^n y^n (xy)^k\\ &= x^{nm} y^{nm} (x^n y^n)^{t-1} (xy)^{n+k} = \ldots = x^{nm} y^{nm} (xy)^{nt+k}.\end{aligned}$$

Similarly,

$$(xy)^{nm+nt+k} = (xy)^{nt+k} x^{nm} y^{nm}.$$

Thus $nm \in WE(S)$. □

We note that the exponent semigroup $E(S)$ of a semigroup S is a subsemigroup of $WE(S)$.

Definition 13.1 *For a fixed integer $m \geq 2$, a semigroup S is called a WE-m semigroup if $m \in WE(S)$. With other words, for every $(a, b) \in S \times S$, there is a non-negative integer k such that*

$$(ab)^{m+k} = a^m b^m (ab)^k = (ab)^k a^m b^m.$$

Semilattice decomposition of WE-m semigroups

Theorem 13.2 *Every WE-m semigroup is a left and right Putcha semigroup.*

Proof. Let S be a WE-m semigroup. If $a, b \in S$ are arbitrary elements with $b \in aS^1$, that is, $b = ay$ for some $y \in S^1$ then there is a positive integer t such that

$$b^{m+t} = (ay)^{m+t} = a^m y^m (ay)^t \in a^2 S^1.$$

Hence S is a left Putcha semigroup. We can prove, in a similar way, that S is a right Putcha semigroup. □

Theorem 13.3 *([51]) Every WE-m semigroup is decomposable into a semilattice of WE-m archimedean semigroups.*

Proof. By Theorem 13.2 and Corollary 2.2, it is obvious. □

Theorem 13.4 *([51]) A semigroup is a 0-simple WE-m semigroup if and only if it is a completely simple E-m semigroup with a zero adjoined.*

Proof. Let S be a 0-simple WE-m semigroup. Then, by Theorem 13.3, it is a semilattice of archimedean semigroups and every non-zero element of S is in the same semilattice component K of S. If 0 was in K then S would be a nil semigroup which is contradicts the assumption that S is 0-simple. Thus $S = K \cup \{0\}$ and K is a simple WE-m semigroup. By Theorem 13.2, K is a left and right Putcha semigroup and so, by Theorem 2.3, it is completely simple. Then, by Theorem 1.25, K is isomorphic with a Rees matrix semigroup $\mathcal{M}(I,G,J;P)$ over a group G with the sandwich matrix P. Assume that P is normalized by $p_{j_0,i} = p_{i,j_0} = e$ for all $i \in I$, $j \in J$ and some $i_0 \in I$, $j_0 \in J$, where e is the identity element of G. Then, for every $g \in G$, $i \in I$, $j \in J$ and a positive integer k,

$$(i, g(p_{j,i}g)^{m+k-1}, j) = (i,g,j)^{m+k} = ((i,g,j_0)(i_0,e,j))^{m+k}$$
$$= (i,g,j_0)^m(i_0,e,j)^m(i,g,j)^k = (i,g^m,j_0)(i_0,e,j)(i,g,j)^k$$
$$= (i,g^m,j)(i,g,j)^k = (i,g^m,j)(i,g(p_{j,i}g)^{k-1},j)$$
$$= (i,g^m p_{j,i}g(p_{j,i}g)^{k-1},j)$$

and so

$$g(p_{j,i}g)^{m+k-1} = g^m p_{j,i}g(p_{j,i}g)^{k-1},$$

that is,

$$(gp_{j,i})^m = g^m p_{j,i}.$$

Then, letting $g = e$, it follows that

$$p_{j,i}^{m-1} = e.$$

Then, for a positive integer t and every $g, h \in G$, we get

$$(i_0, (gh)^{m+t}, j_0) = (i_0, gh, j_0)^{m+t} = ((i_0,g,j_0)(i_0,h,j_0))^{m+t}$$
$$= (i_0,g,j_0)^m(i_0,h,j_0)^m((i_0,g,j_0)(i_0,h,j_0))^t$$
$$(i_0,g^m h^m,j_0)(i_0,(gh)^t,j_0) = (i_0,g^m h^m (gh)^t, j_0),$$

that is,

$$(gh)^{m+t} = g^m h^m (gh)^t$$

from which it follows that

$$(gh)^m = g^m h^m.$$

Hence G is an E-m group and $p_{ji}^{m-1} = e$ for all $i \in I$ and $j \in J$. Then, by Theorem 12.15, K is an E-m semigroup.

As the converse statement is obvious, the theorem is proved. □

Corollary 13.1 *A semigroup is a 0-simple WE-2 semigroup if and only if it is a rectangular abelian group with a zero adjoined.*

Proof. By Theorem 13.4 and Corollary 12.6, it is obvious. □

Theorem 13.5 *([51]) A retract extension of a WE-m semigroup by a WE-m semigroup with zero is a WE-m semigroup.*

Proof. By Definition 1.45, a WE-m semigroup is a W-semigroup with $W = ((ab)^{m+k} = a^m b^m (ab)^k)_{k \geq 0}$ and $W = ((ab)^{m+k} = (ab)^k a^m b^m)_{k \geq 0}$. Hence our assertion follows from Theorem 1.38. □

Theorem 13.6 *([51]) A semigroup is a WE-m archimedean semigroup containing at least one idempotent element if and only if it is a retract extension of a completely simple E-m semigroup by a nil semigroup.*

Proof. If S is a WE-m archimedean semigroup containing at least one idempotent element then it is a left and right Putcha semigroup from which we get that S is a retract extension of a completely simple E-m semigroup by a nil semigroup (see Theorem 2.4, Theorem 13.2 and Theorem 13.4). The converse follows from Theorem 2.2, Theorem 13.5 and the fact that an E-m semigroup is also a WE-m semigroup. □

Theorem 13.7 *([51]) S is an archimedean WE-2 semigroup containing at least one idempotent element if and only if it is an ideal extension of the direct product $K = I \times G \times J$ of a left zero semigroup I, an abelian group G and a right zero semigroup J by a nil semigroup N with product determined by three partial homomorphisms*

$$(\ )\phi : N^* \to I,\ (\ )\omega : N^* \to G,\ (\ )\varphi : N^* \to J$$

in the following manner. If (π, a, μ), $(\eta, b, \nu) \in I \times G \times J$, $s, t \in N^*$ then

$$(\pi, a, \mu)s = (\pi, a((s)\omega, (s)\varphi)$$

$$s(\pi, a, \mu) = ((s)\phi, (s)\omega a, \mu)$$

$$(\pi, a, \mu)(\eta, b, \nu) = (\pi, ab, \nu)$$

$$st = st \text{ in } N \text{ if } st \neq 0 \text{ in } N$$

$$st = ((s)\phi, (s)\omega(t)\omega, (t)\varphi) \text{ if } st = 0 \text{ in } N.$$

Proof. Let S be an archimedean WE-2 semigroup with idempotent elements. Then, by Theorem 13.6, S is a retract extension of a completely simple E-2 semigroup K by a nil semigroup N. By Corollary 12.6, K is isomorphic to the direct product $I \times G \times J$ of a left zero semigroup I, an abelian group G and a right zero semigroup J. Let $\Phi(\ )$ denote the retract homomorphism of S onto K. Then, for every $s \in N^*$, there are elements $a \in G$, $i \in I$ and $j \in J$ such that $\Phi(s) = (i, a, j)$. Consequently Φ induces mappings

$$(\ )\phi : N^* \to I,\ (\ )\omega : N^* \to G,\ (\ )\varphi : N^* \to J$$

such that $i = (s)\phi$, $a = (s)\omega$, $(s)\varphi = j$, that is,

$$\Phi(s) = ((s)\phi, (s)\omega, (s)\varphi).$$

Since Φ is a homomorphism, it follows that all of ϕ, ω, φ are partial homomorphisms. It is a matter of checking case to see that the equations of the theorem hold.

To prove the converse, assume that the semigroup S is an ideal extension of K by N, the partial homomorphisms ϕ, ω, φ are given and the product in S is defined by the equations of the theorem. Denote a mapping $\Phi(\ )$ of S onto K as

$$\Phi(s) = s \text{ if } s \in K \text{ and } ((s)\phi, (s)\omega, (s)\varphi) \text{ if } s \in N^*.$$

We show that Φ is a homomorphism. Let s and t be arbitrary elements of S. Assume $s, t \in N^*$. Then

$$\Phi(s)\Phi(t) = ((s)\phi, (s)\omega(t)\omega, (t)\varphi) = \Phi(st).$$

If $s \in K$ and $t \in N^*$ then, with $s = (\pi, a\mu)$, we have

$$\Phi(st) = st = (\pi, a, \mu)t = (\pi, a(t)\omega, (t)\varphi) = (\pi, a, \mu)\Phi(t) = \Phi(s)\Phi(t).$$

Similarly, $\Phi(st) = \Phi(s)\Phi(t)$ in case $s \in N^*$ and $t \in K$. Assume $s, t \in K$. Then

$$\Phi(st) = st = \Phi(s)\Phi(t).$$

Consequently, Φ is a retract homomorphism of S onto K. Then, by Theorem 2.2 and Theorem 13.5, S is an archimedean WE-2 semigroup containing at least one idempotent element. □

Lemma 13.1 *If S is a WE-2 semigroup then, for every $a \in S$,*

$$S_a = \{x \in S : \ a_i x a^j = a^h \text{ for some positive integers } i, j, k\}$$

is the least reflexive unitary subsemigroup of S containing a.

Proof. Let S be a WE-2 semigroup and $a \in S$ be arbitrary. To show that S_a is a subsemigroup of S, let $x, y \in S_a$ be arbitrary. Then there are positive integers i, j, k, h, m, n such that

$$a^i x a^j = a^k$$

and

$$a^m y a^n = a^h.$$

As S is a WE-2 semigroup, there is a positive integer t such that

$$(xa^{j+m}ya^{n+i})^{2+t} = (xa^{j+m}y)^2 a^{2(n+i)}(xa^{j+m}ya^{n+i})^t.$$

We can suppose that

$$2 + t = 2^r$$

for some positive integer r. By Theorem 13.1,

$$2+t \in WE(S).$$

Thus there is a positive integer s such that

$$(a^i x a^{j+m} y a^{n+i})^{2+t+s} = a^{i(2+t)}(x a^{j+m} y a^{n+i})^{2+t}(a^i x a^{j+m} y a^{n+i})^s.$$

Let $p := k+h$. Then

$$a^{(p+i)(2+t+s)} = (a^i x a^{j+m} y a^{n+i})^{2+t+s}$$

$$= a^{(2+t)i}(x a^{j+m} y a^{n+i})^{2+t}(a^{p+i})^s$$

$$(*) \qquad = a^{(2+t)i}(x a^{j+m} y)^2 a^{2(n+i)}(x a^{j+m} y a^{n+i})^t a^{s(p+i)}$$

$$= a^{(1+t)i} a^i x a^j (a^m y x a^j)(a^m y a^n) a^{n+i}((a^i x a^j)(a^m y a^n))^t a^i a^{s(p+i)}$$

$$= a^{(1+t)i+k+m} y x a^{j+h+n+p(t+s)+i(s+2)}.$$

Hence

$$yx \in S_a,$$

that is, S_a is a subsemigroup of S.

We show that S_a is left unitary. Assume $x, xy \in S_a$ for some $x, y \in S$. Then there are positive integers i, j, k, m, n, h such that

$$a^i x a^j = a^k$$

and

$$a^m x y a^n = a^h.$$

Let r denote a positive integer which satisfies $r \geq max\{i-m, j-h\}$. As S is a WE-2 semigroup, there is a positive integer t such that

$$(a^{r+m} x y a^n)^{2+t} = (a^{r+m} x)^2 (y a^n)^2 (a^{r+m} x y a^n)^t.$$

From this we get

$$a^{(2+t)(r+h)} = (a^{r+h})^{2+t} = (a^{r+m} x y a^n)^{2+t}$$

$$= (a^{r+m} x)^2 (y a^n)^2 (a^{r+m} x y a^n)^t$$

$$= a^{r+m} x a^{r+m} x y a^n y a^n a^{t(r+h)}$$

$$= a^{r+m} x a^{r+h} y a^{t(r+h)+n}$$

$$= a^{m+r-i} a^i x a^j a^{r+h-j} y a^{t(r+k)+n}$$

$$= a^{2r+m+h+k-i-j} y a^{t(r+h)+n}.$$

Hence $y \in S_a$. Consequently, S_a is a left unitary subsemigroup of S. We can prove, in a similar way, that S_a is right unitary in S.

We show that S_a is reflexive in S. Assume $xy \in S_a$ for some $x, y \in S$. As S is a WE-2 semigroup, there is a positive integer k such that

$$(xy)^{3+k} = x(yx)^{2+k}y = xy^2x^2(yx)^ky = (xy)(yx)(xy)^{k+1} \in S_a.$$

As S_a is unitary in S, we have

$$yx \in S_a.$$

Hence S_a is reflexive in S. It is clear that $a \in S_a$. We show that S_a is the least reflexive unitary subsemigroup of S which contains a. Assume, in an indirect way, that S has a reflexive unitary subsemigroup V such that $a \in V$ and $V \subset S_a$. Then there is an element $x \in S_a - V$ such that

$$a^i x a^j = a^k \in V$$

for some positive integers i, j, k. As V is unitary in S, we get $x \in V$ which is impossible. Thus the lemma is proved. □

Theorem 13.8 *Every WE-2 archimedean semigroup without idempotent element has a non-trivial group homomorphic image.*

Proof. Let S be a WE-2 archimedean semigroup without idempotent element. Let $a \in S$ be arbitrary. Then, by Lemma 13.1, S_a is a reflexive unitary subsemigroup of S and so, by Theorem 1.41, the principal right congruence $\mathcal{R}_{S_a}$ is a group congruence on S. If $S_a \neq S$ then $S/\mathcal{R}_{S_a}$ is a non-trivial group homomorphic image of S. Next, we can suppose that $S_a = S$. In this case, for every $x \in S$, there are positive intgeres i, j, k such that $a^i x a^j = a^k$. Assume that

$$a^p x a^q = a^m$$

also holds for some positive integers p, q, m. Then

$$a^{k+p+q} = a^{i+p} x a^{j+q} = a^{m+i+j}$$

from which we get

$$m - (p+q) = k - (i+j),$$

because S does not contains idempotent element. Thus the integer $k - (i+j)$ is well-determined by the element x. Let φ be the following mapping.

$$\varphi: \ x \in S \ \rightarrow k - (i+j),$$

where $k - (i+j)$ is the integer which is determined by x as above. Since $S_a = S$ then φ is defined on S, and it maps S into the additive semigroup of integers. We show that φ is a homomorphism. Let $x, y \in S$ be arbitrary. Assume

$$a^i x a^j = a^k$$

and

$$a^m y a^n = a^h$$

for some positive integers i, j, k, m, n, h. Let $p = k + h$. Then, by (*) of the proof of Lemma 13.1,

$$a^{(p+i)(2+t+s)} = a^{(1+t)i+k+m}yxa^{j+h+n+p(t+s);(s+2)}.$$

(for some positive integers t and s). Since

$$(p+i)(2+t+s) - ((1+t)i + k + m + j + h + n + p(t+s) = i(s+2)) =$$

$$k - (i+j) + k - (m+n),$$

we get

$$\varphi(yx) = \varphi(y) + \varphi(x).$$

Hence φ is a homomorphism of S into the additive semigroup of integers. It is clear that $\varphi(a) = 1$. Thus $\varphi(S)$ equals either the additive semigroup of all integers or the additive semigroup of all non-negative integers or the additive semigroup of all positive integers. Since all of these additive semigroups have non-trivial group homomorphic images, the theorem is proved. □

Theorem 13.9 *([51]) A regular WE-m semigroup is a semilattice of completely simple E-m semigroups.*

Proof. Let S be a regular WE-m semigroup. Then S is a semilattice of archimedean WE-m semigroups. As S is regular, every semilattice component of S contains an idempotent and so is a retract extension of completely simple E-m semigroup by a nil semigroup. From this we can conclude that every semilattice component of S is a completely simple E-m semigroup. □

Theorem 13.10 *([52]) On a semigroup S, the following are equivalent.*

(i) S is a regular WE-2 semigroup.

(ii) S is a regular E-2 semigroup.

(iii) S is an orthodox band of abelian groups.

(iv) S is a spined product of some band and a semilattice of abelian groups.

(v) S is a regular exponential semigroup.

(vi) S is a regular WE-m semigroup for all positive integer $m \geq 2$.

Proof. (i) implies (ii): Let S be a regular WE-2 semigroup. Then, by Theorem 13.3 and Theorem 13.7, we can conclude that S is a semilattice Y of rectangular abelian groups $S_\alpha = I_\alpha \times G_\alpha \times M_\alpha$, where I_α are left zero semigroups, G_α are abelian groups and M_α are right zero semigroups, $\alpha \in Y$. By Theorem 1.27, S is an orthogroup. By Theorem 1.28, for every pair (α, β), $\alpha, \beta \in Y$ with $\alpha \geq \beta$, there exist a left representation $t_{\alpha,\beta}(\ )$ of S_α by transformations of I_β, a right representation $(\ )\tau_{\alpha,\beta}$ of S_α by transformations of M_β and a homomorphism $(\ )\phi_{\alpha,\beta}$ of G_α into G_β such that the product in S is given as follows. Let

$A = (i_\alpha, a_\alpha, \kappa_\alpha) \in S_\alpha$ and $B = (j_\beta, b_\beta, \lambda_\beta) \in S_\beta$ be arbitrary elements of S. Let $\gamma = \alpha\beta$ (in Y), and let

$$(i_\alpha, \kappa_\alpha)(j_\beta, \lambda_\beta) = (\nu_\gamma, \mu_\gamma)$$

be the given product of $(i_\alpha, \kappa_\alpha)$ and (j_β, λ_β) in the band E_S. Then

$$AB = ((t_{\alpha,\gamma}A)\nu_\gamma, a_\alpha\phi_{\alpha,\gamma}b_\beta\phi_{\beta,\gamma}, \mu_\gamma(B\tau_{\beta,\gamma})).$$

Since S is a WE-2 semigroup, there is a positive integer k such that

$$\begin{aligned}&((i_\alpha, a_\alpha, \kappa_\alpha)(j_\beta, b_\beta\lambda_\beta))^{2+k}\\ &= (i_\alpha, a_\alpha, \kappa_\alpha)^2(j_\beta, b_\beta, \lambda_\beta)^2((i_\alpha, a_\alpha, \kappa_\alpha)(j_\beta, b_\beta, \lambda_\beta))^k\\ &((i_\alpha, a_\alpha, \kappa_\alpha)(j_\beta, b_\beta, \lambda_\beta))^k(i_\alpha, a_\alpha, \kappa_\alpha)^2(j_\beta, b_\beta, \lambda_\beta)^2.\end{aligned}$$

Since

$$\begin{aligned}&((i_\alpha, a_\alpha, \kappa_\alpha)(j_\beta, b_\beta, \lambda_\beta))^{2+k}\\ &= ((t_{\alpha,\gamma}A)\nu_\gamma, a_\alpha\phi_{\alpha,\gamma}b_\beta\phi_{\beta,\gamma}, \mu_\gamma(B\tau_{\beta,\gamma}))^{2+k}\\ &= ((t_{\alpha,\gamma}A)\nu_\gamma, (a_\alpha\phi_{\alpha,\gamma}b_\beta\phi_{\beta,\gamma})^{2+k}, \mu_\gamma(B\tau_{\beta,\gamma}))\end{aligned}$$

and

$$\begin{aligned}&= (i_\alpha, a_\alpha, \kappa_\alpha)^2(j_\beta, b_\beta, \lambda_\beta)^2((i_\alpha, a_\alpha, \kappa_\alpha)(j_\beta, b_\beta, \lambda_\beta))^k\\ &= (i_\alpha, a_\alpha^2, \kappa_\alpha)(j_\beta, b_\beta^2, \lambda_\beta)((i_\alpha, a_\alpha, \kappa_\alpha)(j_\beta, b_\beta, \lambda_\beta))^k\\ &= ((t_{\alpha,\gamma}A^2)\nu_\gamma, a_\alpha^2\phi_{\alpha,\gamma}b_\beta^2\phi_{\beta,\gamma}, \mu_\gamma(B^2\tau_{\beta,\gamma}))((t_{\alpha,\gamma}A)\nu_\gamma, a_\alpha\phi_{\alpha,\gamma}b_\beta\phi_{\beta,\gamma}, \mu_\gamma(B\tau_{\beta,\gamma}))^k\\ &= ((t_{\alpha,\gamma}A^2)\nu_\gamma, a_\alpha^2\phi_{\alpha,\gamma}b_\beta^2\phi_{\beta,\gamma}(a_\alpha\phi_{\alpha,\gamma}b_\beta\phi_{\beta,\gamma})^k, \mu_\gamma(B\tau_{\beta,\gamma})),\end{aligned}$$

we have

$$t_{\alpha,\gamma}A = t_{\alpha,\gamma}A^2.$$

We can prove, in a similar way, that

$$B\tau_{\beta,\gamma} = B^2\tau_{\beta,\gamma}.$$

As

$$\begin{aligned}(AB)^2 &= ((i_\alpha, a_\alpha, \kappa_\alpha)(j_\beta, b_\beta, \lambda_\beta))^2\\ &= ((t_{\alpha,\gamma}A)\nu_\gamma, a_\alpha\phi_{\alpha,\gamma}b_\beta\phi_{\beta,\gamma}, \mu_\gamma(B\tau_{\beta,\gamma}))^2\\ &= ((t_{\alpha,\gamma}A)\nu_\gamma, (a_\alpha\phi_{\alpha,\gamma}b_\beta\phi_{\beta,\gamma})^2, \mu_\gamma(B\tau_{\beta,\gamma}))\end{aligned}$$

and

$$\begin{aligned}A^2B^2 &= (i_\alpha, a_\alpha, \kappa_\alpha)^2(j_\beta, b_\beta, \lambda_\beta)^2\\ &= (i_\alpha, a_\alpha^2, \kappa_\alpha)(j_\beta, b_\beta^2, \lambda_\beta)\\ &= ((t_{\alpha,\gamma}A^2)\nu_\gamma, a_\alpha^2\phi_{\alpha,\gamma}b_\beta^2\phi_{\beta,\gamma}, \mu_\gamma(B^2\tau_{\beta,\gamma})),\end{aligned}$$

we get $(AB)^2 = A^2B^2$, because

$$t_{\alpha,\gamma}A = t_{\alpha,\gamma}A^2,$$

$$B\tau_{\beta,\gamma} = B^2\tau_{\beta,\gamma},$$

and G_γ is an abelian group containing the elements $a_\alpha\phi_{\alpha,\gamma}$ and $b_\beta\phi_{\beta,\gamma}$. Hence S is an E-2 regular semigroup and so (i) implies (ii).

Conditions (ii), (iii), (iv) and (v) are equivalent by Theorem 12.21.

(v) implies (vi) and (vi) implies (i) in a trivial way. Thus the theorem is proved. □

Theorem 13.11 *([52]) A semigroup S which is an ideal extension of a regular semigroup K by a nil semigroup N is a WE-2 semigroup if and only if K is an E-2 semigroup and the extension is retract.*

Proof. Let S be a WE-2 semigroup such that it is an ideal extension of a regular semigroup K by a nil semigroup N. Then, by Theorem 13.10, K is an E-2 regular semigroup. By Theorem 13.3 and Theorem 13.7, K is a semilattice Y of rectangular abelian groups $K_\alpha = I_\alpha \times G_\alpha \times M_\alpha$, $(\alpha \in Y)$. By Theorem 1.27, K is an orthogroup and, by Theorem 1.28, the product in K is determined by homomorphisms $(\ )\psi_{\alpha,\beta}$ of G_β into G_α, left representations $t_{\alpha,\beta}(\ )$ of K_α by transformations of I_β and right representations $(\ )\tau_{\alpha,\beta}$ of K_α by transformations of M_β, $\alpha \geq \beta$, $\alpha, \beta \in Y$. If $A = (i_\alpha, g_\alpha, m_\alpha) \in S_\alpha$ and $B = (i_\beta, g_\beta, m_\beta) \in S_\beta$ are arbitrary elements and $\alpha \geq \beta$ then

$$AB = ((t_{\alpha,\beta}A)i_\beta, (g_\alpha\psi_{\alpha,\beta})g_\beta, m_\beta)$$

and

$$BA = (i_\beta, g_\beta(g_\alpha\psi_{\alpha,\beta}), m_\beta(A\tau_{\alpha,\beta})).$$

By the remark before Theorem 12.22, $t_{\alpha,\beta}(A)$ does not depend on g_α. Let θ denote the canonical homomorphism of K onto Y. Since K is weakly reductive then, it is isomorphic with the inner part of the translational hull $\Omega(S)$ and, by Theorem 1.36, there is an extension $\Sigma'(+)$ of $\Omega(K)$ by N such that S is a subsemigroup of $\Sigma'(+)$. Let ϵ denote the identity of $\Omega(K)$. Then χ defined by $\chi(x) = x + \epsilon$ $(x \in \Sigma'(+))$ is a retract homomorphism of $\Sigma'(+)$ onto $\Omega(K)$. We show that the restriction of χ to S is a retract homomorphism of S onto K. Let x be an arbitrary element in $N^* = N - \{0\}$. Let $\chi(x) = (\lambda_x, \rho_x) \in \Omega(K)$. We show that $\chi(x)$ is an inner bitranslation of K. By the remark before Theorem 12.22, λ'_x and ρ'_x defined by $\lambda'_x(\theta(y)) = \theta(\lambda_x(y))$ and $(\theta(y))\rho'_x = \theta((y)\rho_x)$, $y \in K$ are idempotent homomorphisms of Y into itself such that $\lambda' = \rho'$. Then $A_x = \lambda'_x(Y)$ is a retract ideal of Y. By Theorem 12.22,

$$(\lambda_x, \rho_x) = ([k_\alpha(.), a_\alpha, (.)l_\alpha])_{\alpha \in A_x} \in \prod_{\alpha \in A_x} (\mathcal{T}_{I_\alpha} \times G_\alpha \times \mathcal{T}_{M_\alpha}).$$

We note that, for all $\alpha, \beta \in A_x$, $\alpha \geq \beta$,

$$a_\alpha\psi_{\alpha,\beta} = a_\beta$$

and, for all $(i_\alpha, m_\alpha) \in I_\alpha \times M_\alpha$,

$$k_\beta \circ t_{\alpha,\beta}(i_\alpha, m_\alpha) = t_{\alpha,\beta}(k_\alpha(i_\alpha), m_\alpha),$$

$$t_{\alpha,\beta}(i_\alpha, m_\alpha) \circ k_\beta = t_{\alpha,\beta}(i_\alpha, (m_\alpha)l_\alpha),$$

$$l_\beta \circ \tau_{\alpha,\beta}(i_\alpha, m_\alpha) = \tau_{\alpha,\beta}(k_\alpha(i_\alpha), m_\alpha),$$

$$\tau_{\alpha,\beta}(i_\alpha, m_\alpha) \circ l_\beta = \tau_{\alpha,\beta}(i_\alpha, (m_\alpha)l_\alpha).$$

Moreover, for all $\alpha \in Y - A_x$ ($\beta = \lambda'_x(\alpha)$) and all $(i_\alpha, m_\alpha) \in I_\alpha \times M_\alpha$, $k_\beta \circ t_{\alpha,\beta}(i_\alpha, m_\alpha)$, $t_{\alpha,\beta}(i_\alpha, m_\alpha) \circ k_\beta$, $l_\beta \circ \tau_{\alpha,\beta}(i_\alpha, m_\alpha)$, $\tau_{\alpha,\beta}(i_\alpha, m_\alpha) \circ l_\beta$ are all constant functions. Thus we now how (λ_x, ρ_x) acts on K if we now how (λ_x, ρ_x) acts on $\cup_{\beta \in A_x} K_\beta$. Since N is a nil semigroup, there is an integer $n \geq 2$ such that $x^n \in K$ and so $x^n \in K_\alpha$ for some $\alpha \in Y$. Let $x^n = (\eta_\alpha, g_\alpha, \nu_\alpha)$, $\eta_\alpha \in I_\alpha$, $g_\alpha \in G_\alpha$, $\nu_\alpha \in M_\alpha$. We show that $\alpha \in A_x$ and $K_\alpha^x = \{x, x^2, \ldots, x^{n-1}\} \cup K_\alpha$ is a subsemigroup of S. It is sufficient to show that $xy, yx \in K_\alpha$ for every $y \in K_\alpha$. Let $y \in K_\alpha$ be an arbitrary element. Then $xy \in K_\beta$ for some $\beta \in Y$. As the homomorphism χ leaves the elements of K fixed, we have

$$xy = \chi(xy) = \chi(x)y = (\lambda_x, \rho_x)y = \lambda_x(y)$$

and so

$$\beta = \theta(xy) = \theta(\lambda_x(y)) = \lambda'_x(\theta(y)) = \lambda'_x(\alpha).$$

Thus $\beta \in A_x$. Since $x^n y \in K_\alpha$ and

$$x^n y = \chi(x^n y) = \chi(x^{n-1})\chi(xy) = (\chi(x))^{n-1} xy = \lambda_x^{n-1}(xy)$$

then

$$\alpha = \theta(x^n y) = \theta(\lambda_x^{n-1}(xy)) = \lambda'_x(\theta(\lambda_x^{n-2}(xy)) = \ldots$$
$$= (\lambda'_x)^{n-1}(\theta(xy)) = \lambda'_x(\beta) = \beta$$

and so $xy \in K_\alpha$. We can prove, in a similar way, that $yx \in K_\alpha$. Thus K_α^x is a subsemigroup and $\alpha \in A_x$. Let r be a positive integer such that $2^r \geq n$. Since K_α^x is a subsemigroup, then $x^{2^r} \in K_\alpha$ (and so $(\chi(x))^{2^r} \in K_\alpha$). Then $(\chi(x))^{2^r} = (\pi, u, \mu)$ for some $(\pi, u, \mu) \in K_\alpha$. Let $y = (\xi, g, \eta) \in K_\alpha$ be arbitrary. As S is a WE-2 semigroup, it is a WE-2^r semigroup (see Theorem 13.1). Thus there is a positive integer z such that

$$(xy)^{2^r + z} = x^{2^r} y^{2^r} (xy)^z.$$

Since χ is a homomorphism and leaves y fixed, we have

$$(\chi(x)y)^{2^r + z} = (\chi(x))^{2^r} y^{2^r} (\chi(x)y)^z$$

and so

$$(k_\alpha(\xi), (a_\alpha g)^{2^r}, \eta) = (k_\alpha(\xi), a_\alpha g, \eta)^{2^r + z}$$
$$= ((\chi(x))^{2^r} (\xi, g^{2^r}, \eta)(k_\alpha(\xi), a_\alpha g, \eta)^z$$

$$= (\pi, u, \mu)(\xi, g^{2^r}, \eta)(k_\alpha(\xi), (a_\alpha g)^z, \eta) = (\pi, ug^{2^r}(a_\alpha g)^z, \eta)$$

from which we get $k_\alpha(\xi) = \pi$. We can prove, in a similar way, that $(\eta)l = \mu$. Hence k_α and l_α are constant mappings. Then, for arbitrary $(i_\alpha, c_\alpha, j_\alpha) \in K_\alpha$, we have

$$(\eta_\alpha, g_\alpha c_\alpha, j_\alpha) = (\eta_\alpha, g_\alpha, \nu_\alpha)(i_\alpha, c_\alpha, j_\alpha)$$
$$= x^n(i_\alpha, c_\alpha, j_\alpha) = \chi(x^n)(i_\alpha, c_\alpha, j_\alpha) = (\chi(x))^n(i_\alpha, c_\alpha, j_\alpha)$$
$$= (\chi(x))^{n-1}(\chi(x)(i_\alpha, c_\alpha, j_\alpha)) = (\chi(x))^{n-1}((\pi, a_\alpha, \mu)(i_\alpha, c_\alpha, j_\alpha))$$
$$= (\chi(x))^{n-1}(\pi, a_\alpha c_\alpha, j_\alpha) = (\chi(x))^{n-2}(\pi, a_\alpha, \mu)(\pi, a_\alpha c_\alpha, j_\alpha)$$
$$= (\chi(x))^{n-2}(\pi, a_\alpha^2 c_\alpha, j_\alpha) = \ldots = (\pi, a_\alpha^n c_\alpha, j_\alpha)$$

from which we get $\eta_\alpha = \pi$ and $g_\alpha = a_\alpha^n$. We get, in a similar way that $\nu_\alpha = \mu$. Thus

$$x^n = (\pi, a_\alpha^n, \mu).$$

Let $\beta \in A_x$ be an arbitrary element. Then there is an element $\gamma \in Y$ such that $\lambda'_x(\gamma) = \beta \in A_x$. Since

$$x^n K_\gamma = \chi(x^n)K_\gamma = ((\chi(x))^n K_\gamma$$
$$= (\lambda_x^n, \rho_x^n)K_\gamma = \lambda_x^n K_\gamma \subseteq K_\beta,$$

then $\beta = \alpha\gamma = \alpha\alpha\gamma = \alpha\beta$, that is, $\alpha \geq \beta$. Thus the homomorphism $(\ )\psi_{\alpha,\beta}$ of G_β into G_α, the left representation $t_{\alpha,\beta}(\ )$ of K_α by transformations of I_β and the right representation $(\ )\tau_{\alpha,\beta}$ of K_α by transformations of M_β are defined. We note that $(a_\alpha)\psi_{\alpha,\beta} = a_\beta$. Let (i, b, j) be an arbitrary element of K_β. Since S is a WE-2 semigroup, there is a positive integer k such that

$$(x(i,b,j))^{2+k} = x^2(i,b,j)^2((x(i,b,j))^k$$

and so

$$(\chi(x)(i,b,j))^{2+k} = (\chi(x))^2(i,b,j)^2(\chi(x)(i,b,j))^k,$$

that is

$$(k_\beta(i), (a_\beta b)^{2+k}, j) = (k_\beta \circ k_\beta(i), a_\beta^2 b^2 (a_\beta b)^k, j).$$

Thus $k_\beta \circ k_\beta = k_\beta$. We can prove, in a similar way, that $l_\beta \circ l_\beta = l_\beta$. Then

$$((t_{\alpha,\beta}(x^n))(i), a_\beta^n b, j) = ((t_{\alpha,\beta}(x^n))(i), (a_\alpha^n)\psi_{\alpha,\beta} b, j)$$
$$= (\pi, a_\alpha^n, \mu)(i,b,j) = x^n(i,b,j) = \chi(x^n)(i,b,j)$$
$$= (\chi(x))^n(i,b,j) = (\chi(x))^{n-1}(\chi(x)(i,b,j))$$
$$= (\chi(x))^{n-1}(k_\beta(i), a_\beta b, j) = \ldots$$
$$= (k_\beta^n(i), a_\beta^n b, j) = (k_\beta(i), a_\beta^n b, j)$$

from which we get

$$k_\beta = t_{\alpha,\beta}(x^n).$$

Then

$$k_\beta = t_{\alpha,\beta}(\pi, a_\alpha^n, \mu) = t_{\alpha,\beta}(\pi, a_\alpha, \mu).$$

We can prove, in a similar way, that

$$l_\beta = (x^n)\tau_{\alpha,\beta} = (\pi, a_\alpha, \mu)\tau_{\alpha,\beta}.$$

Then

$$\begin{aligned}\chi(x)(i,b,j) &= (k_\beta(i), a_\beta b, j) = ((t_{\alpha,\beta}(x^n))i, a_\beta b, j) = \\ &((t_{\alpha,\beta}(\pi, a_\alpha^n, \mu))i, a_\beta b, j) = ((t_{\alpha,\beta}(\pi, a_\alpha, \mu))i, a_\beta b, j) \\ &= ((t_{\alpha,\beta}(\pi, a_\alpha, \mu))i, (a_\alpha\psi_{\alpha,\beta})b, j) = (\pi, a_\alpha, \mu)(i,b,j).\end{aligned}$$

We can prove, in a similar way, that

$$(i,b,j)\chi(x) = (i,b,j)(\pi, a_\alpha, \mu).$$

Thus $\chi(x)$ acts on $\cup_{\beta\in A_x} K_\beta$ as (π, a_α, μ) acts on $\cup_{\beta\in A_x} K_\beta$. Thus $\chi(x)$ can be identify with the inner bitranslation of K corresponding to (π, a_α, μ), that is, $\chi(x) \in K$. Consequently, the restriction of χ to S is a retract homomorphism of N onto K. Thus the first part of the theorem is proved. The converse follows from Theorem 13.5. □

Corollary 13.2 *([65]) A semigroup S which is an ideal extension of a regular semigroup K by a nilsemigroup N is an E-2 semigroup if and only if K and N are E-2 semigroups and the extension is retract.*

Proof. Let S be an E-2 semigroup which is an ideal extension of a regular semigroup K by a nil semigroup N. It is clear that K and N are E-2 semigroups. Since S is also a WE-2 semigroup, then, by Theorem 13.11, there is a retract homomorphism of S onto K. As a retract extension of an E-2 semigroup by an E-2 semigroup is also an E-2-semigroup, the converse statement is evident. □

Theorem 13.12 *([51]) On a semigroup S, the following are equivalent.*

(i) S is an inverse WE-m semigroup,

(ii) S is a semilattice of E-m groups,

(iii) S is an inverse E-m semigroup.

Proof. Let S be an inverse WE-m semigroup. Then, by Theorem 13.9, S is a semilattice of completely simple E-m semigroups. It is easy to see that the semilattice components are inverse semigroups. As an inverse completely simple semigroup is a group, S is a semilattice of E-m groups. Hence (i) implies (ii). By Theorem 12.20, (iii) follows from (ii). It is obvious that (iii) implies (i). □

Subdirectly irreducible WE-2 semigroups

Theorem 13.13 *([53]) A semigroup S is a subdirectly irreducible WE-2 semigroup with a globally idempotent core if and only if it satisfies one of the following conditions.*

(i) $S \cong G$ or $S \cong G^0$, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime).

(ii) S is a non-trivial subdirectly irreducible band.

Proof. Let S be a subdirectly irreducible WE-2 semigroup with a globally idempotent core K. First, assume that S has no zero element. Then K is simple. By Corollary 13.1, K is a rectangular abelian group, that is, $K = G \times I \times J$, where G is an abelian group, I is a left zero semigroup and J is a right zero semigroup. By Corollary 1.4, we have either $K = G$ or $K = I$ or $K = J$.

Assume $K = G$. Then S is a homogroup and so, by Theorem 1.47, it is a subdirectly irreducible abelian group. Then, by Theorem 3.14, S is a non-trivial subgroup of a quasicyclic p-group (p is a prime) and so (i) is satisfied.

Assume $K = I$, that is, K is a left zero semigroup. It can be easily verified that

$$\delta = \{(a,b) \in S \times S : \ ai = bi \text{ for all } i \in I\}$$

is a congruence on S such that

$$\delta|I = id_I.$$

As I is a dense ideal of S, we get

$$\delta = id_S.$$

Let $i \in I$ and $s \in S$ be arbitrary elements. As S is a WE-2 semigroup, there is a positive integer k such that

$$(si)^{2+k} = s^2 i^2 (si)^k$$

and so

$$si = (si)^{2+k} = s^2 i^2 (si)^k = s^2 isi = s^2 i.$$

Thus

$$(s, s^2) \in \delta.$$

Hence

$$s = s^2.$$

Thus S is a band and so (ii) is satisfied. We can prove, in a similar way, that S is a band if K is a right zero semigroup.

Next, assume that S has a zero element 0. We can prove (as in the proof of Theorem 9.18) that $S' = S - \{0\}$ is a subsemigroup of S. If $|S'| = 1$ then S is a two-element semilattice and so (ii) is satisfied. If $|S'| > 1$ then S' is a subdirectly irreducible WE-2 semigroup without zero. If S' contained a zero $0'$ then $I = \{0, 0'\}$ would be an ideal of S and we would have $\rho_I \cap \rho_{S'} = id_S$ which is a contradiction, because $\rho_I \neq id_S$ and $\rho_{S'} \neq id_S$ (here ρ_I and $\rho_{S'}$ denote the Rees congruence on S modulo the ideal I and S', respectively). Thus the core S' is globally idempotent. If S' is non-trivial subgroup G of a quasicyclic p-group (p is a prime) then $S = G^0$ and so (i) is satisfied. If S' is a band then (ii) is satisfied. As the semigroups listed in the theorem are subdirectly irreducible WE-2 semigroups, the theorem is proved. □

Corollary 13.3 *A semigroup S is a subdirectly irreducible E-2 (exponential) semigroup with a globally idempotent core if and only if it is satisfies one of the following conditions.*

(i) $S \cong G$ or $S \cong G^0$, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime).

(ii) S is a non-trivial subdirectly irreducible band.

Proof. By Theorem 13.13, it is obvious. □

We remark that subdirectly irreducible bands are characterized in Theorem 1.48.

Theorem 13.14 *A WE-2 (E-2, exponential) semigroup with a zero and a non-trivial annihilator is subdirectly irreducible if and only if it has a non-zero disjunctive element.*

Proof. By Theorem 1.49, it is obvious. □

Chapter 14

Weakly exponential semigroups

In the previous chapter we dealt with semigroups in which, for a fixed integer $m \geq 2$ and every elements a and b, there is a non-negative integer k such that $(ab)^{m+k} = a^m b^m (ab)^k = (ab)^k a^m b^m$. In this chapter we deal with semigroups which satisfy this condition for every integer $m \geq 2$. These semigroups are called weakly exponential semigroups. It follows from results of the previous chapter that every weakly exponential semigroup is a semilattice of weakly exponential archimedean semigroups. A semigroup is a weakly exponential archimedean semigroup containing at least one idempotent element if and only if it is a retract extension of a rectangular abelian group by a nil semigroup. It is also proved that every weakly exponential archimedean semigroup without idempotent element has a non-trivial group homomorphic image. We prove that every weakly exponential semigroup is a band of weakly exponential t-archimedean semigroups. As a consequence of the previous chapter, a semigroup is a subdirectly irreducible weakly exponential semigroup with a globally idempotent core if and only if it is isomorphic to either G or G^0 or B, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime) and B is a non-trivial subdirectly irreducible band. At the end of the chapter, we determine the weakly exponential Δ-semigroups. We prove that a semigroup S is a weakly exponential Δ-semigroup if and only if one of the following satisfied. (1) S is isomorphic to either G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group. (2) S is isomorphic to a two-element semilattice. (3) S is isomorphic to either R or R^0 or R^1, where R is a two-element right zero semigroup. (4) S is isomorphic to either L or L^0 or L^1, where L is a two-element left zero semigroup. (5) S is a nil semigroup whose principal ideals form a chain with respect to inclusion. (6) S is a weakly exponential T1 or a T2R or a T2L semigroup. We note that it is have not proved yet that there are weakly exponential T2R and T2L semigroups.

Definition 14.1 *A semigroup S is called a weakly exponential semigroup if it is a WE-m semigroup for every $m \geq 2$. With other words, for every $(a,b) \in S \times S$ and every integer $m \geq 2$, there is a non-negative integer k such that $(ab)^{m+k} = a^m b^m (ab)^k = (ab)^k a^m b^m$.*

Theorem 14.1 *Every weakly exponential semigroup is a left and right Putcha semigroup.*

Proof. By Theorem 13.2, it is obvious. □

Theorem 14.2 *([49]) Every weakly exponential semigroup is a semilattice of weakly exponential archimedean semigroups.*

Proof. By Theorem 13.3, it is obvious. □

Theorem 14.3 *([50]) Every weakly exponential semigroup is a band of weakly exponential t-archimedean semigroups.*

Proof. Let S be a weakly exponential semigroup and $a \in S$, $x, y \in S^1$ be arbitrary elements. By Theorem 1.7, it is sufficient to show that $xay -_t xa^2y$. Since S is weakly exponential, there are positive integers n and m such that

$$(ayx)^{2+n} = a^2(yx)^2(ayx)^m$$

and

$$(yxa)^{2+m} = (yxa)^m(yx)^2a^2.$$

Then

$$\begin{aligned}(xay)^{3+n} &= x(ayx)^{2+n}ay = xa^2(yx)^2(ayx)^n ay \\ &= (xa^2y)xyx(ayx)^n ay\end{aligned}$$

and, similarly,

$$(xay)^{3+m} = xa(yxa)^{2+m}y = xa(yxa)^m(yx)^2a^2y = xa(yxa)^m yxy(xa^2y).$$

Let

$$i = max\ \{3+n, 3+m\}.$$

Then

$$xa^2y|_t(xay)^i.$$

Using the condition that S is weakly exponential, there are positive integers p, s, k such that

$$(ayxa)^{2+p} = a^2(yxa)^2(ayxa)^p,$$

$$(ayxa^2)^{2+s} = a^2(yxa^2)^2(ayxa^2)^s$$

and

$$\begin{aligned}&(ayxa^2(yxa)^2(ayxa)^p ayx)^{s+k} \\ &= (ayxa^2)^s((yxa)^2(ayxa)^p ayx)^s(ayxa^2(yxa)^2(ayxa)^p ayx)^k.\end{aligned}$$

Using the notations

$$w = a^2(yxa^2)^2,$$

$$v = ayxa^2(yxa)^2(ayxa)^p ay,$$

$$u = wv,$$

there is a positive integer n such that

$$(ux)^{s+k+n} = (w(vx))^{s+k+n} = w^{s+k}(vx)^{s+k}(ux)^n.$$

Thus

$$\begin{aligned}(xa^2y)^{6+p} &= xa^2yxa^2y(xa^2y)^{3+p}xa^2y \\ &= xa^2yxa^2yxa(ayxa)^{2+p}ayxaay \\ &= xa^2(yxa^2)yxaa^2(yxa)^2(ayxa)^{1+p}ay \\ &= xa^2(yxa^2)^2ayx(ayxa)^{2+p}ay \\ &= xa^2(yxa^2)^2ayxa^2(yxa)^2(ayxa)^p ay \\ &= xwv = xu\end{aligned}$$

and so

$$\begin{aligned}(xa^2y)^{(6+p)(s+k+n+1)} &= (xu)^{s+k+n+1} \\ &= x(ux)^{s+k+n}u = xw^{s+k}(vx)^{s+k}(ux)^n u \\ &= xw^{s+k}(ayxa^2(yxa)^2(ayxa)^p ayx)^{s+k}(ux)^n u \\ &= xw^{s+k}(ayxa^2)^s((yxa)^2(ayxa)^p ayx)^s(vx)^k(ux)^n u \\ &= xw^{s+k-1}a^2(yxa^2)^2(ayxa^2)^s((yxa)^2(ayxa)^p ayx)^s(vx)^k(ux)^n u \\ &= xw^{s+k-1}(ayxa^2)^{2+s}((yxa)^2(ayxa)^p ayx)^s(vx)^k(ux)^n u \\ &= \ldots = \\ &= x(ayxa^2)^{s+2(s+k)}((yxa)^2(ayxa)^p ayx)^s(vx)^k(ux)^n u \\ &= (xay)xa^2(ayxa^2)^{s+2(s+k)-1}((yxa)^2(ayxa)^p ayx)^s(vx)^k(ux)^n u.\end{aligned}$$

Thus

$$xay|_r(xa^2y)^{(6+p)(s+k+n+1)}.$$

We can prove, in a similar way, that

$$xay|_l(xa^2y)^{(6+q)(z+r+m+1)}$$

for some positive integers q, z, r, m. Let

$$j = max\ \{(6+p)(s+k+n+1),\ (6+q)(z+r+m+1)\}.$$

Then

$$xay|_t(xa^2y)^j.$$

Consequently,

$$xay -_t xa^2y.$$

Thus the theorem is proved. □

Theorem 14.4 *([49]) A semigroup is weakly exponential and 0-simple if and only if it is a rectangular abelian group with a zero adjoined.*

Proof. Let S be a weakly exponential 0-simple semigroup. Then S is a WE-2 semigroup and so, by Corollary 13.1, it is a rectangular abelian group with a zero adjoined. The converse statement is obvious. □

Theorem 14.5 *([49]) A retract extension of weakly exponential semigroup by a weakly exponential semigroup is also weakly exponential.*

Proof. It is easy to see that a weakly exponential semigroup is a W-semigroup (see Definition 1.45) with $W = ((ab)^{m+k} = a^m b^m (ab)^k)_{k \geq 0}$ and $W = ((ab)^{m+k} = (ab)^k a^m b^m)_{k \geq 0}$ for every positive integer $m \geq 2$. Hence our assertion follows from Theorem 1.38. □

Theorem 14.6 *([49]) A semigroup is a weakly exponential archimedean semigroup containing at least one idempotent element if and only if it is a retract extension of a rectangular abelian group by a nil semigroup.*

Proof. Let S be a weakly exponential archimedean semigroup containing at least one idempotent element. As S is a left and right Putcha semigroup, by Theorem 2.4 and Theorem 14.4, it is a retract extension of a rectangular abelian group by a nil semigroup.

The converse follows from Theorem 2.2 and Theorem 14.5. □

Corollary 14.1 *A semigroup is weakly exponential and regular if and only if it is a regular exponential semigroup.*

Proof. See Theorem 13.10. □

Theorem 14.7 *([49]) Every weakly exponential archimedean semigroup without idempotent element has a non-trivial group homomorphic image.*

Proof. Since a weakly exponential semigroup is a WE-2 semigroup, the assertion follows from Theorem 13.8. □

Subdirectly irreducible weakly exponential semigroups

Theorem 14.8 *A semigroup S is a subdirectly irreducible weakly exponential (exponential) semigroup with a globally idempotent core if and only if it satisfies one of the following conditions.*

(i) $S \cong G$ or G^0, where G is a non-trivial subgroup of a quasicyclic p-group, p is a prime.

(ii) S is a non-trivial subdirectly irreducible band.

Proof. Let S be a subdirectly irreducible weakly exponential (exponential) semigroup with a globally idempotent core. Then S is a WE-2 semigroup and so, by Theorem 13.13, either (i) or (ii) is satified. As the semigroups in (i) and (ii) are weakly exponential (exponential) semigroups with globally idempotent core, the theorem is proved. □

Theorem 14.9 *A weakly exponential (exponential) semigroup with a zero and a non-trivial annihilator is subdirectly irreducible if and only if it has a non-zero disjunctive elements*

Proof. By Theorem 1.49, it is obvious. □

Weakly exponential Δ-semigroups

Theorem 14.10 *([54]) A semigroup S is a weakly exponential Δ-semigroup if and only if one of the following satisfied.*

(i) $S \cong G$ *or* G^0, *where* G *is a non-trivial subgroup of a quasicyclic p-group (p is a prime).*

(ii) $S \cong F$, *where* F *is a two-element semilattice.*

(iii) $S \cong R$ *or* R^0 *or* R^1, *where* R *is a two-element right zero semigroup.*

(iv) $S \cong L$ *or* L^0 *or* L^1, *where* L *is a two-element left zero semigroup.*

(v) S *is a nil semigroup whose principal ideals form a chain with respect to inclusion.*

(vi) S *is a weakly exponential T1 or a T2R or a T2L semigroup.*

Proof. Let S be a weakly exponential Δ-semigroup. Then, by Theorem 14.2, it is a semilattice of archimedean weakly exponential semigroups. By Remark 1.2, S is either archimedean or a disjoint union $S = S_0 \cup S_1$ of an ideal S_0 and a subsemigroup S_1 of S which are archimedean and weakly exponential.

First, assume that S is archimedean. If S has a zero element then it is a nil semigroup. By Theorem 1.56, the principal ideals of S form a chain with respect to inclusion. Hence (v) is satisfied.

In the next, we consider the case when S has no zero element. If S is simple then, by Theorem 14.4, it is a rectangular abelian group, that is, a direct product of a left zero semigroup L, a right zero semigroup R and an abelian group G. Then we have either $S = L$ or $S = R$ or $S = G$. In the first case, by Theorem 1.61, S is a two-element left zero semigroup and so (iv) is satisfied. In the second case, by Theorem 1.61, S is a two-element right zero semigroup and so (iii) is satisfied. In the third case, by Theorem 3.22, S is a non-trivial subgroup of a quasicyclic p-group (p is a prime) and so (i) is satisfied.

Consider the case when S is not simple (and S has no zero element). Then, by Theorem 14.7 and Theorem 1.52, S has an idempotent element. By Theorem 14.6, S is a retract extension of a rectangular abelian group K ($|K| > 1$) by a nil semigroup N. Let δ denote the congruence on S determined by the retract homomorphism. Then

$$\delta \cap \rho_K = id_S,$$

where ρ_K denotes the Rees congruence of S defined by the ideal K of S. As S is a Δ-semigroup and $|K| > 1$, we have

$$\delta = id_S.$$

Then $S = K$ which contradicts the assumption for S.

Next, consider the case when S is a disjoint union $S = S_0 \cup S_1$ of an ideal S_0 and a subsemigroup S_1 of S, where S_0 and S_1 are archimedean. By Theorem 1.51 and Remark 1.1, S_1 is an archimedean weakly exponential Δ-semigroup. If S_1 is a nil semigroup then, by Theorem 1.57, $|S_1| = 1$. Thus S_1 is either a two-element left zero semigroup L or a two-element right zero semigroup R or a subgroup G of a quasicyclic p-group (p is a prime).

If $|S_0| = 1$ then either $S = L^0$ or $S = R^0$ or $S = G^0$ (if $|G| = 1$ then S is a two-element semilattice).

Next, we can suppose that $|S_0| > 1$. Recall that S_0 is a weakly exponential archimedean semigroup. By Theorem 1.47 and Theorem 1.52, S_0 has an idempotent element. By Theorem 14.6, S_0 is a retract extension of a rectangular abelian group $K = L \times R \times G$ (L is a left zero semigroup, R is a right zero semigroup, G is an abelian group) by a nil semigroup. By Theorem 1.54, K has no non-trivial group homomorphic images. Hence $K = L \times R$. As $K^2 = K$, by Theorem 1.14, K is an ideal of S. Consider the case when $|K| > 1$. By Corollary 1.3, $K = L$ or $K = R$. Assume that $K = L$. It is easy to see that

$$\alpha = \{(a, b) \in S \times S : \ ax = bx \text{ for all } x \in L\}$$

is a congruence on S such that

$$\alpha | L = id_L.$$

As L is a dense ideal, it follows that

$$\alpha = id_S.$$

Let $x \in L$ and $c \in S$ be arbitrary elements. Then there is a positive integer k such that

$$cx = (cx)^{2+k} = c^2x^2(cx)^k = c^2x$$

which means that

$$(c, c^2) \in \alpha.$$

Then

$$c = c^2.$$

Consequently, S is a band and $S_0 = L$. By Theorem 1.61, $S = S_0^1$ and S_0 is a two-element left zero semigroup. We get, in a similar way, that $S_0 = K$ and S

is a band in that case when K is a right zero semigroup and so, by Theorem 1.61, $S = S_0^1$ and S_0 is a two-element right zero semigroup.

Next, consider the case when $|K| = 1$. Then S_0 is a (non-trivial) nil semigroup.

If $|S_1| = 1$ then S is a weakly exponential T1 semigroup. If S_1 is a two-element left zero semigroup then S is a weakly exponential T2L semigroup. If S_1 is a two-element right zero semigroup then S is a weakly exponential T2L semigroup.

If S_1 was a non-trivial subgroup G of a quasicyclic p-group (p is a prime) then, by Theorem 1.59, S_0 would be trivial which contradicts the assumption that $|S_0| > 1$. Thus the first part of the theorem is proved. As the semigroups listed in the theorem are weakly exponential Δ-semigroups, the proof is complete. □

Corollary 14.2 *([107]) A semigroup S is an exponential Δ-semigroup if and only if one of the following satisfied.*

(i) $S \cong G$ or G^0, where G is a non-trivial subgroup of a quasicyclic p-group.

(ii) $S \cong F$, where F is a two-element semilattice.

(iii) $S \cong R$ or R^0 or R^1, where R is a two-element right zero semigroup.

(iv) $S \cong L$ or L^0 or L^1, where L is a two-element left zero semigroup.

(v) S is an exponential nil semigroup whose principal ideals are chain ordered by inclusion.

(vi) S is an exponential T1 or a T2R or a T2L semigroup.

Proof. Since an exponential semigroup is weakly exponential then, by Theorem 14.10, it is obvious. □

Theorem 14.11 *([54]) S is a weakly exponential T1 semigroup if and only if it is a semilattice of a non-trivial nil Δ-semigroup S_0 and a one-element semigroup $S_1 = \{e\}$ such that $S_0S_1 \subseteq S_0$ and $S^1eS^1 = S$.*

Proof. If S is a weakly exponential T1 semigroup then, by Theorem 1.58, S_0 is a non-trivial nil Δ-semigroup and $S^1eS^1 = S$.

Conversely, let S be a semilattice of a non-trivial nil Δ-semigroup S_0 and a one-element semigroup $S_1 = \{e\}$ such that $S_0S_1 \subseteq S_0$ and $S^1eS^1 = S$. Let a and b be arbitrary elements of S, and let n be a positive integer. We may assume

$$\{a, b\} \neq \{e\}.$$

Then

$$ab \in S_0.$$

Since S_0 is a nil semigroup, there is a positive integer k such that

$$(ab)^k = 0.$$

Then

$$(ab)^{n+k} = 0 = a^n b^n (ab)^k = (ab)^k a^n b^n.$$

Thus S is weakly exponential. By Theorem 1.58, S is also a Δ-semigroup. $\square$

Theorem 14.12 *([107]) A semigroup S is an exponential T1 semigroup if and only if S is a semilattice of an ideal N of S which is a non-trivial exponential nil Δ-semigroup and a one-element semigroup $P = \{e\}$ such that $ea, ae \in N^1 e N^1$ for every $a \in N$ and, for each $a \in N$, either $ea = a$ or $ae = a$ or $ea^2 = a^2 = 0$.*

We note that the weakly exponential and exponential T2R and T2L semigroups are characterized in [54] and [107], but the authors were not able to construct such semigroups.

Chapter 15

(m, n)-commutative semigroups

In the last two chapters we deal with the (m,n)-commutative and the $n_{(2)}$-permutable semigroups, respectively. A semigroup is called an (m,n)-commutative semigroup if it satisfies the identity $(x_1 ... x_m)(y_1 ... y_n) = (y_1 ... y_n)(x_1 ... x_m)$ (m and n are positive integers). For a fixed integer $n \geq 2$, a semigroup S is called an $n_{(2)}$-permutable semigroup if, for any n-tuple $(x_1, x_2, ..., x_n)$ of elements of S, there is a positive integer t with $1 \leq t \leq n-1$ such that $x_1x_2...x_tx_{t+1}...x_n = x_{t+1}...x_nx_1...x_t$. First we deal with the (m,n)-commutative semigroups, because some results about them are necessary in the examinations of $n_{(2)}$-permutable ones. In this chapter the (m,n)-commutative semigroups are examined. In the first part of the chapter we determine all couples (m,n) of positive integers m and n for which a semigroup is (m,n)-commutative. Since an (m,n)-commutative semigroup S is (m',n')-commutative for every $m' \geq m$ and $n' \geq n$, it is sufficient to know the function $f_S(n) = min\{m : S \text{ is } (m,n) - \text{commutative}\}$. As every (m,n)-commutative semigroup is $(1, m+n)$-commutative, f_S is defined for all positive integers. We define a special function, the permutation function, and show that the functions f_S are exactly the permutation functions. In the second part of the chapter, we show that every (m,n)-commutative semigroup is an E-k semigroup for some integer $k \geq 2$. We also show that every $(1,2)$-commutative semigroup is exponential. In the third part of the chapter, we deal with the semilattice decomposition of (m,n)-commutative semigroups. Every (m,n)-commutative semigroup is a semilattice of archimedean (m,n)-commutative semigroups. It is shown that a semigroup is (m,n)-commutative and archimedean containing at least one idempotent element if and only if it is an ideal extension of a commutative group by an (m,n)-commutative nil semigroup. We show that every (m,n)-commutative archimedean semigroup without idempotent has a non-trivial group homomorphic image. We prove that a semigroup is (m,n)-commutative and regular if and only if it is a commutative Clifford semigroup. We also show that a semigroup which is an ideal extension of a regular semigroup K by a nil semigroup N is (m,n)-commutative if and only if K is a commutative Clifford semigroup and N is (m,n)-commutative. In the forth part of the chapter, we deal with the subdirectly irreducible (m,n)-commutative semigroups. A semigroup is a subdirectly irreducible (m,n)-commutative semigroup with a globally idempotent core if and only if it is isomorphic to either G or G^0 or F, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime) and F is a two-element semilattice. An (m,n)-commutative semigroup with a zero and a non-trivial annihilator is subdirectly irreducible if and only if it has a non-zero disjunctive element. Moreover, we show that a subdirectly irreducible (m,n)-commutative semigroup with a trivial annihilator and a nilpotent core is commutative. In the last part of the chapter, the (m,n)-commutative Δ-semigroups are determined. We show that a semigroup is an (m,n)-commutative Δ-semigroup if and only if one of the following conditions is satisfied. (1) S is isomorphic to G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime). (2) S is isomorphic to N or N^1, where N is an (m,n)-commutative nil semigroup whose principal ideals form a chain with respect to inclusion.

Definition 15.1 *For positive integers m and n, a semigroup S is called an (m,n)-commutative semigroup if it satisfies the identity*

$$(x_1...x_m)(y_1...y_n) = (y_1...y_n)(x_1...x_m).$$

We note that if a semigroup is (m,n)-commutative for some m and n then it is (m^*,n^*)-commutative for all $m^* \geq m$ and $n^* \geq n$. Moreover, a semigroup is (m,n)-commutative if and only if it is (n,m)-commutative.

Lemma 15.1 *([1]) If S is an (m,n)-commutative semigroup then it is $(1,m+n)$-commutative.*

Proof. Let $x_1, x_2, \dots, x_m$, $y_1, y_2, \dots, y_n$, z be arbitrary elements of an (m,n)-commutative semigroup S. It is clear that S is $(m+1,n)$-commutative and $(n+1,m)$-commutative. Thus

$$z(x_1x_2\dots x_my_1y_2\dots y_n) = (zx_1x_2\dots x_m)(y_1y_2\dots y_n)$$

$$= (y_1y_2\dots y_n)(zx_1x_2\dots x_m) = (y_1y_2\dots y_nz)(x_1x_2\dots x_m)$$

$$= (x_1x_2\dots x_m)(y_1y_2\dots y_nz) = (x_1x_2\dots x_my_1y_2\dots y_n)z.$$

which means that S is $(1,m+n)$-commutative. □

Theorem 15.1 *A finitely generated periodic (m,n)-commutative semigroup is finite.*

Proof. By Theorem 1.1, it is obvious. □

Permutation functions and (m,n)-commutativity of semigroups

In this section, for an arbitrary semigroup S, we determine all couples (m,n) of positive integers m and n for which the semigroup S is (m,n)-commutative. In our investigation a special type of functions mapping the set of all positive integers N^+ into itself plays an important role. These functions are called permutation functions.

Definition 15.2 *A function $f : N^+ \longrightarrow N^+$ with $Dom f = N^+$ is called a permutation function if it satisfies all of the following four conditions:*

(i) $f(n) = 1$ *for all* $n > f(1)$,

(ii) $n + f(n) = f(1)$ *or* $n + f(n) = f(1) + 1$ *for all* $1 \leq n \leq f(1)$,

(iii) *If* $n + f(n) = m + f(m) = f(1)$ *and* $m < f(n)$ *for some* $1 < n, m < f(1)$ *then* $f(n+m) = f(n) - m$,

(iv) *If* $n + f(n) = f(1)$ *then* $f(f(n)) = n$.

Remark 15.1 From *(ii)* it follows that $f(f(1)) = 1$.

Remark 15.2 If $n + f(n) = m + f(m) = f(1)$ and $m < f(n)$ then, by *(iii)*, $n + m + f(n+m) = f(1)$.

Lemma 15.2 *([60]) If f is a permutation function then, for every $n, t \in N^+$, conditions $n + f(n) = f(1)$ and $tn \leq f(n)$ together imply $f(tn) = f(n) - (t-1)n$ and $tn + f(tn) = f(1)$.*

Proof. Let f be a permutation function. Consider positive integers n and t such that $n + f(n) = f(1)$ and $tn \leq f(n)$. Then

$$1 < n < f(1).$$

We can suppose that $t > 1$. Then

$$n < f(n).$$

Using *(iii)* for $m = n$, we get

$$f(2n) = f(n) - n$$

and so

$$2n + f(2n) = 2n + f(n) - n = n + f(n) = f(1).$$

If $t = 2$ then the lemma is proved. If $t > 2$ then

$$2n < f(n).$$

Using *(iii)* for $m = 2n$, we have

$$f(3n) = f(n) - 2n$$

and

$$3n + f(3n) = f(1).$$

Continuouing this procedure, we get

$$(t-1)n < f(n)$$

and

$$(t-1)n + f((t-1)n) = f(1).$$

Using *(iii)* for $m = (t-1)n$, we get

$$f(tn) = f(n) - (t-1)n$$

and so

$$tn + f(tn) = f(1).$$

□

Lemma 15.3 *([60]) If f is a permutation function then it is monotone decreasing and, in the case $f(1) > 1$, $f(2) < f(1)$ and $f(f(1)-1) = 2$.*

Proof. Let f be a permutation function. To show that f is monotone decreasing, we can suppose that $f(1) > 1$. Let n be an arbitrary positive integer with $n < f(1)$. Then, by *(ii)*, $f(n) = f(1) - n$ or $f(n) = f(1) - n + 1$ and $f(n+1) = f(1) - n - 1$ or $f(n+1) = f(1) - n$. Comparing $f(n)$ and $f(n+1)$, we can conclude that $f(n) \geq f(n+1)$. As $f(n) = 1$ for all $n \geq f(1)$ (see *(i)* and Remark 15.1), f is monotone decreasing.

To prove the second assertion of the lemma, assume $f(1) > 1$. By *(ii)*,

$$2 + f(2) \leq f(1) + 1$$

from which it follows that

$$f(2) \leq f(1) - 1 < f(1).$$

Using again *(ii)*,

$$f(1) - 1 + f(f(1) - 1) = f(1)$$

or

$$f(1) - 1 + f(f(1) - 1) = f(1) + 1.$$

In the first case

$$f(f(1) - 1) = 1$$

and, by *(iv)*,

$$f(f(f(1) - 1)) = f(1) - 1,$$

that is,

$$f(1) = f(1) - 1$$

which is impossible. So

$$f(1) - 1 + f(f(1) - 1) = f(1) + 1$$

which means that

$$f(f(1) - 1) = 2.$$

□

Lemma 15.4 *([60]) If f is a permutation function then, for every $n, m \in N^+$, conditions $n + f(n) = m + f(m) = f(1)$ and $m > f(n)$ imply $f(m - f(n)) = f(m) + f(n)$.*

Proof. Let f be a permutation function. Assume $n + f(n) = m + f(m) = f(1)$ and $m > f(n)$ for some $n, m \in N^+$. Then

$$1 < n,\ m < f(1).$$

By (iv),

$$f(f(n)) = n$$

and

$$f(f(m)) = m.$$

Thus

$$f(n) < f(f(m)).$$

Applying (iii) for $f(m)$ and $f(n)$, we have

$$f(f(m) + f(n)) = f(f(m)) - f(n) = m - f(n).$$

As

$$f(m) + f(n) + f(f(m) + f(n)) = f(1),$$

(iv) implies

$$f(f(f(m) + f(n))) = f(m) + f(n)$$

and so

$$f(m - f(n)) = f(f(f(m) + f(n))) = f(m) + f(n).$$

Thus the lemma is proved. □

For all couples (n, m) $(n, m \in N^+)$, let $p_{(n,m)}$ denote the power

$$p_{(n,m)} = p^n_{(1,n+m-1)},$$

where $p_{(1,n+m-1)}$ is the permutation of $\{1, 2, \ldots, n+m\}$ defined by

$$p_{(1,n+m-1)} = \begin{pmatrix} 1 & 2 & \ldots & n+m-1 & n+m \\ 2 & 3 & \ldots & n+m & 1 \end{pmatrix}.$$

Let f be a permutation function and let p_{id} denote the identical permutation of $\{1,2,\ldots,f(1)\}$. One can check that

$$C_f = \{p_{(n,m)} : n+m = f(1)\} \cup \{p_{id}\}$$

is a cyclic subgroup of the symmetrical group $\mathcal{S}_{f(1)}$ of degree $f(1)$ and the order of C_f is $f(1)$. If $f(1) > 1$ then C_f is generated by the permutation $p_{(1,f(1)-1)}$. For f, define the following sets:

$$P_f = \{p_{(n,f(n))} : n+f(n) = f(1)\} \cup \{p_{id}\}$$

and

$$A_f = \{n \in N^+ : n+f(n) = f(1)\}.$$

It is clear that P_f is a subset of C_f, and $\mid P_f \mid > 1$ if and only if $A_f \neq \emptyset$.

Theorem 15.2 *([60]) If f is a permutation function then P_f is a cyclic subgroup of the symmetric group $\mathcal{S}_{f(1)}$. If $\mid P_f \mid > 1$ then P_f is generated by the permutation $p_{(a,f(a))}$ where a is the minimal element of A_f such that it is a divisor of all $n \in A_f$ and $f(1)$.*

Proof. Let f be a permutation function. We can suppose that $\mid P_f \mid > 1$. Consider arbitrary elements $p_{(n,f(n))}$ and $p_{(m,f(m))}$ of P_f. Then

$$n + f(n) = m + f(m) = f(1).$$

It can be verified that

$$p_{(n,f(n))}p_{(m,f(m))} = \begin{cases} p_{(m-f(n),f(1)-m+f(n))}, & \text{if } m > f(n); \\ p_{id}, & \text{if } m = f(n); \\ p_{(n+m,f(n)-m)}, & \text{if } m < f(n). \end{cases}$$

If $m > f(n)$ then, by Lemma 15.4,

$$f(1) - m + f(n) = f(m) + f(n) = f(m - f(n))$$

and so

$$m - f(n) + f(m - f(n)) = f(1).$$

Thus

$$p_{(n,f(n))}p_{(m,f(m))} = p_{(m-f(n),f(m-f(n))} \in P_f.$$

If $m < f(n)$ then, by (iii) and Remark 15.2,

$$f(n) - m = f(n+m)$$

and

$$n + m + f(n+m) = f(1)$$

from which it follows that

$$p_{(n,f(n))}p_{(m,f(m))} = p_{(n+m,f(n+m))} \in P_f.$$

Thus P_f is closed under the operation of the permutations.

If $p_{(n,f(n))} \in P_f$ then
$$n + f(n) = f(1)$$
and, by (iv),
$$f(f(n)) = n$$
which implies
$$p_{(f(n),n)} \in P_f.$$
As
$$p_{(n,f(n))}p_{(f(n),n)} = p_{id},$$
$p_{(f(n),n)}$ is the inverse of $p_{(n,f(n))}$ in P_f. Thus P_f is a subgroup of the symmetric group $\mathcal{S}_{f(1)}$.

It is clear that C_f is isomorphic to the group Z_k of integers modulo $k = f(1)$, under addition. As P_f is a subgroup of C_f, $A_f \cup \{0\}$ is a subgroup of Z_k. Then, denoting the least element of A_f by a, $A_f \cup \{0\}$ is generated by a, that is, A_f consists of elements ta, where $t \in N^+$ and $1 \leq t \leq \frac{f(a)}{a}$. It is clear that a is a divisor of $f(1)$. Using Lemma 15.2, it can be easily verified that P_f is generated by $p_{(a,f(a))}$. Thus the theorem is proved. □

Corollary 15.1 *([60]) If f is a permutation function such that $A_f \neq \emptyset$ then $A_f = \{ta : \ t \in N^+, \ 1 \leq t \leq \frac{f(a)}{a}\}$ and $\mid P_f \mid = \frac{f(a)}{a} + 1$ for a divisor a of $f(1)$ with $1 < a < f(1)$.*

Corollary 15.2 *([60]) If f is a permutation function such that $f(1)$ is a prime then $A_f = \emptyset$.*

Corollary 15.3 *([60]) If f is a permutation function then, for all positive integers n, m and their greatest common divisor d, the condition $m \geq f(n)$ implies $n + m - d \geq f(d)$.*

Proof. Let f be a permutation function. Consider positive integers n and m such that $m \geq f(n)$. Let d denote the greatest common divisor of n and m. If $d \geq f(1)$ then $f(d) = 1$ and so $n + m - d \geq f(d)$. Assume $d < f(1)$. Then
$$f(1) \leq d + f(d) \leq f(1) + 1.$$
As $m \geq f(n)$, we have
$$n + m \geq f(1).$$
If $n + m > f(1)$ then
$$n + m - d \geq f(1) - d + 1 \geq f(d).$$
If $n + m = f(1)$ then, from $m \geq f(n)$, we get
$$n + f(n) \leq n + m = f(1)$$

which implies

$$n + f(n) = f(1)$$

and

$$m = f(n).$$

Then

$$n, m \in A_f.$$

By Theorem 15.2, $a = minA_f$ is a common divisor of n and m. Evidently, $d \in A_f$. Thus

$$n + m - d = f(1) - d = f(d)$$

and so the corollary is proved. □

Let $k \geq 1$ be an arbitrary integer. Let P be a cyclic subgroup of the symmetric group $\mathcal{S}_k$ of degree k such that if $\mid P \mid > 1$ then P is generated by a permutation $p_{(a,k-a)}$, where a is a divisor of k with $1 < a < k$. Define a function $f_P^* : N^+ \to N^+$ by the following way:

$(i)^*$ $f_P^*(n) = 1$ for all $n \geq k$,

$(ii)^*$ $f_P^*(n) = k - n$ if $\mid P \mid > 1$, $n < k$ and $n = at$ for some $t \in N^+$,

$(iii)^*$ $f_P^*(n) = k - n + 1$ if either $\mid P \mid > 1$, $n < k$ and $n \neq at$ for all $t \in N^+$ or $\mid P \mid = 1$ and $n < k$.

Remark 15.3 From $(iii)^*$ it follows that $f_P^*(1) = k$ and, supposing $\mid P \mid > 1$, a is a divisor of $f_P^*(1)$. From $(ii)^*$ it follows that $f_P^*(a) = k - a$ if $\mid P \mid > 1$.

Remark 15.4 If $n + f_P^*(n) = f_P^*(1)$ then ($\mid P \mid > 1$ and) a is a divisor of n, because $n + f_P^*(n) = f_P^*(1) = k$ implies (by $(ii)^*$) that $n = at$ for some $t \in N^+$.

Theorem 15.3 *([60]) A function f is a permutation function if and only if $f = f_P^*$ for some function f_P^* constructed as above.*

Proof. Let $k \geq 1$ be an integer and let P be a cyclic subgroup of the symmetric group $\mathcal{S}_k$ as above. We show that f_P^* is a permutation function. We can consider only that case when P is generated by a permutation $p_{(a,k-a)}$, where a is a divisor of k with $1 < a < k$. (If $\mid P \mid = 1$ then the proof is trivial.)

Condition (i) follows from condition $(i)^*$ and Remark 15.3, and condition (ii) is an immediate consequence of conditions $(ii)^*$ and $(iii)^*$.

To prove (iii), let n and m be positive integers with $1 < n, m < f_P^*(1)$, $n + f_P^*(n) = m + f_P^*(m) = f_P^*(1)$ and $m < f_P^*(n)$. Then, by Remark 15.4, $n = al$ and $m = aj$ for some positive integers $l, j \leq \frac{f_P^*(a)}{a}$. As $m < f_P^*(n)$, we get

$$n + m < n + f_P^*(n) = f_P^*(1) = k.$$

Thus

$$f_P^*(n) - m = f_P^*(1) - n - m = f_P^*(1) - (n + m) = f_P^*(n + m).$$

So (iii) is satisfied.

To show (iv), let n be a positive integer with $n + f_P^*(n) = f_P^*(1)$. Then $n = la$ for some positive integer $l \leq \frac{f_P^*(a)}{a}$. As $k = f_P^*(1)$ and $n = la$, we get that a is a divisor of $f_P^*(n)$ and $f_P^*(n) < f_P^*(1) = k$. Then, by $(ii)^*$, we have

$$f_P^*(f_P^*(n)) = k - f_P^*(n) = f_P^*(1) - f_P^*(n) = n.$$

Thus (iv) is satisfied. Consequently f_P^* is a permutation function.

Conversely, let f be an arbitrary permutation function. Then, by Theorem 15.2, P_f is a cyclic subgroup of $\mathcal{S}_{f(1)}$ such that either $|P_f| = 1$ or P_f is genetared by the permutation $p_{(a,f(a))}$, where $a = min\{A_f\}$ and a is a divisor of $f(1)$ with $1 < a < f(1)$. Consider the function $f_{P_f}^*$ defined by $(i)^*$, $(ii)^*$ and $(iii)^*$ under choosing $k = f(1)$. We show that $f = f_{P_f}^*$. Let n be an arbitrary positive integer. If $n \geq f(1) = k$ then

$$f(n) = f_{P_f}^*(n) = 1.$$

Assume $n < f(1)$. If $f(n) = f(1) - n$ then $\mid P_f \mid > 1$ and a is a divisor of n. Thus

$$f_{P_f}^*(n) = f(1) - n$$

and so

$$f(n) = f_{P_f}^*(n).$$

If $f(n) = f(1) - n + 1$ then either $\mid P_f \mid > 1$ and a is not a divisor of n or $\mid P_f \mid = 1$. Thus

$$f_{P_f}^*(n) = f(1) - n + 1$$

and so

$$f(n) = f_{P_f}^*(n).$$

Consequently,

$$f = f_{P_f}^*$$

and the theorem is proved. □

Corollary 15.4 *([60]) If f is a permutation function then $f = f_{P_f}^*$. If P is a cyclic permutation group of a symmetric group $\mathcal{S}_k$ ($k \geq 1$ is an integer) such that $\mid P \mid = 1$ or P is generated by a permutation $p_{(a,k-a)}$ (a is a divisor of k with $1 < a < k$) then $P_{f_P^*} = P$.*

Proof. The first assertion is proved in the last part of the proof of Theorem 15.3. To prove the second assertion, let P be a cyclic subgroup of a symmetric group $\mathcal{S}_k$ satisfying the condition of the corollary. If $|P| = 1$ then $A_{f_P^*} = \emptyset$ and so $P_{f_P^*} = P$. Consider the case when P is generated by a permutation $p_{(a,k-a)}$, where a is a divisor of k with $1 < a < k$. Then, by Theorem 15.3, f_P^* is a permutation function such that the conditions $n < f_P^*(1)$ and a is a divisor of n together hold if and only if $n \in A_{f_P^*}$. By Theorem 15.2, $P_{f_P^*}$ is a cyclic group which is generated by the permutation $p_{(a,f_P^*(a))}$, because a is the minimal element of $A_{f_P^*}$. As $f_P^*(a) = k - a$ (see Remark 15.3), we get $P_{f_P^*} = P$. □

Remark 15.5 For an arbitrary permutation function f, $\mid C_f \mid = \mid P_f \mid\mid C_f : P_f \mid$, where $\mid C_f : P_f \mid$ denotes the index of P_f in C_f. By Corollary 1,

$$\mid P_f \mid = \begin{cases} 1, & \text{if } A_f = \emptyset; \\ 1 + \frac{f(a)}{a}, & \text{if } A_f \neq \emptyset \end{cases},$$

where $a = minA_f$. As $a + f(a) = f(1)$, we get $a(1 + \frac{f(a)}{a}) = f(1)$ and $a = \mid C_f : P_f \mid$. By these equations we introduce the following notions and notations.

Definition 15.3 *By the degree of a permutation function f we shall mean the positive integer $f(1)$. If $A_f = \emptyset$ then $f(1)$ is also said to be the index of f. If $A_f \neq \emptyset$ then the index of a permutation function f is defined by $a = minA_f$. The order of P_f will be called the order of a permutation function f. The degree, the index and the order of f will be denoted by d_f, i_f and o_f, respectively.*

By Definition 15.3 and Remark 15.5, the following lemma can be proved easily.

Lemma 15.5 *([60]) For an arbitrary permutation function f, the following equations hold: $i_f = \mid C_f : P_f \mid$ and $d_f = o_f\, i_f$.*

For an arbitrary semigroup S, let $f_S : N^+ \longrightarrow N^+$ denote the function whose domain is

$$Domf_S = \{n \in N^+ : \ (\exists m \in N^+)\ S \text{ is } (n,m) - \text{commutative}\}$$

and, for all $n \in Domf_S$,

$$f_S(n) = min\{m \in N^+ : \ S \text{ is } (n,m) - \text{commutative}\}.$$

Remark 15.6 If we know the function f_S then we know all couples (m,n) of positive integers m and n for which the semigroup S is (m,n)-commutative. In the next we describe f_S for all semigroups S.

Lemma 15.6 *([60]) For every semigroup S, $Domf_S = \emptyset$ or $Domf_S = N^+$.*

Proof. Let S be an arbitrary semigroup with $Domf_S \neq \emptyset$. Then S is (m,n)-commutative for some $m,n \in N^+$. By Lemma 15.1, S is $(1,m+n)$-commutative which implies that S is $(k,m+n)$-commutative for all $k \in N^+$. So $Domf_S = N^+$. □

Theorem 15.4 *([60]) A function f is a permutation function if and only if $f = f_S$ for some semigroup S with $Domf_S = N^+$.*

Proof. Let S be a semigroup such that $Dom f_S = N^+$. We show that f_S is a permutation function.

To prove (i), consider a positive integer n such that $n > f_S(1)$. As S is $(1, f_S(1))$-commutative, it is also $(1, n)$-commutative and so $(n, 1)$-commutative. Thus $f_S(n) = 1$.

To prove (ii), let n be an arbitrary positive integer with $1 \leq n \leq f_S(1)$. As S is $(1, f_S(1))$-commutative, it is $(n, f_S(1) - n + 1)$-commutative from which we get $f_S(n) \leq f_S(1) - n + 1$, that is, $n + f_S(n) \leq f_S(1) + 1$. From the condition that S is $(n, f_S(n))$-commutative it follows, by Lemma 15.1, that S is $(1, n + f_S(n))$-commutative. Thus $f_S(1) \leq n + f_S(n)$. This and $n + f_S(n) \leq f_S(1) + 1$ together imply that $n + f_S(n) = f_S(1)$ or $n + f_S(n) = f_S(1) + 1$. Thus (ii) is satisfied.

To prove (iii), consider two positive integers n and m with $1 < n, m < f(1)$ such that $n + f_S(n) = m + f_S(m) = f_S(1)$ and $m < f_S(n)$. As S is $(n, f_S(n))$-commutative and $(m, f_S(m))$-commutative, it follows that S is $(n + m, f_S(n) - m)$-commutative. So $f_S(n + m) \leq f_S(n) - m$. Thus

$$n + m + f_S(n + m) \leq n + m + f_S(n) - m \leq n + f_S(n) = f_S(1).$$

Evidently, $n + m < f_S(1)$. Applying (ii) for $n + m$, we get

$$n + m + f_S(n + m) \geq f_S(1).$$

This and $n + m + f_S(n + m) \leq f_S(1)$ together imply that

$$n + m + f_S(n + m) = f_S(1).$$

So

$$f_S(n + m) = f_S(1) - n - m = f_S(n) - m.$$

Thus (iii) is satisfied.

To prove (iv), let n be an arbitrary positive integer with $n + f_S(n) = f_S(1)$. As S is $(n, f_S(n))$-commutative, we get

$$f_S(f_S(n)) \leq n.$$

Let $f_S(f_S(n))$ be denoted by k. Then $k \leq n$. Assume $k < n$. As S is $(k, f_S(k))$-commutative, it is $(n, f_S(k))$-commutative. So $f_S(n) \leq f_S(k)$. As S is $(k, f_S(n))$-commutative, we get $f_S(k) \leq f_S(n)$. So $f_S(k) = f_S(n)$ which implies

$$k + f_S(k) < n + f_S(k) = n + f_S(n) = f_S(1).$$

This is impossible (see (ii)). Thus $k = n$ and so

$$f_S(f_S(n)) = n.$$

Thus (iv) is satisfied and the first part of the theorem is proved.

Conversely, let f be a permutation function. By Theorem 15.2, P_f is a subgroup of the symmetric group $S_{f(1)}$. Let X be an arbitrary set with $|X| \geq f(1)$. Consider the free semigroup $\mathcal{F}_X$ over X Let

$$I = \{\omega \in \mathcal{F}_X : \ l(\omega) \geq f(1) + 1\},$$

where $l(\omega)$ denotes the length of ω. We define a relation α on $\mathcal{F}_X$ in the following way: $(\omega_1,\omega_2) \in \alpha$ for $\omega_1,\omega_2 \in \mathcal{F}_X$ iff $\omega_1 = x_1x_2\dots x_{f(1)}$, $\omega_2 = y_1y_2\dots y_{f(1)}$ (x_i, $y_j \in X$; $i,j = 1,2,\dots,f(1)$) and there is a permutation $p \in P_f$ such that

$$y_1y_2\dots y_{f(1)} = x_{p(1)}x_{p(2)}\dots x_{p(f(1))}$$

in $\mathcal{F}_X$. With the help of α, we define a relation β by

$$\beta = \{(\omega_1,\omega_2) \in \mathcal{F}_X \times \mathcal{F}_X : \omega_1 = \omega_2 \ \textit{or}\ \omega_1,\omega_2 \in I \ \textit{or}\ (\omega_1,\omega_2) \in \alpha\}.$$

As P_f is a group, β is an equivalence relation. As I is an ideal of $\mathcal{F}_X$, β is compatible. So β is a congruence on $\mathcal{F}_X$. We shall denote the β-class of $\mathcal{F}_X$ containing the word ω by $[\omega]$. Let S be the factor semigroup of $\mathcal{F}_X$ modulo β. We show that $f_S = f$. First we show that $f_S(1) = f(1)$. By the construction, S is $(1,f(1))$-commutative. So $f_S(1) \leq f(1)$. Assume $f_S(1) < f(1)$. Then $1+f_S(1) < 1+f(1)$. If $1+f_S(1) < f(1)$ then S is not $(1,f_S(1))$-commutative (for example $(ab^{f_S(1)}, b^{f_S(1)}a) \notin \beta$ for arbitrary a and b of X with $a \neq b$) which is impossible. If $1+f_S(1) = f(1)$ then $f_S(1) = f(1)-1$ and so S is $(1,f(1)-1)$-commutative. Let $x_1,x_2,\dots,x_{f(1)}$ be pair-wise distinct elements of X. Then

$$[x_1][x_2]\dots[x_{f(1)}] = [x_2][x_3]\dots[x_{f(1)}][x_1],$$

that is,

$$(x_1x_2\dots x_{f(1)}, x_2x_3\dots x_{f(1)}x_1) \in \beta.$$

Then $|P_f| > 1$ and there is a permutation $p_{(z,f(z))}$ in P_f such that

$$x_2x_3\dots x_{f(1)}x_1 = x_{p_{(z,f(z))}(1)}\dots x_{p_{(z,f(z))}(f(1))} = x_{z+1}\dots x_{f(1)}x_1\dots x_z$$

in $\mathcal{F}_X$. Then $z = 1$ and so $f(1) = z + f(z) = 1 + f(1)$ which is a contradiction. Consequently, $f_S(1) = f(1)$.

If $n \geq f(1) = f_S(1)$ then $f(n) = f_S(n) = 1$. Let n be an arbitrary positive integer with $n < f(1)$. Assume $n + f(n) = f(1)$. Then $p_{(n,f(n))} \in P_f$. So S is $(n,f(n))$-commutative which implies that $f_S(n) \leq f(n)$. If $f_S(n) < f(n)$ then $n+f_S(n) < n+f(n) = f(1) = f_S(1)$ which is a contradiction. So $f(n) = f_S(n)$. Consider the case when $n+f(n) = f(1)+1$. Then S is $(n,f(n))$-commutative. Thus $f_S(n) \leq f(n)$. Assume $f_S(n) < f(n)$. Then $n + f_S(n) < n + f(n) = f(1) + 1 = f_S(1) + 1$ and so $n + f_S(n) = f_S(1) = f(1)$. Evidently, S is $(n,f_S(n))$-commutative. Let $x_1,x_2,\dots,x_n,x_{n+1},\dots,x_{f(1)}$ be pair-wise distinct elements of X. Then

$$[x_1][x_2]\dots[x_n][x_{n+1}]\dots[x_{f(1)}] = [x_{n+1}]\dots[x_{f(1)}][x_1]\dots[x_n],$$

that is,

$$(x_1x_2\dots x_nx_{n+1}\dots x_{f(1)}, x_{n+1}\dots x_{f(1)}x_1\dots x_n) \in \beta.$$

Then $\mid P_f \mid > 1$ and there is a permutation $p_{(z,f(z))} \in P_f$ such that

$$x_{n+1}\dots x_{f(1)}x_1\dots x_n = x_{p_{(z,f(z))}(1)}\dots x_{p_{(z,f(z))}(f(1))} = x_{z+1}\dots x_{f(1)}x_1\dots x_z$$

in F_X. Then $z = n$ and so $f(1) = z + f(z) = n + f(n) = f(1) + 1$ which is a contradiction. So $f(n) = f_S(n)$. Consequently, $f = f_S$ and the theorem is proved. □

Corollary 15.5 *([1],[60]) A semigroup is (m,n)-commutative if and only if it is $(d, m+n-d)$-commutative, where d is the greatest common divisor of m and n.*

Proof. Let S be an (m,n)-commutative semigroup and d the greatest common divisor of m and n. Then, by Lemma 15.6, $Domf_S = N^+$. By Theorem 15.4, f_S is a permutation function. Evidently, $n \geq f_S(m)$. So, by Corollary 15.3, $m+n-d \geq f_S(d)$. Thus S is $(d, m+n-d)$-commutative. As the proof of the converse statement is trivial, the corollary is proved. □

Corollary 15.6 *([1],[60]) If S is an (m,n)-commutative semigroup such that m, n are relatively primes then S is $(k, m+n-k)$-commutative for all $1 \leq k < m+n$.*

Proof. Let S be an (m,n)-commutative semigroup such that m and n are relatively primes. Then, by Corollary 15.5, S is $(1, m+n-1)$-commutative from which it follows that S is $(k, m+n-k)$-commutative for all $1 \leq k < m+n$. □

Corollary 15.7 *([1],[60]) If S is an (m,n)-commutative semigroup such that $m+n$ is a prime then S is $(k, m+n-k)$-commutative for all $1 \leq k < m+n$.*

Lemma 15.7 *([61]) If $Domf_S = N^+$ for a semigroup S then S is $(k, p-k+1)$-commutative for all integers $p \geq f_S(1)$ and $k = 1, 2, \ldots, p$.*

Proof. Let S be a semigroup such that $Domf_S = N^+$. Then S is $(1, f_S(1))$-commutative. Let $p \geq f_S(1)$ be arbitrary. Then S is $(1,p)$-commutative from which it follows that S is $(k, p-k+1)$-commutative for every $k = 1, 2, \ldots, p$. □

Connection of (m,n)-commutative semigroups and $E-k$ semigroups

Next, we prove that every (m,n)-commutative semigroup is an $E-k$ semigroup for some k.

Theorem 15.5 *([61]) If $Domf_S = N^+$ for a semigroup S then S is an $E-m$ semigroup for all $m \geq \frac{q+1}{2}$, where q is the least odd number satisfying $q > f_S(1)$.*

Proof. Let S be a semigroup such that $Domf_S = N^+$. Let q be the least odd number satisfying $q > f_S(1)$ and let $m \geq \frac{q+1}{2}$ be an arbitrary integer. Then there is an odd number $p \geq q$ such that $m = \frac{p+1}{2}$. As $p-1 \geq f_S(1)$, Lemma 15.7 implies that S is $(k, p-k)$-commutative for every $k = 1, 2, \ldots, p-1$. Then S is $(1, 2m-2)$-commutative and $(2, 2m-3)$-commutative. Using again Lemma 15.7, S is $(k, p-k+1)$-commutative for every $k = 1, 2, \ldots, p$ and so S is $(1, 2m-1)$-commutative and $(2, 2m-2)$-commutative. Then, for arbitrary $a, b \in S$, we get

$$(ab)^m = ((ab)^{m-1}a)b = (a(ab)^{m-1})b = a^2(ba)^{m-2}b^2$$

$$= (ba)^{m-2}b^2a^2 = b((ab)^{m-3}ab^2a^2) = b(a^2(ab)^{m-3}ab^2)$$

$$= b(a^3(ba)^{m-4}bab^2) = a^3(ba)^{m-3}b^3 = \ldots = a^m b^m$$

which means that S is an $E-m$ semigroup. □

Lemma 15.8 *([61]) If S is an (m,n)-commutative semigroup such that $m+n$ is a prime then $f_S(1) < m+n$.*

Proof. By Corollary 15.7, it is obvious. □

As the $(1,1)$-commutative semigroups are commutative, in the next theorem we consider only such (m,n)-commutative semigroups where $m+n \geq 3$.

Theorem 15.6 *([61]) If S is an (m,n)-commutative semigroup with $m+n \geq 3$ then it is an $E-k$ semigroup for all $k \geq \frac{q+1}{2}$, where q is the least odd number satisfying $q > m+n$. Especially, if $m+n$ is a prime then $q = m+n$.*

Proof. Let S be an (m,n)-commutative semigroup with $m+n \geq 3$. Then, by Lemma 15.1, it is $(1, m+n)$-commutative which implies that $f_S(1) \leq m+n$. Let p be an arbitrary odd number with $p > m+n$. Then $p > f_S(1)$ and so, by Theorem 15.5, S is an $E-k$ semigroup, where $k = \frac{p+1}{2}$. Consequently S is an $E-k$ semigroup for all $k \geq \frac{q+1}{2}$, where q is the least odd number satisfying $q > m+n$.

Consider the case when $m+n \geq 3$ is a prime. By Lemma 15.8, $f_S(1) < m+n$. As $m+n$ is an odd number, Theorem 15.5 implies that S is an $E-k$ semigroup for all $k \geq \frac{q+1}{2}$ where $q = m+n$. Thus the theorem is proved. □

We note that, by Theorem 15.6, if S is an (m,n)-commutative semigroup for some m and n then it is an $E-k$ semigroup for some k. The converse is not true. For example, a non trivial right zero semigroup is an $E-k$ semigroup for all k, but there is no positive integers m and n such that it is (m,n)-commutative. (We note that the idempotent elements of an (m,n)-commutative semigroup must be central.)

Corollary 15.8 *([36],[61]) Every $(1,2)$-commutaive semigroup is an exponential semigroup.*

Proof. Let S be an $(1,2)$-commutative semigroup. Then, by Theorem 15.6, S is an $E-k$ semigroup for all $k \geq 2$ which means that S is an exponential semigroup. □

Semilattice decomposition of (m,n)-commutative semigroups

Theorem 15.7 *Every (m,n)-commutative semigroup is a left and right Putcha semigroup.*

Proof. Let S be an (m,n)-commutative semigroup and a,b be arbitrary elements of S with $b \in aS^1$, that is $b = ax$ for some $x \in S^1$. As

$$b^{m+n+2} = (ax)^{m+n+2} = (ax)^{m+1}(ax)^{n+1}$$

$$= a(xa)^m xa(xa)^n x$$
$$= a((xa)^m x)(a(xa)^n)x$$
$$= a(a(xa)^n)((xa)^m x)x \in a^2S^1,$$

we get that S is a left Putcha semigroup. We can prove, in a similar way, that every (m,n)-commutative semigroup is a right Putcha semigroup. □

Theorem 15.8 *([56]) Every (m,n)-commutative semigroup is a semilattice of archimedean (m,n)-commutative semigroups.*

Proof. By Theorem 15.7 and Corollary 2.2, it is obvious. □

Theorem 15.9 *([56]) A semigroup is 0-simple and (m,n)-commutative if and only if it is a commutative group with a zero adjoined.*

Proof. Let S be a 0-simple (m,n)-commutative semigroup. By Theorem 15.8, S is a semilattice of (m,n)-commutative archimedean semigroups. As $S^1aS^1 = S$ for all non-zero elements a of S, the non-zero elements of S are in the same semilattice component A of S. The zero 0 of S is not in A. If 0 was in A, then S would be a nil semigroup and this would contradict the assumption that S is 0-simple. Consequently, $S = A^0$ and A is a simple (m,n)-commutative semigroup. As A is a left and right Putcha semigroup, it is completely simple (see Theorem 2.3). Then, by Theorem 1.25, A is isomorphic to a Rees matrix semigroup $\mathcal{M}(I,G,J;P)$ over a group G with the sandwich matrix P. By Theorem 1.24, we may assume that P is normalized, that is, $p_{j_0,i} = p_{j,i_0} = e$ for some $i_0 \in I$ and $j_o \in J$ and for all $i \in I$ and $j \in J$ (here e denotes the identity element of G). Then, for arbitrary elements $i \in I$ and $j \in J$, we get

$$(i,e,j_0) = (i,e,j)(i_0,e,j)^m(i_0,e,j_0)^n$$
$$= (i_0,e,j_0)^n(i,e,j)(i_0,e,j)^m = (i_0,e,j).$$

So $i = i_0$ and $j = j_0$ for all $i \in I$ and $j \in J$. Thus A is isomorphic to G. As A is (m,n)-commutative, for all $a,b \in A$, we get

$$ab = e^{m-1}ae^{n-1}b = e^{n-1}be^{m-1}a = ba.$$

So A is a commutative group. Thus the first part of the theorem is proved.
As the converse statement is obvious, the theorem is proved. □

Theorem 15.10 *([56]) A semigroup is (m,n)-commutative archimedean and has an idempotent element if and only if it is an ideal extension of a commutative group by an (m,n)-commutative nil semigroup.*

Proof. Let S be an (m,n)-commutative archimedean semigroup with an idempotent element f. By Theorem 2.2, S is an ideal extension of the simple semigroup $G = SfS$ by the nil semigroup $Q = S/G$. As G is also (m,n)-commutative, by Theorem 15.9, it is a commutative group. It is clear that Q is (m,n)-commutative.

Conversely, let S be a semigroup such that S is an ideal extension of a commutative group G by an (m,n)-commutative nil semigroup Q. Then, by Theorem 2.2, S is an archimedean semigroup with an idempotent element. Since an ideal extension of a group (by a semigroup with zero) is a retract extension and the (m,n)-commutative semigroups form a variety then, by Theorem 1.40, S is (m,n)-commutative. □

Lemma 15.9 *([61]) A semigroup S is regular and satisfies a permutation identity*

$$s_1 s_2 \dots s_n = s_{\sigma(1)} s_{\sigma(2)} \dots s_{\sigma(n)}$$

for some $\sigma \in S_n$ $(n \geq 2)$ with $\sigma(1) \neq 1$, $\sigma(n) \neq n$ if and only if S is a commutative Clifford semigroup.

Proof. Let S be a regular semigroup which satisfies a permutation identity mentioned in the lemma. Then, by Theorem 2 of [81], there is a positive integer k such that S satisfies the permutation identity

$$s_1 s_2 \dots s_l = s_{\alpha(1)} s_{\alpha(2)} \dots s_{\alpha(l)}$$

for every $l \geq k$ and every $\alpha \in S_l$. From this it follows that the idempotent elements of S are central. Then S is a Clifford semigroup. By Theorem 1.21, S is a strong semilattice of its subgroups. It is easy to see that these subgroups are commutative and so S is a commutative Clifford semigroup. As the converse statement is evident, the lemma is proved. □

Corollary 15.9 *([61]) A semigroup is regular and (m,n)-commutative for some m and n if and only if it is a commutative Clifford semigroup.*

Proof. By Lemma 15.9, it is trivial. □

Definition 15.4 *For an integer $n \geq 2$, let Σ_n be a non-empty subset of permutations of the symmetric group S_n of all permutations of $\{1, 2, \dots, n\}$. We shall say that a semigroup S has the permutation property P_n with respect to Σ_n if, for every n-tuple $(s_1, s_2, \dots, s_n)$ of elements of S, there is a non-identity permutation σ in Σ_n such that*

$$s_1 s_2 \dots s_n = s_{\sigma(1)} s_{\sigma(2)} \dots s_{\sigma(n)}.$$

Definition 15.5 *If $\mathcal{T}$ is a subclass of the class of all regular semigroups then a subset Σ_n of the symmetric group S_n is called a $\mathcal{T}$-subset of S_n if every regular semigroup having the permutation property P_n with respect to Σ_n belongs to $\mathcal{T}$.*

Let CC denote the class of all commutative Clifford semigroups. Next we deal with CC-subsets of S_n. First we prove the following lemma.

Lemma 15.10 *Every ideal extension of a Clifford semigroup by a nil semigroup is a retract extension.*

Proof. Let S be a semigroup which is an ideal extension of a Clifford semigroup K by a nil semigroup N. Let e be an arbitrary idempotent element of K. Then, for every $s \in S$, we have

$$es = e(es) = (es)e = e(se) = (se)e = se.$$

By Theorem 1.21, K is a semilattice Y of groups G_i $(i \in Y)$. Let η denote the corresponding semilattice congruence on K. Since $N = S/K$ is a nil semigroup then, for every $a \in S$, there is a least positive integer n such that $a^n \in G_i$ for some $i \in Y$. If e denotes the identity element of G_i then

$$(ae)^n = a^n e = a^n \in G_i$$

from which we get $ea = ae \in G_i$. Then, for every integer $m \geq n$, we get

$$a^m = a^{m-n}a^n = a^{m-1}a^n e = (ae)^m \in G_i.$$

Consequently, for every $a \in S$, there is a subgroup G_i of K which contains all powers of a belonging to K. Let $\phi(a) = ae$. Then ϕ is a well-defined mapping of S onto K. It is clear that ϕ leaves the elements of K fixed. We show that ϕ is a homomorphism. First of all, we note that if f is an idempotent element of K then, for arbitrary $x, y \in S$, we have

$$fxy = f(fx)y = (fx)(fy)\ \eta\ (fy)(fx) = (fy)fx = f(fy)x = fyx.$$

Let $a, b \in S$ be arbitrary elements with $a^r \in G_i$, $b^s \in G_j$, $(ab)^t \in G_k$ for some positive integers r, s, t and elements $i, j, k \in Y$. We can suppose that $t \geq max\{r, s\}$. Let e, f and g denote the identity element of G_i, G_j and G_k, respectively. As $(ab)^t = g(ab)^t\ \eta\ ga^tb^t \in G_{ijk}$, we have $ijk = k$, because $(ab)^t \in G_k$. From this we get $g = efg$. As $ef(ab)^t\ \eta\ (ea)^t(fb)^t \in G_{ij}$, we have $ijk = ij$, because $ef(ab)^t \in G_{ijk}$. From this we get $efg = ef$. Consequently, $ef = g$. Thus

$$\phi(a)\phi(b) = (ae)(bf) = ae(bf) = abfe = abg = \phi(ab).$$

Hence ϕ is a retract homomorphism of S onto K. □

Theorem 15.11 *([61]) Let Σ_n be a CC-subset of the symmetric group $\mathcal{S}_n$. Assume that a semigroup S is an ideal extension of a regular semigroup K by a nil semigroup Q. Then S has the permutation property P_n with respect to Σ_n if and only if K is a commutative Clifford semigroup and Q has the permutation property P_n with respect to Σ_n.*

Proof. Assume that S has the permutation property P_n with respect to Σ_n. Then K and Q have the same property. As Σ_n is a CC-subset of $\mathcal{S}_n$, K is a commutative Clifford semigroup. Thus the first part of the theorem is proved.

Conversely, assume that K is a commutative Clifford semigroup and Q has the permutation property P_n with respect to Σ_n. By Lemma 15.20, there is a retract homomorphism ϕ of S onto K. Let $x_1, x_2, \ldots, x_n$ be arbitrary elements of S.

If $x_1x_2\ldots x_n \notin K$ then $x_1, x_2, \ldots, x_n \notin K$ and so, using the assumption that Q has the permutation property P_n with respect to Σ_n, there is a non-identity permutation $\sigma \in \Sigma_n$ such that

$$x_1x_2\ldots x_n = x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)}$$

in Q and so in S.

If $x_1x_2\ldots x_n \in K$ and $x_1, x_2, \ldots, x_n \notin K$ then, using again the assumption that Q has the permutation property P_n with respect to Σ_n,

$$0 = x_1x_2\ldots x_n = x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)}$$

in Q for some non-identity $\sigma \in \Sigma_n$. Thus, in S,

$$x_1x_2\ldots x_n = \phi(x_1x_2\ldots x_n) = \phi(x_1)\phi(x_2)\ldots\phi(x_n)$$

$$= \phi(x_{\sigma(1)})\phi(x_{\sigma(2)})\ldots\phi(x_{\sigma(n)}) = \phi(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)})$$

$$= x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)},$$

because K is commutative and $x_1x_2\ldots x_n$, $x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)} \in K$ are fixed under ϕ.

If $x_1x_2\ldots x_n \in K$ and $x_j \in K$ for some $j = 1, 2, \ldots, n$ then, for all $\sigma \in \Sigma_n$, we get

$$x_1x_2\ldots x_n = \phi(x_1x_2\ldots x_n) = \phi(x_1)\phi(x_2)\ldots\phi(x_n)$$

$$= \phi(x_{\sigma(1)})\phi(x_{\sigma(2)})\ldots\phi(x_{\sigma(n)}) = \phi(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)})$$

$$= x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)}.$$

Consequently S has the permutation property P_n with respect to Σ_n. Thus the theorem is proved. □

Corollary 15.10 *([61]) Let the semigroup S be an ideal extension of a regular semigroup K by a nil semigroup Q. Then S is (m,n)-commutative if and only if K is a commutative Clifford semigroup and Q is (m,n)-commutative.*

Proof. A semigroup is (m,n)-commutative if and only if it has the permutation property P_{m+n} with respect to $\Sigma_{m+n} = \{p_{m,n}\}$, where $p_{m,n} = p_{1,m+n-1}^m$ is defined after Lemma 15.4. By Corollary 15.9, $\{p_{m,n}\}$ is a $\mathcal{CC}$-subset. Thus the assertion follows from Theorem 15.11. □

Theorem 15.12 *([56]) An (m,n)-commutative archimedean semigroup without idempotent element has a non-trivial group homomorphic image.*

Proof. Let S be an (m,n)-commutative archimedean semigroup without idempotent element, and let a be an arbitrary element of S. We show that

$$S_a = \{x \in S : \ a^t x a^r = a^s \text{ for some positive integers } t, r, s\}$$

is a reflexive unitary subsemigroup of S and S_a is minimal among reflexive unitary subsemigroups of S containing the element a.

Let $x, y \in S_a$ be arbitrary. Then there are positive integers t, r, s, i, j, k such that

$$a^t x a^r = a^s$$

and

$$a^i y a^j = a^k.$$

We may assume that $t, r, i, j \geq max\{m,n\}$. Then

$$(**) \qquad a^{s+k} = a^t x(a^{r+i})(ya^j) = a^t xya^{j+r+i}$$

and so

$$xy \in S_a.$$

Thus S_a is a subsemigroup of S.

Assume $x, xb \in S_a$ for some $x, b \in S$. Then there are positive integers t, r, s, i, j and k such that

$$a^t x a^r = a^s$$

and

$$a^i xba^j = a^k.$$

We may assume that $t, r \geq max\{m,n\}$. Then

$$a^s = a^{r+t}x.$$

Let p be a positive integer such that $p \geq max\{i, r+t\}$. Then

$$a^p x = a^{s+p-(r+t)}$$

and so

$$a^{k+p-i} = a^p xba^j = a^{s+p-(r+t)}ba^j.$$

So

$$b \in S_a,$$

that is S_a is left unitary in S.

We can prove, in a similar way, that S_a is right unitary in S. So S_a is an unitary subsemigroup of S.

To prove the reflexivity of S_a, assume $xy \in S_a$ for some $x, y \in S$. Then

$$a^t xya^r = a^s$$

for some positive integers $t, r, s \geq max\{m,n\}$. As S is (m,n)-commutative,

$$a^s = ya^r a^t x$$

from which we get

$$a^{s+m+n} = a^m y a^r a^t x a^n = a^r a^m y x a^n a^t.$$

So

$$yx \in S_a,$$

that is, S_a is reflexive in S.

Let U be a reflexive unitary subsemigroup of S such that U contains a. If x is an arbitrary element of S_a then

$$a^t x a^r = a^s \in U$$

for some positive integers t, r, s. As U is unitary in S and $a^t, a^r \in U$, we have

$$x \in U.$$

Thus

$$S_a \subseteq U.$$

So S_a is the minimal reflexive unitary subsemigroup of S containing the element a.

If $S_a \neq S$ for some $a \in S$ then, by Theorem 1.41, the principal right congruence on S determined by S_a is a (non-trivial) group congruence.

If $S_a = S$ for all elements a of S then, by (**), it can be proved (as in the proof of Theorem 13.8) that there is a homomorphism of S onto the additive semigroup of either the integers or the non-negative integers or the positive integers. These semigroups have non-trivial group homomorphic images. Thus the theorem is proved. □

Subdirectly irreducible (m,n)-commutative semigroups

Theorem 15.13 *([56]) S is a subdirectly irreducible (m,n)-commutative semigroup with a globally idempotent core if and only if it satisfies one of the following conditions.*

- *(i) S is isomorphic to either G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime).*
- *(ii) S is a two-element semilattice.*

Proof. Let S be a subdirectly irreducible (m,n)-commutative semigroup with a globally idempotent core K.

Consider the case when S has no zero element. Then K is simple and (m,n)-commutative. By Theorem 15.9, K is a commutative group. So S is a homogroup without zero which implies, by Theorem 1.47, that S is a commutative group. Thus, by Theorem 3.14, S is isomorphic to a non-trivial subgroup of a quasicyclic p-group (p is a prime).

Consider case when S has a zero element. We can prove, as in Theorem 9.18, that $S^* = S - \{0\}$ is a subsemigroup of S. If $| S^* |= 1$, then S is a two-element semilattice. If $| S^* |> 1$, then S^* has no zero element. Thus S^* is a subdirectly irreducible (m, n)-commutative semigroup with globally idempotent core and so it is isomorphic to a non-trivial subgroup G of a quasicyclic p-group, p is a prime. Consequently S is isomorphic to G^0.

As the semigroups listed in the theorem are subdirectly irreducible (m, n)-commutative semigroups with globally idempotent core, the theorem is proved. □

Theorem 15.14 *([56]) An (m, n)-commutative semigroup with a zero and a non-trivial annihilator is subdirectly irreducible if and only if it has a non-zero disjunctive element.*

Proof. By Theorem 1.44, it is obvoius. □

Theorem 15.15 *([56]) If S is a subdirectly irreducible (m, n)-commutative semigroup with a trivial annihilator ($|A_S| = 1$) and a nilpotent core then S is commutative.*

Proof. Let S be a subdirectly irreducible (m, n)-commutative semigroup such that $| A_S |= 1$ and the core of S is nilpotent. Define the following subsets of S:

$$R = \{r \in S : \; Kr = \{0\}\}$$

and

$$L = \{l \in S : \; lK = \{0\}\}.$$

We show that $R = L$. Let $r \in R$ be arbitrary. Assume, in an indirect way, that $r \notin L$. Then $rK = rK \cup Kr \cup KrK$ is a non-trivial ideal of S. So $K \subseteq rK$, that is $K = rK$. So $K = r^t K$ for all positive integers t. Using the (m, n)-commutativity of S and that $r \in R$,

$$K = r^{m+n-1}K = r^m r^{n-1}K = r^{n-1}Kr^m = \{0\}$$

which is a contradiction. So $r \in L$, that is, $R \subseteq L$. We can prove, in a similar way, that $L \subseteq R$. So $R = L$.

Let $B = S - R$. As $|A_S| = 1$, $B \neq \emptyset$. We show that B is a subsemigroup of S. Assume, in an indirect way, that there are elements $a, b \in B$ such that $ab \notin B$. As $b \in B$,

$$Kb \cup bK = Kb \cup bK \cup KbK = K.$$

Using the indirect assumption $ab \in R$,

$$aK = a(bK \cup Kb) = abK \cup aKb = aKb$$

and so

$$aK = aKb^{m+n-1} = a(Kb^{m-1})b^n = ab^n Kb^{m-1} = \{0\}$$

which contradicts the assumption $a \in B$. So B is a subsemigroup of S.

Let k_1 be an arbitrary non-zero element of K. Then

$$K = Sk_1 \cup k_1S \cup Sk_1S = Bk_1 \cup k_1B \cup Bk_1B \cup \{0\}.$$

So $k_1 = ek_1$ or $k_1 = k_1f$ or $k_1 = gk_1h$ for some $e, f, g, h \in B$.

If $k_1 = gk_1h$, $g, h \in B$ then $g^{mn}k_1h^{mn} = k_1$ and so $k_1 = k_1h^{mn}g^{mn}$, because S is (m,n)-commutative. Thus we may consider only the first two cases.

Assume $k_1 = ek_1$ for some $e \in B$. We note that $k_1 = e^tk_1$ for all positive integers t. Let

$$Z = \{a \in S : \ e^ta = a \text{ for some positive integer } t\}.$$

It is evident that Z is a non-trivial right ideal of S. We show that Z is a two-sided ideal. Let $a \in Z$, $s \in S$ be arbitrary elements. Then $e^ta = a$ for some positive integer t. Then $e^{it}a = a$ for all positive integers i. Choose i and j such that $it, jt \geq m, n$. Then

$$sa = se^{it+jt}a = (se^{it})e^{jt}a = e^{jt}se^{it}a = e^{jt}sa.$$

Thus $sa \in Z$ and so Z is a (non-trivial) two-sided ideal of S. As K is the core of S, $K \subseteq Z$. Consequently, for all $k \in K$, there is a positive integer j such that $e^jk = k$. Define a relation α on S as follows:

$$\alpha = \{(a,b) \in S \times S : \ e^ja = e^jb \text{ for some positive integer } j\}.$$

It can be easily verified that α is a right congruence. We show that α is also left compatible. Let x, a, b be arbitrary elements of S with $(a,b) \in \alpha$. Then $e^ja = e^jb$ for some positive integer j. Then $e^ra = e^rb$ for all positive integers $r \geq j$. Let r be a positive integer with $r \geq max\{j, m\}$. Then

$$e^{r+n}xa = e^r(e^nx)a = (e^nx)e^ra = e^nxe^{r-j}e^ja$$

$$= e^nxe^{r-j}e^jb = (e^nx)e^rb = e^re^nxb = e^{r+n}xb$$

which means that $(xa, xb) \in \alpha$. Thus α is a congruence on S.

Let k_1 and k_2 be arbitrary elements of K such that $(k_1, k_2) \in \alpha$. Then $e^jk_1 = e^jk_2$ for some positive integer j and so $e^tk_1 = e^tk_2$ for all positive integers $t \geq j$. As it was proved above, there are positive integers i_1 and i_2 such that $e^{i_1}k_1 = k_1$ and $e^{i_2}k_2 = k_2$. Let t be a positive integer such that $t \geq j$ and $t = i_1i_2h$, where h is a positive integer. Then $k_1 = e^tk_1 = e^tk_2 = k_2$ and so the restriction of α to K is the equality relation on K. As S is subdirectly irreducible, α is the equality relation on S.

As $ee^2 = e^2e$ (that is, $(e, e^2) \in \alpha$), we get $e^2 = e$. As $es = e^me^ns = e^nse^m = ese$ (that is, $(s, se) \in \alpha$) for all $s \in S$, we get $s = se$ and so e is a right identity element of S. As $s = se = se^me^n = e^nse^m = ese = es$ for all $s \in S$, e is a left identity element of S. Then, for all $a, b \in S$, $ab = ae^me^nb = e^nbae^m = ba$. Consequently S is a commutative semigroup.

In case $k_1 = k_1 f$, $f \in B$ we can prove, in a similar way, that S is commutative. Thus the theorem is proved. □

(m,n)-commutative Δ-semigroups

Theorem 15.16 *([56]) A semigroup is an (m,n)-commutative Δ-semigroup if and only if one of the following conditions is satisfied.*

(i) S is isomorphic to G or G^0, where G is a non-trivial a subgroup of a quasicyclic p-group (p is a prime).

(ii) S is isomorphic to N or N^1, where N is an (m,n)-commutative nil semigroup whose principal ideals form a chain with respect to inclusion.

Proof. Let S be an (m,n)-commutative Δ-semigroup. Then, by Remark 1.2, S is either semilattice indecomposable or a semilattice of two semilattice indecomposable subsemigroups S_1 and S_0 of S ($S_0S_1 \subseteq S_0$). First, assume that S is semilattice indecomposable. Then, by Theorem 15.8, S is archimedean. If S has a zero element then S is a nil semigroup and so, by Theorem 1.56, (ii) is satisfied. Assume that S has no zero element. If S is simple then, by Theorem 15.9, S is a non-trivial commutative group. Then, by Theorem 3.22, S is isomorphic to a non-trivial subgroup of a quasicyclic p-group (p is a prime) and so (i) is satisfied.
Consider the case when S has a proper two-sided ideal (and does not contain zero element). Then, by Theorem 15.12 and Theorem 1.52, S has an idempotent element. By Theorem 15.10, S is an ideal extension of a commutative group G by an (m,n)-commutative nil semigroup. By Theorem 1.52, $|G| = 1$ or $G = S$ which contradicts the assumption for S.

Let us suppose that S is an (m,n)-commutative semilattice decomposable Δ-semigroup, that is, S is a semilattice of two archimedean (m,n)-commutative semigroups S_0 and S_1, $S_0S_1 \subseteq S_0$. By Theorem 1.52, the Rees factor semigroup $S_0^1 = S/S_0$ is a Δ-semigroup. By Remark 1.1, S_1 is a Δ-semigroup. As S_1 is archimedean and (m,n)-commutative, it is either a non-trivial subgroup of a quasicyclic p-group (p is a prime) or an (m,n)-commutative nil semigroup whose principal ideals form a chain with respect to inclusion. By Theorem 1.57, $|S_1| = 1$ if S_1 is a nil semigroup. Hence S_1 may be only a subgroup of a quasicyclic p-group (p is a prime). If $|S_0| = 1$ then $S = S_1^0$; in case $|S_1| > 1$ (i) is satisfied, in case $|S_1| = 1$ S is a two element semilattice and so (ii) is satisfied.

Assume $|S_0| > 1$. By Theorem 15.12 and Theorem 1.52, S_0 has an idempotent element. Then, by Theorem 15.10, S_0 is an ideal extension of a group G by an (m,n)-commutative nil semigroup. By Theorem 1.52, $|G| = 1$ and so S_0 is an (m,n)-commutative nil semigroup. Then, by Theorem 1.59, $|S_1| = 1$. Let $S_1 = \{e\}$. Then, for all $a \in S$,

$$ea = e^{m+n-1}a = e^m e^{n-1} a = e^{n-1} a e^m = eae$$

$$= e^m a e^{n-1} = a e^{n-1} e^m = ae.$$

So

$$S = SeS \cup eS \cup Se = eS = Se,$$

that is e is a two-sided identity element of S. Consequently S is isomorphic to S_0^1, where S_0 is an (m,n)-commutative nil semigroup whose principal ideals form a chain with respect to inclusion. In this case (ii) is satisfied. Thus the first part of the theorem is proved. The converse is obvious. □

Chapter 16

$n_{(2)}$-permutable semigroups

In this chapter we deal with the $n_{(2)}$-permutable semigroups. It is proved that every $n_{(2)}$-permutable semigroup is $(1, 2n-4)$ commutative. Denoting the assertion "If S is an arbitrary $n_{(2)}$-permutable semigroup then there exist positive integers r and t with $r + t = m$ such that S is (r, t)-commutative." by $\mathcal{P}_{m,n}$, consider $\varphi(n) = min\{m; \mathcal{P}_{m,n}$ is true$\}$. It is evident that $\varphi(2) = 2$. We show that $\varphi(3) = 3$, $\varphi(4) = 5$ and $2n - 4 \leq \varphi(n) \leq 2n - 3$ for $n \geq 5$. We deal with the semilattice decompositons of $n_{(2)}$-permutable semigroups. We show that every $n_{(2)}$-permutable semigroup is a semilattice of $n_{(2)}$-permutable archimedean semigroups. It is proved that a semigroup is 0-simple and $n_{(2)}$-permutable if and only if it is a commutative group with a zero adjoined. Moreover, a semigroup is archimedean and $n_{(2)}$-permutable containing at least one idempotent element if and only if it is an ideal extension of a commutative group by an $n_{(2)}$-permutable nil semigroup. We prove that a semigroup is regular and $n_{(2)}$-permutable if and only if it is a commutative Clifford semigroup. Finally, it is shown that a semigroup which is an ideal extension of a regular semigroup K by a nil semigroup N is $n_{(2)}$-permutable if and only if K is a commutative Clifford semigroup and N is $n_{(2)}$-permutable. At the end of the chapter we formulate some theorems about subdirectly irreducible $n_{(2)}$-permutable semigroups and $n_{(2)}$-permutable Δ-semigroups.

Definition 16.1 *For a fixed integer $n \geq 2$, a semigroup S is called an $n_{(2)}$-permutable semigroup if, for any n-tuple $(x_1, x_2, ..., x_n)$ of elements of S, there is a positive integer t with $1 \leq t \leq n-1$ such that*

$$x_1 x_2 \dots x_t x_{t+1} \dots x_n = x_{t+1} \dots x_n x_1 \dots x_t.$$

Theorem 16.1 *Every finitely generated periodic $n_{(2)}$-permutable semigroup is finite.*

Proof. By Theorem 1.1, it is obvious. □

Lemma 16.1 *([58]) An $n_{(2)}$-permutable semigroup is $(n+1)_{(2)}$-permutable.*

Proof. Let S be an $n_{(2)}$-permutable semigroup and $y, x_1, \dots x_n \in S$ be arbitrary elements. Then there is a positive integer $2 \geq t \geq n-1$ such that

$$(yx_1)x_2 \dots x_n = x_t \dots x_n y x_1 \dots x_{t-1}$$

which means that S is $(n+1)_{(2)}$-permutable. □

For an integer $n \geq 2$, let σ_1 denote the permutation of $\{1, 2, \dots, n\}$ defined by

$$\sigma_1(i) = \begin{cases} i+1, & \text{if } i = 1, 2, \dots, n-1 \\ 1, & \text{if } i = n. \end{cases}$$

For $k = 1, 2, \dots n-1$, let $\sigma_k = \sigma_1^k$. Denote σ_{id} the identity permutation of $\{1, 2, \dots, n\}$. It is easy to see that

$$G_n = \{\sigma_k : \ k = 1, 2, \dots, n-1\} \cup \{\sigma_{id}\}$$

is a subgroup of the group of all permutations of $\{1, 2, \dots, n\}$.

It is clear that a semigroup S is $n_{(2)}$-permutable if and only if, for any n-tuple $(x_1, x_2, \dots, x_n)$ of elements of S, there is an element $\sigma_k \in G_n$ such that

$$(1) \qquad x_1 x_2 \dots x_n = x_{\sigma_k(1)} x_{\sigma_k(2)} \dots x_{\sigma_k(n)}.$$

Moreover, (1) is satisfied for all elements $x_1, x_2, \dots, x_n$ of a semigroup S and a fixed σ_k of G_n iff S is $(k, n-k)$-commutative. Thus every $(k, n-k)$-commutative semigroup $(1 \leq k < n)$ is $n_{(2)}$-permutable. Lemma 16.1 shows that the converse statement is not true if $n \geq 4$.

Lemma 16.2 *([58]) For every integer $n \geq 4$, there is a semigroup which is $n_{(2)}$-permutable but not $(k, n-k)$-commutative for all positive integers $k < n$.*

Proof. Consider a two-element set $X = \{x_1, x_2\}$ and the free semigroup $\mathcal{F}_X$ over X. Let $n \geq 4$ be an arbitrary integer. Consider the following subsets of $\mathcal{F}_X$:

$$A_i = \{x_1^i x_2^{n-i}; x_2^{n-i} x_1^i\}, \quad i = 1, 2, ..., n-1,$$

and

$$B = \{\omega \in \mathcal{F}_X : l(\omega) \geq n, \quad \omega \notin \cup A_i\},$$

where $l(\omega)$ denotes the length of the word ω.

Define an equivalence relation α on $\mathcal{F}_X$ by

$$\alpha = \{(\omega_1, \omega_2) \in \mathcal{F}_X \times \mathcal{F}_X : \omega_1 = \omega_2 \ \textit{or} \ (\exists i) \ \omega_1, \omega_2 \in A_i \ \textit{or} \ \omega_1, \omega_2 \in B\}.$$

It can be easily verified that α is a congruence on $\mathcal{F}_X$.

Let $(\omega_1, \omega_2, \dots, \omega_n)$ be an arbitrary n-tuple of elements of $\mathcal{F}_X$. To prove that $S = \mathcal{F}_X / \alpha$ is $n_{(2)}$-permutable, we must show that there is a permutation $\sigma_k \in G_n$ such that

$$(\omega_1 \omega_2 \dots \omega_n, \omega_{\sigma_k(1)} \omega_{\sigma_k(2)} \dots \omega_{\sigma_k(n)}) \in \alpha.$$

If $l(\omega_1\omega_2\dots\omega_n) > n$ then, for all $\sigma_k \in G_n$,

$$(\omega_1\omega_2\dots\omega_n, \omega_{\sigma_k(1)}\omega_{\sigma_k(2)}\dots\omega_{\sigma_k(n)}) \in \alpha.$$

If $l(\omega_1\omega_2\dots\omega_n) = n$, then $\omega_j \in X$ for all $j = 1, 2, \dots, n$. In this case we have two subcases.

If $\omega_1\omega_2\dots\omega_n \in A_i$ for some $i = 1, \dots, n-1$ then

$$\omega_{\sigma_i(1)}\omega_{\sigma_i(2)}\dots\omega_{\sigma_i(n)} \in A_i$$

or

$$\omega_{\sigma_{n-i}(1)}\omega_{\sigma_{n-i}(2)}\dots\omega_{\sigma_{n-i}(n)} \in A_i$$

which means that

$$(\omega_1\omega_2\dots\omega_n, \omega_{\sigma_k(1)}\omega_{\sigma_k(2)}\dots\omega_{\sigma_k(n)}) \in \alpha$$

for $k = i$ or $k = n - i$.
If $\omega_1\omega_2\dots\omega_n \notin A_i$ for all $i = 1, 2, \dots, n-1$ then

$$\omega_1\omega_2\dots\omega_n \in B.$$

Assume $\omega_{\sigma_k(1)}\omega_{\sigma_k(2)}\dots\omega_{\sigma_k(n)} \notin B$ for all $k = 1, 2, \dots, n-1$. Then

$$a = \omega_{\sigma_1(1)}\omega_{\sigma_1(2)}\dots\omega_{\sigma_1(n)} \in A_i$$

for some $i = 1, 2, \dots, n-1$. We may suppose that $a = x_1^i x_2^{n-i}$ (in case $a = x_2^{n-i}x_1^i$ the proof is similar). As $n \geq 4$, there is an integer j from $\{1, 2, \dots, n-1\}$ such that $j \neq i$, $j \neq n-1$ (so $\sigma_j\sigma_1 \neq \sigma_{id}$) and

$$b = \omega_{\sigma_j\sigma_1(1)}\omega_{\sigma_j\sigma_1(2)}\dots\omega_{\sigma_j\sigma_1(n)} \notin A_i.$$

As a and b must contain x_1 the same times, we get $b \notin A_r$ for all $r = 1, 2, \dots, n-1$. So $b \in B$ which is a contradiction. Thus

$$\omega_{\sigma_k(1)}\omega_{\sigma_k(2)}\dots\omega_{\sigma_k(n)} \in B$$

and so

$$(\omega_1\omega_2\dots\omega_n, \omega_{\sigma_k(1)}\omega_{\sigma_k(2)}\dots\omega_{\sigma_k(n)}) \in \alpha$$

for some $k = 1, 2, \dots, n-1$.

Thus it has been proved that in all cases

$$(\omega_1\omega_2\dots\omega_n, \omega_{\sigma_k(1)}\omega_{\sigma_k(2)}\dots\omega_{\sigma_k(n)}) \in \alpha$$

for some $\sigma_k \in G_n$. Consequently S is $n_{(2)}$-permutable.

Let $k < n$ be a positive integer. As

$$(x_1^{k+1}x_2^{n-k-1}, x_1x_2^{n-k-1}x_1^k) \notin \alpha$$

or

$$(x_1^{k-1}x_2^{n-k+1}, x_2^{n-k}x_1^{k-1}x_2) \notin \alpha,$$

S is not $(k, n-k)$-commutative. Thus the lemma is proved. □

Lemma 16.3 *([58]) A semigroup is $3_{(2)}$-permutable iff it is $(1,2)$-commutative (or $(2,1)$-commutative).*

Proof. It is clear that $(1,2)$-commutativity and $(2,1)$-commutativity are equivalent, and $(1,2)$-commutativity implies $3_{(2)}$-permutability. Assume that S is a $3_{(2)}$-permutable semigroup. Then, for arbitrary elements $a, b, c \in S$,

$$abc = bca \text{ or } cab$$

and

$$bca = cab \text{ or } abc$$

from which we can conclude that

$$abc = bca.$$

Thus S is $(1,2)$-commutative. □

Remark 16.1 From Lemma 16.2, it follows that, for every integer $n \geq 4$, there is a semigroup which is not (t,r)-commutative for all t and r with $t + r \leq n$.

We have the following question: "Does $n_{(2)}$-permutability ($n \geq 4$) of a semigroup S imply (t,r)-commutativity (for some t and r) of S?"

This and other related problems are examined in the next section.

On (r,t)-commutativity of $n_{(2)}$-permutable semigroups

Theorem 16.2 *([28]) If a semigroup is $n_{(2)}$-permutable then it is $(1,2n-4)$-commutative.*

Proof. Let S be an $n_{(2)}$-permutable semigroup. For every integer $1 \leq k \leq n-1$, let

$$T_k = \{(x_1, \ldots, x_k, x_{k+1} \ldots, x_n) \in S^n : x_1 \ldots x_k x_{k+1} \ldots x_n = x_{k+1} \ldots x_n x_1 \ldots x_k\}.$$

As S is $n_{(2)}$-permutable,

$$S^n = \cup_{k=1}^{n-1} T_k.$$

Consider the elements $x_1, \ldots, x_{2n-3}$ of S and

$$p_1 = x_1 x_2 \ldots x_{2n-3},$$

$$p_2 = x_2 x_3 \ldots x_{2n-3} x_1,$$

$$\ldots$$

$$p_{2n-3} = x_{2n-3} x_1 \ldots x_{2n-4}.$$

Let

$$I = \{i \in \{1, 2, \ldots, 2n-3\} : p_1 = p_i\}.$$

We prove that $|I| \geq n-1$. Let $A \subset \{2,\ldots,2n-3\}$ such that $|A| = n-1$. Assume $A = \{i_1,\ldots i_{n-1}\}$ and $2 \leq i_1 < i_2 \ldots < i_{n-1} \leq 2n-3$. Then

$$p_1 = (x_1 \ldots x_{i_1-1})(x_{i_1} \ldots x_{i_2-1}) \ldots (x_{i_{n-1}} \ldots x_{2n-3}) \in S^n.$$

As S is $n_{(2)}$-permutable,

$$(x_1 \ldots x_{i_1-1}, x_{i_1} \ldots x_{i_2-1} \ldots x_{i_{n-1}} \ldots x_{2n-3}) \in \cup_{k=1}^{n-1} T_k.$$

Thus

$$I \cap A \neq \emptyset.$$

Assume that $\{1,\ldots,2n-3\} - I$ has a subset B such that $|B| = n-1$. Then it can be proved (as above for A) that

$$I \cap B \neq \emptyset$$

which is a contradiction. Hence

$$|I| \geq n-1.$$

We can prove, in a similar way, that, for arbitrary $k \in \{2,\ldots,2n-3\}$, we have $|J_k| \geq n-1$, where

$$J_k = \{j \in \{1,\ldots,2n-3\} : \ p_k = p_j\}.$$

Thus $I \cap J_k \neq \emptyset$ for every $k \in \{2,\ldots,2n-3\}$. Consequently, $p_1 = p_2 = \ldots p_{2n-3}$ and so the theorem is proved. □

Denoting the assertion "If S is an arbitrary $n_{(2)}$-permutable semigroup then there exist r and t in N^+ with $r+t=m$ such that S is (r,t)-commutative." by $\mathcal{P}_{m,n}$ $(m,n \in N^+,\ n \geq 2)$, consider

$$\varphi(n) = min\{m \in N^+; \mathcal{P}_{m,n} \text{ is true}\}.$$

It is evident that $\varphi(2) = 2$ and, by Lemma 16.3, $\varphi(3) = 3$. By Theorem 16.2, $\varphi(n) \leq 2n-3$ if $n \geq 4$ and so $\varphi(4) = 5$ (see also Lemma 16.2). The problem is: find $\varphi(n)$ for $n \geq 5$. It is evident that $\varphi(n) \geq n$ (see Lemma 16.2).

We show that $2n-4 \leq \varphi(n) \leq 2n-3$ for $n \geq 5$.

For a product $s_1 s_2 \cdots s_n$ of elements s_i $(i = 1,2,\ldots,n)$ of a semigroup S let $p_i = s_i \cdots s_n s_1 \cdots s_{i-1}$ and $I_{p_i} = \{j \in \{1,2,\ldots,n\};\ p_i = p_j\}$. We note that s_0 denotes the identity element of S^1.

The following lemma plays an important role in our investigation.

Lemma 16.4 *([3]) If S is an $n_{(2)}$-permutable semigroup then, for every non-negative integer k and $p_1 = s_1 s_2 \cdots s_{n+k} \in S^{n+k}$, the cardinality of I_{p_1} is at least $k+2$.*

Proof. By induction for k. Let $|I_{p_1}|$ denote the cardinality of I_{p_1}. If $k = 0$ then $|I_{p_1}| \geq 2$ for every $p_1 \in S^n$, because S is $n_{(2)}$-permutable. Assume that $|I_{p_1}| \geq k+2$ for some nonnegative integer k and every $p_1 \in S^{n+k}$. Let $s_1, s_2, \dots, s_{n+k+1}$ be arbitrary elements of S. As S is an $n_{(2)}$-permutable semigroup, by Lemma 16.1, S is also $(n+k+1)_{(2)}$-permutable. Hence there is an index $i \in \{2, \dots, n+k+1\}$ such that $p_1 = p_i$ Consider the product $q = s_1 s_2 \cdots (s_{i-1} s_i) \cdots s_{n+k+1} \in S^{n+k}$. By the assumption,

$$|I_q| \geq k+2.$$

As $|I_q| < |I_{p_1}|$, therefore

$$|I_{p_1}| \geq k+3.$$

□

Construction 16.1 *Let $\mathcal{F}_X$ be the free semigroup (without the empty word) over the set $X = \{a, b\}$. If $\omega \in \mathcal{F}_X$ then $l(\omega)$ denotes the length of ω. Let n be a fixed integer with $n \geq 4$. For an arbitrary non-negative integer i, consider the subsets $A_{n,i}$, $B_{n,i}$ $C_{n,i}$, $D_{n,i}$ of $\mathcal{F}_X$ defined as follows. Let*

$$A_{n,0} = \{a^n\},$$

$$B_{n,0} = \left\{a^{n-(2g-1)} b a^{2g-2}; \quad g = 1, 2, \dots \left[\frac{(n+1)}{2}\right]\right\},$$

$$C_{n,0} = \left\{a^{n-2h} b a^{2h-1}; \quad h = 1, 2, \dots \left[\frac{n}{2}\right]\right\},$$

$$D_{n,0} = \{\omega \in \mathcal{F}_X; \quad l(\omega) = n\} - (A_{n,0} \cup B_{n,0} \cup C_{n,0}),$$

and, for $i \geq 1$, let

$$A_{n,i} = \{a^{n+i}\},$$

$$B_{n,i} = aB_{n,i-1} \cup C_{n,i-1}a,$$

$$C_{n,i} = aC_{n,i-1} \cup B_{n,i-1}a,$$

$$D_{n,i} = \{\omega \in \mathcal{F}_X; \quad l(\omega) = n+i\} - (A_{n,i} \cup B_{n,i} \cup C_{n,i})$$

($[x]$ denotes the integer part of x). It is evident that these subsets are pairwise disjoint. Consider the relation $\alpha_{n,k}$ (k is a non-negative integer) defined by

$$\alpha_{n,k} = \{(\omega_1, \omega_2) \in \mathcal{F}_X \times \mathcal{F}_X : \quad \omega_1 = \omega_2 \quad or \quad l(\omega_1), l(\omega_2) > n+k \quad or$$

$$\exists\ 0 \leq i, j, t, \leq k \quad (\omega_1, \omega_2 \in B_{n,i} \quad or \quad \omega_1, \omega_2 \in C_{n,j} \quad or \quad \omega_1, \omega_2 \in D_{n,t})\}.$$

It is easy to see that $\alpha_{n,k}$ is congruence on $\mathcal{F}_X$. Let $S_{n,k} = \mathcal{F}_X / \alpha_{n,k}$.

Theorem 16.3 *([3]) The factor semigroup $S_{n,k}$ is $n_{(2)}$-permutable if and only if $k \leq n-4$.*

Proof. Assume that $S_{n,k}$ is $n_{(2)}$-permutable. As the lenght of the elements of $B_{n,k}$ and $C_{n,k}$ is $n+k$, both of $B_{n,k}$ and $C_{n,k}$ have at least $k+2$ elements (see Lemma 16.4). Hence $|B_{n,k} \cup C_{n,k}| \geq 2k+4$. On the other hand $|B_{n,k} \cup C_{n,k}| = n+k$. Therefore, $2k+4 \leq n+k$ from wich we get $k \leq n-4$.

Conversely, assume that $k \leq n-4$. Let $s_1, s_2, \ldots, s_n \in S_{n,k}$ be arbitrary elements. Consider words $q_i \in \mathcal{F}_X$ such that $\kappa_{n,k}(q_i) = s_i$ $(i = 1, 2, \ldots, n)$, where $\kappa_{n,k}$ denotes the canonical homomomorhism of $\mathcal{F}_X$ onto $S_{n,k}$. If $l(q_1 q_2 \cdots q_n) > n+k$ then $(q_1 q_2 \cdots q_n,\ q_2 \cdots q_n q_1) \in \alpha_{n,k}$ and so $s_1 s_2 \cdots s_n = s_2 \cdots s_n s_1$. Assume $l(q_1 q_2 \cdots q_n) \leq n+k$. Then there is an integer $i \in \{0, 1, \ldots, k\}$ such that $l(q_1 q_2 \cdots q_n) = n+i$. If $q_1 q_2 \cdots q_n \in D_{n,i}$ then $(q_1 q_2 \cdots q_n,\ q_2 \cdots q_n q_1) \in \alpha_{n,k}$ and so $s_1 s_2 \cdots s_n = s_2 \cdots s_n s_1$. Assume $q_1 q_2 \cdots q_n \in B_{n,i}$. Then there is an index $j \in \{1, 2, \ldots, n\}$ such that the word q_j contains the letter b as a factor (and so $q_1, q_2, \ldots, q_{j-1}, q_{j+1}, \ldots q_n$ do not contain b). Assume that $l(q_1 q_2 \cdots q_t)$ and $l(q_r \cdots q_n)$ are odd numbers for all $t \in \{1, 2, \ldots j-1\}$ and $r \in \{j+1, \ldots, n\}$. If $j = 1$ then $l(q_n)$ is odd and $l(q_r)$ $r = 2, 3, \ldots, n-1$ is even. Hence

$$l(q_1 q_2 \cdots q_n) \geq 2(n-2) + 2 > 2n - 4.$$

This is a contradiction. In case $j = n$ we can get a contradiction, in a similar way. Assume $j \notin \{1, n\}$. Then $l(q_1)$ and $l(q_n)$ are odd and $l(q_r)$ is even for every $r = 2, 3, \ldots j-1, j+1, \ldots, n-1$. Therefore,

$$l(q_1 q_2 \cdots q_n) \geq 2(n-3) + 3 > 2n - 4$$

which is a contradiction. Consequently, $l(q_1 q_2 \cdots q_t)$ or $l(q_r \cdots q_n)$ is even for some $t \in \{1, 2, \ldots j-1\}$ and $r \in \{j+1, \ldots, n\}$. Thus $q_{t+1} \cdots q_n q_1 \cdots q_t \in B_{n,i}$ or $q_r \cdots q_n q_1 \cdots q_{r-1} \in B_{n,i}$ and so

$$s_1 s_2 \cdots s_n = s_{t+1} \cdots s_n s_1 s_2 \cdots s_t \quad \text{or} \quad s_1 s_2 \cdots s_n = s_r \cdots s_n s_1 s_2 \cdots s_{r-1}$$

for some $t \in \{1, 2, \ldots j-1\}$ and $r \in \{j+1, \ldots, n\}$. We get a similar result in case $q_1 q_2 \ldots q_n \in C_{n,i}$. If $q_1 q_2 \cdots q_n \in A_{n,i}$ then $s_1 s_2 \cdots s_n = s_2 \cdots s_n s_1$. Thus $S_{n,k}$ is $n_{(2)}$-permutable. □

Corollary 16.1 *([3]) The semigroup $S_{n,k}$ $(4 \leq n,\ 0 \leq k \leq n-4)$ is $(1, n+k)$-commutative, but not $(1, n+k-1)$-commutative.*

Proof. It is clear that $S_{n,k}$ is $(1, n+k)$-commutative. As $B_{n,k} \cap C_{n,k} = \emptyset$, the semigroup $S_{n,k}$ is not $(1, n+k-1)$-commutative. □

Definition 16.2 *By the degree of $n_{(2)}$-permutability of a semigroup S we shall mean an integer $p(S) \geq 2$ such that S is $p(S)_{(2)}$-permutable but not $(p(S)-1)_{(2)}$-permutable.*

By Lemma 15.1, every (r, t)-commutative semigroup is $(1, r+t)$-commutative. Thus we can define the *degree of commutativity* of a semigroup S as an integer $c(S) \in N^+$ such that S is $(1, c(S))$-commutative but not $(1, c(S)-1)$-commutative. We note that $c(S) = f_S(1)$, where f_S was definedin Chapter 15 (f_S is a permutation function).

Next we deal with the connection between $p(S)$ and $c(S)$ for an arbitrary semigroup.

Theorem 16.4 *([3]) For every integers n and c with $n \geq 3$ and $n-1 \leq c \leq 2n-4$, there is a semigroup S such that $p(S) = n$ and $c(S) = c$.*

Proof. By Lemma 16.3, a semigroup is $3_{(2)}$-permutable if and only if it is $(1,2)$-commutative. Assume $n \geq 4$. It is evident that every $(1,t)$-commutative semigroup is $(t+1)_{(2)}$-permutable. From this it follows that $c(S) < p(S)$ implies $c(S) = p(S) - 1$. For the factor semigroup $S = \mathcal{F}_X/\beta$ constructed in the proof of Theorem 15.4, $p(S) = f(1)$ and $c(S) = f(1) - 1$ and $f(1)$ may be any positive integer n. Therefore, we can suppose that $n \leq c \leq 2n-4$. Then, for the semigroup $S_{n,c-n}$ defined above, $p(S) = n$ and $c(S) = c$. □

Theorem 16.5 *([3]) For every integers n and m with $n \geq 5$ and $2 \leq m < 2n-4$, there is a semigroup which is $n_{(2)}$-permutable but not (r,t)-commutative for all r and t such that $r+t = m$.*

Proof. Let n be an arbitrary integer with $n \geq 5$. If m is an integer with $2 \leq m \leq n$ then the assertion is true (see Lemma 16.2). Assume that $n < m < 2n-4$ for some integer m. Then $n \geq 6$. Let k be a positive integer such that $m = n+k-1$. Clearly, $2 \leq k \leq n-4$. Consider the semigroup $S_{n,k}$ defined in the Construction. By Theorem 16.3, $S_{n,k}$ is $n_{(2)}$-permutable. Assume that $S_{n,k}$ is (r,t)-commutative for some positive integers r and t with $r+t = m = n+k-1$. If r is odd then $a^{n+k-2}b,\ a^{n+k-r-2}ba^r \in B_{n,k-1}$. Therefore, the parity of $n+k-2$ and $n+k-r-2$ must be the same. But this is impossible. If r is even then $a^{n+k-1}b,\ a^{n+k-r}ba^2a^{r-1} \in B_{n,k}$ and so the parity of $n+k-1$ and $n+k-r$ must be the same. This is also impossible. Consequently $S_{n,k}$ is not (r,t)-commutative for all r and t with $r+t = m$. □

Corollary 16.2 *([3]) For every integer $n \geq 5$, $2n-4 \leq \varphi(n) \leq 2n-3$.*

Proof. By Theorem 16.5, if $\mathcal{P}_{m,n}$ is true for some positive integers m and n with $n \geq 5$ then $m \geq 2n-4$. Thus $\varphi(n) \geq 2n-4$. This and the fact $\varphi(n) \leq 2n-3$ (see Theorem 16.2) together imply our assertion. □

The following lemma is an addendum to the problem of finding the exact value of $\varphi(n)$.

Lemma 16.5 *([3]) If an $n_{(2)}$-permutable semigroup is (r,t)-commutative for some n, r and t with $r+t = 2n-4$ then it is (r',t')-commutative for some even r' and t' with $r'+t' = 2n-4$.*

Proof. Assume that S is a semigroup such that it is $n_{(2)}$-permutable and (r,t)-commutative for some integers n, r and t with $n \geq 4$, $r+t = 2n-4$. We can suppose that S is not $(1, 2n-5)$-commutative. In the opposite case S is $(2, 2n-6)$-commutative. Let d denote the greatest common divisor of t and r. By Corollary 15.5, S is $(d, 2n-4-d)$-commutative and so it is $(hd, 2n-4-hd)$-commutative for every $h = 1, 2, \ldots, \frac{2n-4}{d} - 1$. We can suppose that $d > 2$. As S is not $(1, 2n-5)$-commutative, there are elements $s_1, s_2, \ldots s_{2n-4}$ of S such that

$$p_1 = s_1 s_2 \cdots s_{2n-4} \neq s_2 \cdots s_{2n-4} s_1 = p_2.$$

By Lemma 16.4, $|I_{p_1}|, |I_{p_2}| \geq n-2$. As $I_{p_1} \cap I_{p_2} = \emptyset$, $|I_{p_1}| = |I_{p_2}| = n-2$. For $i = 0, 1, \ldots d-1$, let

$$J_i = \{(h-1)d + i + 1; \quad h = 1, 2, \ldots, \frac{2n-4}{d}\}.$$

It is easy to see that J_i contained in either I_{p_1} or I_{p_2} for every $i = 0, 1, \ldots d-1$. Moreover

$$\cup_{i=0}^{d-1} J_i = \{1, 2, \ldots, 2n-4\}.$$

Therefore, $n-2 = \frac{2(n-2)}{d} g$ for some positive integer g. From this it follows that $d = 2g$. Thus r and t are even. □

We note that, from Lemma 16.5, it follows that if a semigroup S is $n_{(2)}$-permutable and (r,t)-commutative such that $n-2$ is a prime, $n \geq 4$ and $r+t = 2n-4$ then S is $(2, 2n-6)$-commutative.

Semilattice decomposition of $n_{(2)}$-permutable semigroups

Lemma 16.6 *([58]) If S is an $n_{(2)}$-permutable semigroup then $(xy)^n = (yx)^n$ for all $x, y \in S$.*

Proof. Let S be a semigroup and x, y be arbitrary elements of S. Then, for every positive integer n,

$$(xy)^n = x(yx)^{n-1}y.$$

If S is $n_{(2)}$-permutable then there is an integer t with $0 \leq t \leq n-1$ such that

$$x(yx)^{n-1}y = ((yx)^t y)(x(yx)^{n-1-t}) = (yx)^n$$

and so

$$(xy)^n = (yx)^n.$$

□

Theorem 16.6 *([58]) Every $n_{(2)}$-permutable semigroup is decomposable as a semilattice of $n_{(2)}$-permutable archimedean semigroups.*

Proof. By Lemma 16.6, every $n_{(2)}$-permutable semigroup is weakly commutative and so, by Theorem 4.3, it is a semilattice of archimedean semigroups. It is clear that the archimedean components are $n_{(2)}$-permutable. □

Theorem 16.7 *([58]) A semigroup is 0-simple and $n_{(2)}$-permutable if and only if it is a commutative group with a zero adjoined.*

Proof. Let S be a 0-simple $n_{(2)}$-permutable semigroup. By Theorem 16.2, S is $(1, 2n-4)$-commutative. Then, by Theorem 15.9, it is a commutative group with a zero adjoined. □

Theorem 16.8 *([58]) A semigroup is an $n_{(2)}$-permutable archimedean semigroup with an idempotent element if and only if it is an ideal extension of a commutative group by an $n_{(2)}$-permutable nil semigroup.*

Proof. Let S be an archimedean $n_{(2)}$-permutable semigroup with an idempotent element. Since S is $(1, 2n-4)$-commutative then, by Theorem 15.10, it is an ideal extension of a commutative group G by a nil semigroup Q. We show that Q is $n_{(2)}$-permutable. Let $a_1, a_2, ..., a_n$ be arbitrary elements of Q. We can suppose that $a_i \neq 0$ for all $i = 1, ..., n$. Then $a_i \in (S - G)$. As S is $n_{(2)}$-permutable, there is an integer t $(1 \leq t \leq n-1)$ such that

$$a_1 a_2 ... a_n = a_{t+1} ... a_n a_1 ... a_t$$

in S and so

$$a_1 a_2 ... a_n = a_{t+1} ... a_n a_1 ... a_t$$

in also Q. So Q is $n_{(2)}$-permutable. Thus the first part of the theorem is proved.

Conversely, assume that the semigroup S is an ideal extension of a commutative group G by an $n_{(2)}$-permutable nil semigroup Q. By Theorem 2.2, S is archimeden and contains an idempotent. It is easy to see that $\phi : s \mapsto es$ (e is the identity of G) is a retract homomorphism of S onto G. To show that S is $n_{(2)}$-permutable, consider arbitrary elements $s_1, s_2, ..., s_n$ of S. There are two cases.

In case $s_1 s_2 ... s_n \notin K$, $s_1 s_2 ... s_n \neq 0$ in Q and so there is an integer t with $1 \leq t \leq n-1$ such that

$$s_1 s_2 ... s_n = s_{t+1} ... s_n s_1 ... s_t$$

in Q and so in S.

In case $s_1 s_2 ... s_n \in K$, $s_1 s_2 ... s_n = 0$ in Q. If there is an index i such that $s_i \in K$, then

$$s_{\sigma(1)} s_{\sigma(2)} ... s_{\sigma(n)} \in K$$

for all permutations σ of $\{1, 2, ..., n\}$ and so

$$s_1 s_2 ... s_n = \phi(s_1 s_2 ... s_n) = \phi(s_1)\phi(s_2)...\phi(s_n) = \phi(s_{\sigma(1)})\phi(s_{\sigma(2)})...\phi(s_{\sigma(n)}) =$$

$$\phi(s_{\sigma(1)} s_{\sigma(2)} ... s_{\sigma(n)}) = s_{\sigma(1)} s_{\sigma(2)} ... s_{\sigma(n)}.$$

If $s_i \notin K$ for all index i then $s_i \neq 0$ in Q (for all i) and so

$$s_1 s_2 ... s_n = s_{t+1} ... s_n s_1 ... s_t$$

in Q for some t with $1 \leq t \leq n-1$. As $s_1 s_2 ... s_n \in K$, we get (in S) that

$$s_1 s_2 ... s_n = \phi(s_1 s_2 ... s_n) = \phi(s_1)\phi(s_2)...\phi(s_n) =$$

$$\phi(s_{t+1})...\phi(s_n)\phi(s_1)...\phi(s_t) = \phi(s_{t+1}...s_n s_1 ... s_t) = s_{t+1}...s_n s_1 ... s_t.$$

So S is $n_{(2)}$-permutable. Thus the theorem is proved. □

Lemma 16.7 *([61]) A semigroup is regular and $n_{(2)}$-permutable for some n if and only if it is a commutative Clifford semigroup.*

Proof. Since an $n_{(2)}$-permutable semigroup is $(1, 2n-4)$-commutative then, by Corollary 15.9, our statement is obvious. □

Corollary 16.3 *([61]) Let the semigroup S be an ideal extension of a regular semigroup K by a nil semigroup Q. Then S is $n_{(2)}$-permutable if and only if K is a commutative Clifford semigroup and Q is $n_{(2)}$-permutable.*

Proof. A semigroup is $n_{(2)}$-permutable if and only if it has the permutation property P_n with respect to $\Sigma_n = G_n$, where G_n is defined after Lemma 16.1. By Lemma 16.7, G_n is a CC subset. Thus our assertion follows from Theorem 15.11. □

Subdirectly irreducible $n_{(2)}$-permutable semigroups

Theorem 16.9 *S is a subdirectly irreducible $n_{(2)}$-permutable semigroup with a globally idempotent core if and only if it satisfies one of the following conditions.*

(i) S is isomorphic to either G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime).

(ii) S is a two-element semilattice.

Proof. By Theorem 16.2 and Theorem 15.13, it is obvious. □

Theorem 16.10 *An $n_{(2)}$-permutable semigroup with a zero and a non-trivial annihilator is subdirectly irreducible if and only if it has a non-zero disjunctive element.*

Proof. By Theorem 1.44, it is obvoius. □

Theorem 16.11 *If S is a subdirectly irreducible $n_{(2)}$-permutable semigroup with a trivial annihilator ($|A_S| = 1$) and a nilpotent core then S is commutative.*

Proof. By Theorem 16.2 and Theorem 15.15, it is obvious. □

$n_{(2)}$-permutable Δ-semigroups

Theorem 16.12 *A semigroup is an $n_{(2)}$-permutable Δ-semigroup if and only if one of the following conditions is satisfied.*

(i) S is isomorphic to G or G^0, where G is a non-trivial a subgroup of a quasicyclic p-group (p is a prime).

(ii) S is isomorphic to N or N^1, where N is an $n_{(2)}$-permutable nil semigroup whose principal ideals form a chain with respect to inclusion.

Proof. By Theorem 16.2 and Theorem 15.16, it is obvious. □

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