Finite permutable Putcha semigroups

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Abstract

A semigroup $S$ is called a permutable semigroup if $\alpha \circ \beta = \beta \circ \alpha$ is satisfied for all congruences $\alpha$ and $\beta$ of $S$. A semigroup is called a Putcha semigroup if it is a semilattice of archimedean semigroups. In this paper we deal with finite permutable Putcha semigroups. We describe the finite permutable archimedean semigroups and finite permutable semigroups which are semilattices of a group and a nilpotent semigroup.

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1 Introduction

The notion of the permutable semigroup was introduced in [9]. A semigroup $S$ is called a permutable semigroup if $\alpha \circ \beta = \beta \circ \alpha$ is satisfied for all congruences $\alpha$ and $\beta$ of $S$. In [9], H. Hamilton proved some general results about permutable semigroups and described the commutative permutable semigroups. In [1], [2], [3], [4], [5], [6], [8], [10], [11], [15], [16] the permutable semigroups are examined in special classes of semigroups. In the present paper we deal with finite permutable Putcha semigroups, that is, finite semigroups which are semilattices of archimedean semigroups (a semigroup $S$ is called archimedean if, for every $a, b \in S$, there are positive integers $n$ and $m$ such that $a^n \in SbS$ and $b^m \in SaS$). We show that the finite archimedean permutable semigroups are exactly the finite cyclic nilpotent semigroups and the finite completely simple permutable semigroups. Dealing with the non-archimedean case, we describe such finite permutable semigroups which are semilattices of a group $G$ and a nilpotent semigroup $N$ with $GN \subseteq N$.

First of all we cite some earlier results which will be used in our investigation. For notations and notions not defined here, we refer to [7] and [13].

Lemma 1 Every group is permutable.

Lemma 2 ([9]) The ideals (equivalently, the principal ideals) of a permutable semigroup form a chain with respect to inclusion.

Lemma 3 ([15]) A nil semigroup $S$ is permutable if and only if the ideals of $S$ form a chain with respect to inclusion.

Lemma 4 ([9]) If a permutable semigroup $S$ contains a proper ideal $K$ then neither $S$ nor $K$ has a non-trivial group homomorphic image.
Lemma 5 ([10]) If \( \rho \) is an arbitrary congruence and \( K \) is an arbitrary ideal of a permutable semigroup then \( K \) is in an \( \alpha \)-class or \( K \) is a union of \( \alpha \)-classes.

Lemma 6 ([6]) A Rees matrix semigroup \( S = M(G, I, J; P) \) is permutable if and only if \( |I| \leq 2 \) and \( |J| \leq 2 \).

Lemma 7 ([9]) Every homomorphic image of a permutable semigroup is permutable.

Lemma 8 ([9]) A semilattice \( F \) is permutable if and only if \( |F| \leq 2 \).

As a Putcha semigroup is a semilattice of archimedean semigroups, Lemma 7 and Lemma 8 together imply that a permutable Putcha semigroup is either archimedean or a semilattice of two archimedean semigroups \( S_0 \) and \( S_1 \) such that, for example, \( S_0S_1 \subseteq S_0 \).

2 Finite permutable archimedean semigroups

First of all we formulate some lemmas which will be used in our examinations several times.

A semigroup \( S \) with a zero \( 0 \) is called a nil semigroup if, for every \( a \in S \), there is a positive integer \( n \) such that \( a^n = 0 \). \( S \) is called nilpotent if, \( S^m = \{0\} \) for some positive integer \( m \).

It is clear, that every archimedean semigroup with a zero is a nil semigroup.

Lemma 9 Every finite nil semigroup is nilpotent.

It is obvious, that every finite semigroup has a kernel \( K \) which is completely simple. Thus we have the following lemma.

Lemma 10 A finite semigroup is archimedean if and only if it is an ideal extension of a completely simple semigroup by a nilpotent semigroup.

Theorem 1 A finite semigroup is an archimedean permutable semigroup if and only if it is either a cyclic nilpotent semigroup or a permutable completely simple semigroup.

Proof. Let \( S \) be a finite permutable archimedean semigroup. By Lemma 10, \( S \) is an ideal extension of a completely simple semigroup \( K \) (\( K \) is the kernel of \( S \)) by the nilpotent semigroup \( N = S/K \). By Lemma 7 of this paper, Lemma 2 of [15] and Lemma 2 of [14], \( N \) is a cyclic nilpotent semigroup. If \( |K| = 1 \) then \( S \) is isomorphic to \( N \). Consider the case when \( |K| > 1 \). We show that \( S = K \). Assume, in an indirect way, that \( K \neq S \). As \( K \) is completely simple, the Green’s relations \( R \) and \( L \) are congruences on \( K \). It is easy to see that \( R \cup 1_S \) and \( L \cup 1_S \) are congruences on \( S \) and the kernels of the quotient semigroups are, respectively, the left zero semigroup \( K/R \) and the right zero semigroup \( K/L \).
By Lemma 4, \( K/\mathcal{R} \) or \( K/\mathcal{L} \) is non-trivial. By symmetry, it can be assumed without loss of generality that \( K \) is a non-trivial right zero semigroup. Let \( a \in S - K \) and let \( f = a^n \in K \), so \( fa^i = f \) for all positive integers \( i \) and \( xf = f \) for all \( x \in S \). Let \( b \in K \), \( b \neq f \). Applying Lemma 5, \( b \) is related to \( f \) under the congruence \( \rho \) on \( S \) generated by \((a, f)\), so there exists a sequence of elementary \( \rho \)-transitions from \( b \) to \( f \) that begins either \( b = sat \mapsto sft \) or \( b = sft \mapsto sat \) \((s, t \in S^1)\), where the right hand side is distinct from \( b \). In addition, since \( b = bb \) and \( f = bf \), we can assume without loss of generality that \( s = bs \in K \). If \( t = 1 \) then \( b = sa \) (since \( b \neq bf = f \)); otherwise, since \( K \) is right zero, \( t \notin K \), so \( t = a^t \) for some \( i < n \) and \( b = sa^{i+1} \), since again \( b \neq sfa^i = f \). In either case, \( b = ca \) for some \( c \in K \), \( c \neq f \). Now the same argument applies to \( c \) and iterating the argument leads to \( b = xax = xf = f \), a contradiction. Thus the first part of the theorem is proved. As the converse is obvious, the theorem is proved. \( \square \)

3 Finite permutable non-archimedean Putcha semigroups

Lemma 11 If \( S \) is a finite non-archimedean Putcha permutable semigroup then it is a semilattice of a completely simple semigroup \( S_1 = M(G; I, J; P) \) such that \(|I|, |J| \leq 2 \) and a semigroup \( S_0 \) such that \( S_1S_0 \subseteq S_0 \) and \( S_0 \) is an ideal extension of a completely simple semigroup \( K \) by a nilpotent semigroup.

Proof. Let \( S \) be a finite permutable non-archimedean Putcha-semigroup. Then, by Lemma 7 and Lemma 8, \( S \) is a semilattice of two archimedean semigroups \( S_0 \) and \( S_1 \) such that \( S_0S_1 \subseteq S_0 \). As the Rees factor \( S_1/0 = S/S_0 \) is permutable by Lemma 7, \( S_1 \) is a permutable archimedean semigroup. By Lemma 8 of [9] and Theorem 1 of this paper, \( S_1 \) is completely simple. Then \( S_1 \) is a Rees matrix semigroup \( S_1 = (G; I, J; P) \) and \(|I|, |J| \leq 2 \) by Lemma 6. By Lemma 10, \( S_0 \) is an ideal extension of a completely simple semigroup \( K \) by the nilpotent Rees factor semigroup \( N = S_0/K \). \( \square \)

In this paper we deal with only that case when \( S_1 \) is a group.

Lemma 12 If a finite permutable semigroup \( S \) is a semilattice of a group \( G \) and a nilpotent semigroup \( N \) such that \( NG \subseteq N \) then the identity element of \( G \) is a left identity element or a right identity element of \( S \).

Proof. Let \( a \in N \) be an arbitrary element. Then \( J(a) \subseteq J(e) \), where \( e \) denotes the identity element of \( G \). Then there are elements \( x, y \in S^3 \) such that \( a = xey \). So \( N = eN \cup Ne \cup NeN \). Since \( N \) is an ideal, \( Ne \cup NeN \) and \( eN \cup NeN \) are ideals of \( S \) and so, by hypothesis, one is included in the other. Suppose \( eN \subseteq Ne \cup NeN \), so that \( N = Ne \cup NeN = Ne \cup (Ne)(eN) \subseteq Ne \cup (Ne)(Ne \cup NeN) \subseteq Ne \cup (Ne)^2N \).

Inductively, \( N \subseteq Ne \cup (Ne)^iN \) for all positive integers \( i \), and since \( N \) is nilpotent, \( N = Ne \), as required. In case \( Ne \not\subseteq eN \cup NeN \), we get \( N = eN \). \( \square \)
Lemma 13 Let $S$ be a finite permutable semigroup which is a semilattice of a group $G$ and a nilpotent semigroup $N$ such that $GN \subseteq N$. Let $e$ denote the identity element of $G$. If $Ne = N$ then $eN = \{0\}$ or $eN = N$. Similarly, if $eN = N$ then $Ne = \{0\}$ or $Ne = N$.

Proof: By the symmetry, we deal with only the first assertion of the lemma. Assume $N = Ne$. Then $SeN = eN \cup NeN$, which is an ideal of $S$. If $SeN = N$, then $N = eN \cup NeN$ from which we get $N = eN$ as in the proof of Lemma 12.

If $SeN \neq N$, then consider the equivalence
\[
\alpha = \{(a, b) \in S \times S : ea = eb\}.
\]
It is obvious that $\alpha$ is a right congruence. Let $a, b, s$ be arbitrary elements of $S$ such that $(a, b) \in \alpha$. As $e$ is a right identity element of $S$, we get
\[
sa = (se)a = s(ea) = s(eb) = (se)b = sb
\]
and so
\[
e(sa) = e(sb).
\]
Thus $\alpha$ is also a left congruence of $S$, and so it is a congruence of $S$. Let $x \in N$ be an arbitrary element. As $(x, ex) \in \alpha$ and $ex \in SeN$, by Lemma 5, the ideal $SeN$ is contained by the $\alpha$-class of $x$, and so $(0, x) \in \alpha$, that is, $0 = e0 = ex$. Hence $eN = \{0\}$. Thus the lemma is proved.

Lemma 14 Let $S$ be a finite non-archimedean permutable semigroup which is a semilattice of a group $G$ and an archimedean semigroup $S_0$ such that $GS_0 \subseteq S_0$.

Then $S_0$ is either

1. completely simple,
2. or a non-trivial null semigroup $N$ such that the identity element of $G$ is a right identity element of $S$ and $SN = \{0\}$,
3. or a non-trivial null semigroup $N$ such that the identity element of $G$ is a left identity element of $S$ and $NS = \{0\}$,
4. or an ideal extension of a completely simple semigroup $K$ by a non-trivial nilpotent semigroup $N$ such that the identity element of $G$ is an identity element of the factor semigroup $S/K$.

Proof. By Lemma 10, $S_0$ is an ideal extension of a completely simple semigroup $K$ by the Rees factor semigroup $N = S_0/K$ which is nilpotent. If $S_0 = K$ then (1) is satisfied.

Assume $S_0 \neq K$. As $K = K^2$ is an ideal of $S_0$ and $S_0$ is an ideal of $S$, we have that $K$ is an ideal of $S$ (see Exercise 4(a) for §2.9 of [7]). Consider the Rees factor semigroup $S/K$ which is a semilattice of $G$ and a nilpotent semigroup which is isomorphic to the non-trivial semigroup $N = S_0/K$. By Lemma 12, the identity element of $G$ is the right identity element or the left identity element of $S/K$. 

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First consider the case when the identity element $e$ of $G$ is the right identity element but not a left identity element of $S/K$. Then, by Lemma 13, $eS_0 \subseteq K$. Now without loss of generality, if $K$ is non-trivial it can be assumed to be either left zero or right zero, but the two cases must be treated separately because of the asymmetry of the hypothesis on $S$. In either case, let $a \in S - K - G$, such that $f = a^2 \in K$, and suppose $b \in K$, $b \neq f$. By Lemma 5, $b$ is related to $f$ under the congruence $\rho$ on $S$ generated by $(a,f)$, so there exists a sequence of elementary $\rho$-transitions from $b$ to $f$ that begins either $b = sat \mapsto sft$ or $b = sft \mapsto sat$ $(s,t \in S^1)$, where the right hand side is distinct from $b$. First suppose that $K$ is right zero. Then again $t \not\in K$. If $t \in N$ then $at \in K$ and so $at = a(at) = ft$, giving $sat = sft$, a contradiction. So $t \in G$ and therefore $b = be$. Hence $K = K e$. As in the proof of Theorem 1, without loss of generality, $s \in K$ and so $s = se$. Also $ea \in K$. Then

$$sa = (se)a = s(ea) = ea = a(ea) = (ae)a = a^2 = f = sf,$$

again giving the contradiction $sat = sft$. Next suppose $K$ is left zero. Now, without loss of generality, $t \in K$ and $s \not\in K$. If $s \neq 1$ then since $S \subseteq K$, $sa = sasa = sa^2 = sf$ (since $sas = sa$), a contradiction. So $s = 1$ and since $b \neq f = ft$, $b = at$. But $t \in K$ and $t \neq f$ (since $af = f$) so similarly $t = at'$ for some $t'$, yielding the contradiction $b = a^2t' = ft' = f$. From this it follows, that $|K| = 1$ and so $S_0 = N$. Let $a \in S_0 = N$ be arbitrary. As $eN = \{0\}$ and $ea \in eN$, we get $ea = 0$ and so, for every $s \in S$, $sa = sea = 0$. Thus $SN = \{0\}$ and so (2) is satisfied.

If the identity element of $G$ is a left identity element but not a right identity element of $S/K$ then (3), the dual of (2) is satisfied.

If the identity element of $G$ is the identity element of $S/K$, then (4) is satisfied. Thus the lemma is proved.

\[ \square \]

**Remark 1** If $|S_0| = 1$ is satisfied in case (1) of Lemma 14 then $S$ is a group with a zero adjoined and so $S$ is permutable.

**Remark 2** Condition (4) of Lemma 14 has two subcases:

- (4a): $|K| = 1$ and so $S_0$ is a non-trivial nilpotent semigroup such that the identity element of $G$ is an identity element of $S$.
- (4b): $|K| > 1$, but $K \neq S_0$.

In this paper we describe only those finite permutable non-archimedean Putcha semigroups which are semilattice of a group $G$ and a semigroup $S_0$ with $GS_0 \subseteq S_0$, where $S_0$ satisfies either condition (2) or condition (3) of Lemma 14 or condition (4a) of Remark 2.

### 3.1 When the identity element of $G$ is only a one-sided identity element of $S$

In this section we deal with only the right side case, but the main theorem (Theorem 2) will be formulated for both right and left cases.
For a non-trivial nil semigroup $N$, let $N^*$ denote $N - \{0\}$.

**Lemma 15** Let $S$ be a finite permutative semigroup which is a semilattice of a group $G$ and a non-trivial nilpotent semigroup $N$ such that the identity element of $G$ is a right identity element of $S$ and $SN = \{0\}$. Then $aG = N^*$ for every $a \in N^*$.

**Proof.** It is clear that $N^2 = \{0\}$. Let $a \in N^*$ be arbitrary. If $ag = 0$ for some $g \in G$ then $a = ae = agg^{-1} = 0$ which is a contradiction. Thus $aG \subseteq N^*$. As $S$ is finite and the ideals of $S$ form a chain, there is an element $b \in N^*$ such that $S^1 b S^1 = N$. Thus $N = S^1 b S^1 = (S \cup 1)b(G \cup N \cup 1) = bG \cup bN = bG \cup \{0\}$. Thus $bG = N^*$. From this it follows that, for an arbitrary $a \in N^*$ and some $x \in G$, $aG = bxG = bG = N^*$.

**Remark 3** If a semigroup $S$ satisfies the conditions of Lemma 15 then $N^*$ is a right $G$-set ([12]) and $G$ acts on $N^*$ transitively.

**Lemma 16** If an arbitrary semigroup $S$ is a semilattice of a group $G$ and a non-trivial nilpotent semigroup $N$ such that $GN = \{0\}$ and $aG = N^*$ for every $a \in N^*$ then, for every non-universal congruence $\alpha$ of $S$, $[0]_\alpha$ is either $\{0\}$ or $N$, and $[g]_\alpha \subseteq G$ for every $g \in G$.

**Proof.** If $g \in [0]_\alpha$ for some $g \in G$ then $G \subseteq [0]_\alpha$ and so $N^* = aG \subseteq [0]_\alpha$, where $a \in N^*$ is an arbitrary element. Then $[0]_\alpha = S$. If $\alpha$ is not a universal congruence of $S$ then $[0]_\alpha \subseteq N$. Assume $[0]_\alpha \neq \{0\}$. Then there is an element $a \in N^*$ such that $a \in [0]_\alpha$, and so $N^* = aG \subseteq I$. Hence $I = N$.

Assume $(a,g) \in \alpha$ for some $a \in N, g \in G$ and a non-universal congruence $\alpha$ of $S$. Then $(ea,g) \in \alpha$, where $e$ is the identity element of $G$. As $ea = 0$, we get $(0,g) \in \alpha$ and so $(0,h) \in \alpha$ for every $h \in G$ and so $\alpha$ is the universal congruence of $S$ by the above. It is a contradiction. Thus $[g]_\alpha \subseteq G$ for every $g \in G$. □

**Remark 4** By Lemma 4.20 of [12], if $X$ is a right $G$-set such that the group $G$ acts on the non-empty set $X$ transitively then the congruence lattice $\Con(X)$ of the $G$-set $X$ is isomorphic to the interval $[\Stab_G(x), G]$ for every $x \in X$, where $\Stab_G(x) = \{g \in G : xg = x\}$. The corresponding isomorphisms are

$$\phi : \alpha \mapsto H_\alpha = \{g \in G : xg \alpha x\} (\alpha \in \Con(X))$$

and

$$\psi : H \mapsto \alpha_H = \{(xg, xh) \in X \times X : Hg = Hh\} (H \in [\Stab_G(x), G])$$

(which are inverses of each other).

As two right congruences of a group $G$ determined by subgroups $H$ and $K$ of $G$ commute with each other if and only if $HK = KH$ it is easy to see (by Remark 4) that the following lemma is true.
Lemma 17 Let $X$ be a right $G$-set such that $G$ acts on $X$ transitively. Let $x \in X$ be an arbitrary fixed element. Then $\alpha \circ \beta = \beta \circ \alpha$ is satisfied for some congruences $\alpha, \beta \in \text{Con}(X)$ if and only if $H_\alpha H_\beta = H_\beta H_\alpha$ is satisfied for $H_\alpha, H_\beta \in [\text{Stab}_G(x), G]$. □

Remark 5 As the congruence lattice $\text{Con}(X)$ of a $G$-set $X$ is isomorphic to the interval $[\text{Stab}_G(x), G]$ for all $x \in X$. Lemma 17 implies that if the subgroups of $G$ belonging to $[\text{Stab}_G(x), G]$ commute with each other for some $x \in X$ then the subgroups of $G$ belonging to $[\text{Stab}_G(y), G]$ commute with each other for all $y \in X$.

Construction 1 Let $G$ be a group and $G_a$ be a subgroup of $G$ such that $HK = KH$ is satisfied for all subgroups $H, K$ of $G$ containing $G_a$. Let $N^*$ denote the right quotient set $G/G_a$, that is, the set of all right cosets $G_a g$ ($g \in G$) of $G$ defined by $G_a$. Let $S = G \cup N^* \cup \{0\}$, where $0$ is a symbol not contained in $G \cup N^*$. On $S$ we define an operation as follows. If $g, h \in G$ then let $gh$ be the original product of $g$ and $h$ in $G$. If $a \in N$ then, for arbitrary $s \in S$, let $sa = 0$. For arbitrary $g \in G$ and arbitrary $G_a h \in N^*$, let $(G_a h)g = G_a (hg)$. It is easy to check that $S$ is a semigroup.

Theorem 2 A finite semigroup $S$ is a permutable semigroup which is a semilattice of a group $G$ and a nil semigroup such that the identity element of $G$ is a right [left] identity element of $S$ and $SN = \{0\}$ [NS = \{0\}] if and only if it is isomorphic to a semigroup defined in Construction 1 [the dual of Construction 1].

Proof. First of all we show that the semigroup $S$ defined in Construction 1, is a permutable semigroup. It is clear that $S$ is a semilattice of the group $G$ and the null semigroup $N = N^* \cup \{0\}$ such that $SN = \{0\}$ and the identity element $e$ of $G$ is a right identity element of $S$. Moreover, $(G_a g)G = N^*$ for all $G_a g \in N^*$. Thus $N^*$ is a right $G$-set and $G$ acts on $N^*$ transitively. By Lemma 4.20 of [12], the congruence lattice $\text{Con}(N^*)$ of the $G$-set $N^*$ is isomorphic to the interval $[\text{Stab}_G(G_a), G]$, where $\text{Stab}_G(G_a) = \{g \in G; G_a g G_a = G_a\} = G_a$. Let $\alpha$ be a non-universal congruence of $S$. Then, by Lemma 16, $[g]_\alpha \subseteq G$ for every $g \in G$ and $[0]_\alpha$ is either $\{0\}$ or $N$. As $N^*$ is a right $G$-set and $G$ acts on $N^*$ transitively, moreover the restriction $\alpha^*$ of $\alpha$ to $N^*$ is in $\text{Con}(N^*)$, there is a subgroup $H_{\alpha^*} \in [G_a, G]$ which determines $\alpha$ on $N^*$.

Let $\alpha$ and $\beta$ be arbitrary congruences of $S$. We show that $\alpha \circ \beta = \beta \circ \alpha$. We can suppose that $\alpha$ and $\beta$ are not the universal relations of $S$. Assume $(b, c) \in \alpha \circ \beta$ for arbitrary elements $b$ and $c$ of $S$. Then there is an element $x \in S$ such that $(b, x) \in \alpha$, $(x, c) \in \beta$. We have two cases.

Case 1: $x \in G$. Then, by Lemma 16, $b, c \in G$. As every group is permutable, there is an element $y \in G$ such that $(b, y) \in \beta$ and $(y, c) \in \alpha$. Hence $(b, c) \in \beta \circ \alpha$.

Case 2: $x \in N$. Then, by Lemma 16, $b, c \in N$. If $[0]_\alpha = N$ or $[0]_\beta = N$ then $(b, c) \in \alpha$ or $(b, c) \in \beta$ and so $(b, c) \in \beta \circ \alpha$. Consider the case when $[0]_\alpha = [0]_\beta = \{0\}$. Then $N^*$ is saturated by both $\alpha$ and $\beta$. If $x = 0$ then
Let the finite semigroup $b = c = 0$ and so $(b, c) \in \beta \circ \alpha$. If $x \in N^*$ then $b, c \in N^*$. If $\alpha^*$ and $\beta^*$ denote the restriction of $\alpha$ and $\beta$ to $N^*$, respectively, then $H_{\alpha^*}, H_{\beta^*} \supseteq G_a$. As $H_{\alpha^*} H_{\beta^*} = H_{\beta^*} H_{\alpha^*}$, we get $\alpha^* \circ \beta^* = \beta^* \circ \alpha^*$ by Lemma 17. Hence $(b, c) \in \beta \circ \alpha$.

Thus we have $(b, c) \in \beta \circ \alpha$ in both cases. Consequently, $\alpha \circ \beta \subseteq \beta \circ \alpha$. By the symmetry, we get $\alpha \circ \beta = \beta \circ \alpha$. Thus $S$ is a permutable semigroup.

Conversely, assume that $S$ is a permutable semigroup which is a semilattice of a group $G$ and a non-trivial nil semigroup $N$ such that the identity element of $G$ is a right identity element of $S$ and $SN = \{0\}$. Then $N$ is a null semigroup and $aG = N^*$ for every $a \in N^*$ by Lemma 15. Thus $N^*$ is a right $G$-set and $G$ acts on $N^*$ transitively. Fix an element $a$ in $N^*$ and consider $G_a = \text{Stab}_G(a) = \{g \in G : ag = a\}$. It is easy to check that $ag = ah$ for some $g, h \in G$ if and only if $G_a g = G_a h$. Thus $|N^*| = |G : G_a|$. Let $\phi$ be the bijection of $N^*$ to the factor set $G/G_a$ defined by $\phi(b) = G_a b$ if $b = ag$. It is clear that $\phi$ is well defined. Moreover, for all $g, h \in G$, $(G_a g) h = G_a (gh)$ implies $(\phi(b)) h = \phi(bh)$. If we identify every $b \in N^*$ with $\phi(b)$ then $N^*$ can be considered as the set of all right cosets of $G$ defined by $G_a$, and the operation on $S$ is defined as in the Construction 1. Let $H$ and $K$ be arbitrary subgroups of $G$ containing the subgroup $G_a$. Let $\alpha^*_H = \alpha_H \cup 1_S$ and $\alpha^*_K = \alpha_K \cup 1_S$, where $\alpha_H = \psi(H)$ and $\alpha_K = \psi(K)$ are congruences of the right $G$-set $N^*$ defined by $H$ and $K$, respectively (for $\psi$, we refer to Remark 4). It is easy to see that $\alpha^*_H$ and $\alpha^*_K$ are congruences of $S$. As $S$ is permutable, they commute with each other from which we get $\alpha_H \circ \alpha_K = \alpha_K \circ \alpha_H$. Hence $HK = KH$ by Lemma 17. Thus the theorem is proved. 

\[\square\]

3.2 When the identity element of $G$ is the two-sided identity element of $S$

**Lemma 18** If $S$ is a permutable semigroup which is a semilattice of a group $G$ and a non-trivial nilpotent semigroup $N$ of nilpotency degree $t$ such that the identity element of $G$ is an identity element of $S$ then, for all $a \in N^k - N^{k+1}$, $GaG = N^k - N^{k+1}$ is satisfied for every $k = 1, \ldots, t - 1$.

**Proof.** Let $a \in N^k - N^{k+1}$ be arbitrary. It is clear that $GaG \subseteq N^k - N^{k+1}$. As the ideals of $S$ form a chain, $N^{k+1} \subseteq SaS$. It is clear that $SaS \subseteq N^k$. Assume $SaS \neq N^k$ for every $a \in N^k - N^{k+1}$. As $S$ is finite, there is an element $b \in N^k - N^{k+1}$ such that $SaS \subseteq SbS \neq N^k$. Let $c \in N^k - SbS$ be arbitrary. Then $SbS \subset ScS$ which is a contradiction. Consequently, $SaS = N^k$ for some $a \in N^k - N^{k+1}$. Thus $N^k = SaS = GaG \cup GaN \cup NaG \cup NaN \subseteq GaG \cup N^{k+1}$ which implies $GaG = N^k - N^{k+1}$. \[\square\]

**Lemma 19** Let the finite semigroup $S$ be a semilattice of a group $G$ and a non-trivial nilpotent semigroup $N$ of nilpotency degree $t$ such that, for every $a \in N^i - N^{i+1}$ ($i = 1, \ldots, t - 1$), $GaG = N^i - N^{i+1}$ is satisfied. Then the ideals of $S$ are $S, N, N^2, \ldots, N^t = \{0\}$.

**Proof.** It is clear that $S, N, N^2, \ldots, N^t = \{0\}$ are ideals of $S$. Let $I$ be an arbitrary ideal of $S$. Let $j$ be the least positive integer such that $I \cap N^j \neq \emptyset$. 8
If \( a \in I \cap (N^j - N^{j+1}) \) then \( N^j - N^{j+1} = GaG \subseteq I \). Let \( b \in N^{j+1} - N^{j+2} \) supposing that \( N^{j+1} \neq \{0\} \). There are elements \( x_1, \ldots, x_{j+1} \in N - N^2 \) such that \( b = x_1 \ldots x_{j+1} \). It is clear that \( x_1 \ldots x_j \in N^j - N^{j+1} \subseteq I \) and so \( b \in I \) which implies that \( N^{j+1} - N^{j+2} \subseteq I \). Continuing this procedure, we get that \( N^j = I \cap N \). If \( I \cap G = \emptyset \) then \( I = N^j \). Assume that \( I \cap G \neq \emptyset \). Then \( G \subseteq I \). Moreover, for all \( i = 1, \ldots, t-1 \), and every \( a \in N^i - N^{i+1} \), \( N^i - N^{i+1} = GaG \subseteq I \) which implies that \( I = S \). Thus the lemma is proved. 

\[ \square \]

**Lemma 20** Let \( S \) be a semigroup which is a semilattice of a group \( G \) and a nilpotent semigroup \( N \) of nilpotency degree \( t \) such that, for every \( i \in \{1, \ldots, t-1\} \) and for some (and so for every) \( a \in N^i - N^{i+1} \), \( GaG = N^j - N^{j+1} \) is satisfied. Then, for every non-universal congruence \( \alpha \) of \( S \), \( [0]_\alpha = N^j \) for some positive integer \( j = 1, \ldots, t \) and \( [g]_\alpha \subseteq G \) for every \( g \in G \), moreover \( [a]_\alpha \subseteq N^i - N^{i+1} \) for every \( a \in N^i - N^{i+1} \) \( (i = 1, \ldots, j - 1) \).

**Proof.** Let \( \alpha \) be a non-universal congruence of \( S \). If \( (g, a) \in \alpha \) for some \( g \in G \) and \( a \in N \) then \( (g', a') \in \alpha \). As \( a' = 0 \), and \( (ugv, 0) \in \alpha \) for all \( u, v \in G \), we get \( G \subseteq [0]_\alpha \). Let \( a \in N \) be an arbitrary element. Then \( (gah, 0) \in \alpha \) for all \( g, h \in G \) and so \( N^i - N^{i+1} \subseteq [0]_\alpha \) for everi \( i = 1, \ldots, t - 1 \). Thus \( S = [0]_\alpha \) which is a contradiction. Hence \( [g]_\alpha \subseteq G \) for every \( g \in G \). By Lemma 19, the ideals of \( S \) are \( S, N, N^2, \ldots, N^t = \{0\} \). Then there is a least positive integer \( j \in \{1, 2, \ldots, t\} \) such that \( [0]_\alpha = N^j \). If \( j = 1 \) or \( j = 2 \) then the assertion is true for \( \alpha \). Assume \( j \geq 3 \). Let \( a \in N^{j-1} - N^j \) be arbitrary. It is clear that \( (a, b) \notin \alpha \) for every \( b \in N^j \). Assume \( (a, b) \in \alpha \) for some \( b \in N^{k-1} - N^k \) for some \( k < j \). There are elements \( x_1, \ldots, x_{j-1} \in N - N^2 \) such that \( a = x_1 \ldots x_{j-1} \). It is clear that \( x_1 \ldots x_{j-2} \in N^{j-2} \), \( x_1 \ldots x_{j-3} \in N^{j-3} - N^{j-2} \), and finally, \( c = x_1 \ldots x_k - 1 \in N^{k-1} - N_k = GbG \). Then \( c = gbb \) for some \( g, h \in G \). Thus \( (c, gah) \in \alpha \). As \( gah \in N^{j-1} - N^j \), \( d = gahx \ldots x_{j-1} \in N^j \) and so \( a = cx_{k+1} \ldots x_{j-1} \) implies \( (a, d) \in \alpha \) which is impossible. Hence \( [a]_\alpha \subseteq N^{j-1} - N^j \). Thus the lemma is proved. 

\[ \square \]

For an arbitrary group \( G \), let \( G^* \) denote the dual of \( G \), that is, \( xy = u \) in \( G^* \) if and only if \( yx = u \) in \( G \).

**Theorem 3** Let \( S \) be a finite semigroup which is a semilattice of a group \( G \) and a non-trivial nilpotent semigroup \( N \) of nilpotency degree \( t \) such that the identity element of \( G \) is the identity element of \( S \). Then \( S \) is permutable if and only if, for all \( i = 1, \ldots, t - 1 \), there is an element \( a_i \) in \( N^i - N^{i+1} \) such that \( Ga_iG = N^i - N^{i+1} \), and \( HK = KH \) is satisfied for all subgroups \( H, K \supseteq G_{a_i} = \{ (g, h) \in G^* \times G : ghah = a_i \} \).

**Proof.** Let \( S \) be a finite semigroup which is a semilattice of a group \( G \) and a nilpotent semigroup \( N \) of nilpotency degree \( t \) such that the identity element of \( G \) is the identity element of \( S \). First assume that \( S \) is permutable. Let \( i \in \{1, \ldots, t - 1\} \) be arbitrary. Then, for every \( a_i \in N^i - N^{i+1} \), \( Ga_iG = N^i - N^{i+1} \) is satisfied by Lemma 18. It is a matter of checking to see that this result implies that \( N^i - N^{i+1} \) is a right \( (G^* \times G) \)-set \( (a(g, h)) = gah \) for
every $a \in N^i$ and every $(g,h) \in G^* \times G$ and $G^* \times G$ acts on $N^i \setminus N^{i+1}$ transitively. Let $G_{a_i} = \text{Stab}_{G^* \times G}(a_i) = \{(g,h) \in G^* \times G : ga_ih = a_i\}$. By Lemma 4.20 of [12] the congruence lattice $\text{Con}(N^i \setminus N^{i+1})$ of the right $(G^* \times G)$-set $N^i \setminus N^{i+1}$ is isomorphic to $[\text{Stab}_{G^* \times G}(a_i), G^* \times G]$. The corresponding isomorphisms $\phi : \alpha_i \mapsto H_{a_i}$ ($\alpha_i \in \text{Con}(N^i \setminus N^{i+1})$ and $\psi : H \mapsto \alpha_H^i$ ($H \in [\text{Stab}_{G^* \times G}(a_i), G^* \times G]$) defined as in Remark 4. Let $H$ be an arbitrary subgroup of $G^* \times G$ containing the subgroup $G_{a_i}$. Let $\alpha_H^i$ be the relation of $S$ defined by $(a,b) \in \alpha_H^i$ if and only if $a = b$ or $a,b \in N^i \setminus N^{i+1}$ or $a,b \in N^i \setminus N^{i+1}$ and $(a,b) \in \alpha_H^i$. It is clear that $\alpha_H^i$ is an equivalence relation. We show that it is a congruence of $S$. Assume $(a,b) \in \alpha_H^i$ for some $a,b \in S$. We can suppose that $a \neq b$. If $a,b \in N^i \setminus N^{i+1}$ then $sa, sb, as, bs \in N^i \setminus N^{i+1}$ and so $(sa, sb) \in \alpha_H^i$ and $(as, bs) \in \alpha_H^i$. Consider the case when $a,b \in N^i \setminus N^{i+1}$.

Then $(a,b) \in \alpha_H^i$ and so, for every $x \in G$, we have $(a(x,e), b(x,e)) \in \alpha_H^i$ and $(a(x,e), b(x,e)) \in \alpha_H^i$ because $\alpha_H^i$ is a congruence of the right $(G^* \times G)$-set $N^i \setminus N^{i+1}$. Thus $(a(x,e), b(x,e)) \in \alpha_H^i$ and $(a(x,e), b(x,e)) \in \alpha_H^i$. Hence $(a(x,e), b(x,e)) \in \alpha_H^i$ and $(a(x,e), b(x,e)) \in \alpha_H^i$. If $u \in N$ then $ua, ub, au, bu \in N^i \setminus N^{i+1}$ and so $(au, bu) \in \alpha_H^i$ and $(ua, ub) \in \alpha_H^i$. Consequently, $\alpha_H^i$ is a congruence on $S$. Let $H$ and $K$ be arbitrary subgroups of $G^* \times G$ containing the subgroup $G_{a_i}$. Let $\alpha_H^i$ and $\alpha_K^i$ be the congruences of $S$ defined by $H$ and $K$ (see above). As $S$ is permutable, $\alpha_H^i \circ \alpha_K^i = \alpha_K^i \circ \alpha_H^i$ from which we get $\alpha_H^i \circ \alpha_K^i = \alpha_K^i \circ \alpha_H^i$. Then $HK = KH$ by Lemma 17. Thus the necessity of the permutability of $S$ is proved.

Conversely, assume that, for all $i = 1, \ldots, t - 1$, there is an element $\alpha_i$ in $N^i \setminus N^{i+1}$ such that $G_{a_i} G = N^i \setminus N^{i+1}$ and $HK = KH$ is satisfied for all subgroups $H,K \supseteq G_{a_i}$. Let $\alpha$ be a non-universal congruence on $S$. Then, by Lemma 20, $[0] = N^i$ for some positive integer $j \in \{1, \ldots, t\}$. Let $\{g \in G : \{a_i, a_j\} \subseteq N^i \setminus N^{i+1} \text{ for every } i \in N^t \setminus N^{t+1}\}$. Let $\alpha_i$ denote the restriction of $\alpha$ to $N^i \setminus N^{i+1}$, and let $H_{a_i}^i = \phi(\alpha_i) = \{(g,h) \in G^* \times G : \{a_i, a_j\} \subseteq N^i \setminus N^{i+1}, \text{ for every } i \in N^t \setminus N^{t+1}\}$. $H_{a_i}^i$ is a subgroup of $G^* \times G$ and $G_{a_i} \subseteq H_{a_i}^i$. Let $\beta$ be an arbitrary non-universal congruence on $S$. As $G_{a_i} \subseteq H_{a_i}^i$, we have $H_{a_i}^i H_{a_j}^i = H_{a_i}^i H_{a_j}^i$. As $\alpha_i$ and $\beta_i$ are in the congruence lattice $\text{Con}(N^i \setminus N^{i+1})$ of the right $(G^* \times G)$-set $N^i \setminus N^{i+1}$, we have $\alpha_i \circ \beta_i = \beta_i \circ \alpha_i$. We show that $\alpha \circ \beta = \beta \circ \alpha$. Assume $(a,b) \in \alpha \circ \beta$ for some $a,b \in S$. Then there is an element $c \in S$ such that $(a,c) \in \alpha$ and $(c,b) \in \beta$. If $c \in G$ then $a,b \in G$ by Lemma 20. As every group is permutative, we get $(a,b) \in \beta \circ \alpha$. By Lemma 2, $[0] \subseteq [0] \beta$ or $[0] \beta \subseteq [0]$. Assume $[0] \subseteq [0] \beta$. If $c \in [0] \beta$ then $a,b \in [0] \beta$ and so $(a,b) \in \beta$. If $c \in [0] \beta$ implies $(a,b) \in \beta \circ \alpha$. Assume $c \not\in [0] \beta$. If $c \in N^i \setminus N^{i+1}$, then, by Lemma 20, $a,b \in N^i \setminus N^{i+1}$. Thus $(a,b) \subseteq \alpha_i \circ \beta_i = \beta_i \circ \alpha_i$ (see also Lemma 17). Thus $(a,b) \in \beta \circ \alpha$. The proof of
\((a,b) \in \beta \circ \alpha\) is similar in that case when \([0]_\beta \subseteq [0]_\alpha\). Thus \(\alpha \circ \beta \subseteq \beta \circ \alpha\). The proof of \(\beta \circ \alpha \subseteq \alpha \circ \beta\) is similar. Thus \(\alpha \circ \beta = \beta \circ \alpha\). Hence \(S\) is permutable. \(\Box\)

By Remark 5, if a semigroup satisfies the conditions of Theorem 3 then \(Ga_iG = N^i - N^{i+1}\) and \(KH = KH\) is satisfied for all subgroups \(H, K \supseteq G_{a_i} = \{(g,h) \in (G^* \times G) : ga_ih = a_i\}\), for all elements \(a_i\) in \(N^i - N^{i+1}\).

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References


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