Complex numbers - Exercises

1. Find the algebraic form of the complex number $\frac{\overline{z}_1}{\overline{z}_2}$, if $z_1 = 3 - 2i$ and $z_2 = 2 + i$.
2. Find the algebraic form of the following complex numbers:
   a) $3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ b) $\frac{2+i}{i(1-4i)}$
3. Find the trigonometric form of the following complex numbers:
   a) $\sqrt{6} - i\sqrt{2}$ b) $-4i$ c) 8
4. Find the trigonometric and algebraic form of the following complex numbers:
   a) $\sqrt[3]{1}$ b) $\sqrt[4]{-16}$ c) $\sqrt[3]{1 + i\sqrt{3}}$
5. Calculate the following powers:
   a) $\left(1 + i\sqrt{3}\right)^3$ b) $(1+i)^8$ c) $(1-i)^4$
6. Solve the following equations on the set of complex numbers:
   a) $z^3 = 1 + i$ b) $|z| - z = 1 + 2i$ c) $z^2 = \overline{z}$
7. Solve the following equations on the set of complex numbers. Give the result in algebraic form.
   a) $z^2 + (1+i)\overline{z} + 4i = 0$ b) $2iz^3 = (1+i)^8$
8. Give the algebraic form of all complex solutions of the following equation whose real part is positive and whose imaginary part is negative.
   $\frac{7i+3}{7-3i}z^4 + 8\left(\sqrt{3} + i\right) = 0$
9. Assume that the imaginary part of the complex number z is not zero, but the imaginary part of the complex number $z + \frac{1}{z}$ is zero. Find $|z|$, the absolute value of z.
10. Give the algebraic form of all complex solutions of the following equation whose real and imaginary parts are both negative.
    $iz^6 = (7+i)^2 + \frac{2-30i}{1-i}$

Homework

11. Solve the following equation on the set of complex numbers:
    $z^2 = z + 3\overline{z}$
12. Find those solutions z of the following equation for which $\text{Re}(z) > 0$ and $\text{Im}(z) < 0$.
    Give these solutions in algebraic form.
    $z^6 + 7z^3 - 8 = 0$
13. Find the algebraic form of $\frac{z^2 - |z^2|}{z - \overline{z}}$ if $z = \sqrt{3} + i$.
14. Give all the solutions of the following equation in algebraic form:
    $iz^3 = \frac{1}{2}(1-i)^8$.

Solutions

1. Find the algebraic form of the complex number $\frac{\overline{z}_1}{\overline{z}_2}$, if $z_1 = 3 - 2i$ and $z_2 = 2 + i$.

Solution. $\frac{\overline{z}_1}{\overline{z}_2} = \frac{\overline{3-2i}}{\overline{2+i}} = \frac{3+2i}{2-i} = \frac{3+2i}{2-i} \cdot \frac{2+i}{2+i} = \frac{6+4i+3i+2i^2}{4-i^2} = \frac{6+4i+3i+2(-1)}{4-(-1)} = \frac{4+7i}{5} = \frac{4}{5} + \frac{7}{5} i$

2. Find the algebraic form of the following complex numbers:

a) $3\left(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}\right)$ b) $\frac{2+i}{i(1-4i)}$

Solution. *a*) $3\left(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}\right) = 3\left(-\frac{1}{2} + i \frac{\sqrt{3}}{2}\right) = -\frac{3}{2} + i \frac{3\sqrt{3}}{2}$

b) $\frac{2+i}{i(1-4i)} = \frac{2+i}{i-4i^2} = \frac{2+i}{i-4(-1)} = \frac{2+i}{4+i} = \frac{2+i}{4+i} \cdot \frac{4-i}{4-i} = \frac{8-i^2+4i-2i}{16-i^2} = \frac{8-(-1)+2i}{16-(-1)} = \frac{9+2i}{17} = \frac{9}{17} + \frac{2}{17} i$

3. Find the trigonometric form of the following complex numbers:

a) $\sqrt{6} - i\sqrt{2}$ b) $-4i$ c) 8

Solution. The trigonometric form of the complex number $z = a + bi$ is

$z = r(\cos\varphi + i \sin\varphi)$, where $r = \sqrt{a^2 + b^2}$.

a) $z = \sqrt{6} - i\sqrt{2} \implies r = |z| = \sqrt{\left(\sqrt{6}\right)^2 + \left(-\sqrt{2}\right)^2} = \sqrt{8} = 2\sqrt{2}$

$\implies z = 2\sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right)$

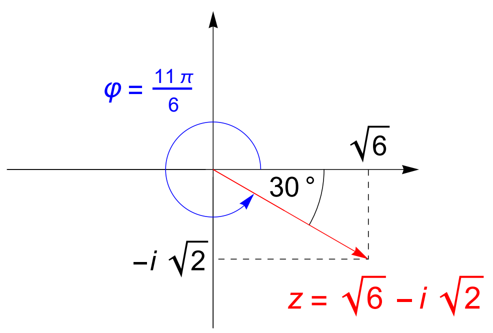
$\implies$ the argument is: $\cos\varphi = \frac{\sqrt{3}}{2}$, $\sin\varphi = -\frac{1}{2} \implies \varphi = \frac{11\pi}{6}$

$\implies z = 2\sqrt{2}\left(\cos\frac{11\pi}{6} + i \sin\frac{11\pi}{6}\right)$

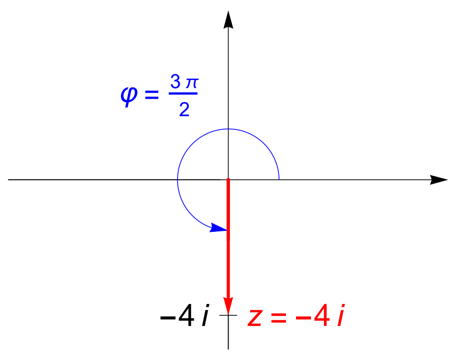
b) $z = -4i = 4(0 + (-1) \cdot i) = 4\left(\cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2}\right)$

c) $z = 8 = 8(1 + 0 \cdot i) = 8(\cos 0 + i \sin 0)$

a)



b)



4. Find the trigonometric and algebraic form of the following complex numbers:

a) $\sqrt[3]{1}$ b) $\sqrt[4]{-16}$ c) $\sqrt[3]{1 + i\sqrt{3}}$

Solution.

a) The values of $\sqrt[3]{1}$ can be obtained from the equation $z^3 = 1$.

The trigonometric form of 1 is: $1 = \cos 0 + i \sin 0$

$\Longrightarrow z_k = \cos \frac{0 + k \cdot 2\pi}{3} + i \sin \frac{0 + k \cdot 2\pi}{3}$, where $k = 0, 1, 2$.

The roots are:

If $k = 0$: $z_0 = \cos 0 + i \sin 0 = 1 + i \cdot 0 = 1$

If $k = 1$: $z_1 = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + i \frac{\sqrt{3}}{2}$

If $k = 2$: $z_2 = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - i \frac{\sqrt{3}}{2}$

b) The values of $\sqrt[4]{-16}$ can be obtained from the equation $z^4 = -16$.

The trigonometric form of -16 is: $-16 = 16(\cos \pi + i \sin \pi)$

$\Longrightarrow z_k = \sqrt[4]{16} \left(\cos \frac{\pi + k \cdot 2\pi}{4} + i \sin \frac{\pi + k \cdot 2\pi}{4}\right)$, where $k = 0, 1, 2, 3$.

The roots are:

If $k = 0$: $z_0 = 2\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}\right) = \sqrt{2} + i \sqrt{2}$

If $k = 1$: $z_1 = 2\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}\right) = -\sqrt{2} + i \sqrt{2}$

If $k = 2$: $z_2 = 2\left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}\right) = -\sqrt{2} - i \sqrt{2}$

If $k = 3$: $z_3 = 2\left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}\right) = \sqrt{2} - i \sqrt{2}$

c) The values of $\sqrt[3]{1 + i \sqrt{3}}$ can be obtained from the equation $z^3 = 1 + i \sqrt{3}$.

The trigonometric form of $1 + i \sqrt{3}$ is: $1 + i \sqrt{3} = 2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

$\Longrightarrow z_k = \sqrt[3]{2} \left(\cos \frac{\frac{\pi}{3} + k \cdot 2\pi}{3} + i \sin \frac{\frac{\pi}{3} + k \cdot 2\pi}{3}\right)$, where $k = 0, 1, 2$.

The roots are:

If $k = 0$: $z_0 = \sqrt[3]{2} \left(\cos \frac{\pi}{9} + i \sin \frac{\pi}{9}\right) \approx 1.18394 + 0.430918\, i$

If $k = 1$: $z_1 = \sqrt[3]{2} \left(\cos \frac{7\pi}{9} + i \sin \frac{7\pi}{9}\right) \approx -0.965156 + 0.809862\, i$

If $k = 2$: $z_2 = \sqrt[3]{2} \left(\cos \frac{13\pi}{9} + i \sin \frac{13\pi}{9}\right) \approx -0.218783 - 1.24078\, i$

5. Calculate the following powers:

a) $\left(1 + i \sqrt{3}\right)^3$ b) $(1 + i)^8$ c) $(1 - i)^4$

Solution.

a) $1 + i \sqrt{3} = 2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

$\Longrightarrow \left(1 + i \sqrt{3}\right)^3 = 2^3 \left(\cos \frac{3\pi}{3} + i \sin \frac{3\pi}{3}\right) = 8(\cos \pi + i \sin \pi) = 8(-1 + i \cdot 0) = -8$

b) $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$\Longrightarrow (1+i)^8=\left(\sqrt{2}\right)^8\left(\cos\frac{8\pi}{4}+i\sin\frac{8\pi}{4}\right)=16(\cos 2\pi+i\sin 2\pi)=16(1+i\cdot 0)=16$

or:

$(1+i)^2=1+2i+i^2=1+2i-1=2i$

$\Longrightarrow (1+i)^8=\left((1+i)^2\right)^4=(2i)^4=16\cdot\left(i^2\right)^2=16\cdot(-1)^2=16$

c) $1-i=\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$

$\Longrightarrow (1-i)^4=\left(\sqrt{2}\right)^4\left(\cos\left(-\frac{4\pi}{4}\right)+i\sin\left(-\frac{4\pi}{4}\right)\right)=4(\cos(-\pi)+i\sin(-\pi))=4(-1+i\cdot 0)=-4$

or:

$(1-i)^2=1-2i+i^2=1-2i-1=-2i$

$\Longrightarrow (1-i)^4=\left((1-i)^2\right)^2=(-2i)^2=4\cdot i^2=4\cdot(-1)=-4$

6. Solve the following equations on the set of complex numbers:
a) $z^3=1+i$ b) $|z|-z=1+2i$ c) $z^2=\bar{z}$

Solution. a) The trigonometric form of $1+i$ is: $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$\Longrightarrow z_k=\sqrt[3]{1+i}=\sqrt[3]{\sqrt{2}}\left(\cos\frac{\frac{\pi}{4}+k\cdot 2\pi}{3}+i\sin\frac{\frac{\pi}{4}+k\cdot 2\pi}{3}\right)$, where $k=0, 1, 2$.

The arguments are: $k=0 \Longrightarrow \arg(z)=\frac{\pi}{12}$

$k=1 \Longrightarrow \arg(z)=\frac{\pi}{12}+\frac{2\pi}{3}=\frac{9\pi}{12}=\frac{3\pi}{4}$

$k=2 \Longrightarrow \arg(z)=\frac{\pi}{12}+\frac{4\pi}{3}=\frac{17\pi}{12}$

The roots are:

If $k=0$: $z_0=\sqrt[6]{2}\left(\cos\left(\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right)\right)\approx 1.08422+0.290515\,i$

If $k=1$: $z_1=\sqrt[6]{2}\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)\approx -0.793701+0.793701\,i$

If $k=2$: $z_2=\sqrt[6]{2}\left(\cos\left(\frac{17\pi}{12}\right)+i\sin\left(\frac{17\pi}{12}\right)\right)\approx -0.290515-1.08422\,i$

6. b) $|z|-z=1+2i$

Solution. b) Let $z=x+y\,i$, where $x, y\in\mathbb{R}$. Then $|z|=\sqrt{x^2+y^2}$, and thus the equation is:

$$|z|-z=1+2i \iff \sqrt{x^2+y^2}-(x+y\,i)=1+2i$$
$$\iff \left(\sqrt{x^2+y^2}-x\right)-y\,i=1+2i$$

Two complex numbers are equal if and only if their real and imaginary parts are respectively equal. Therefore, the equation above is equivalent with the following equation system:

(1) $\sqrt{x^2+y^2} - x = 1$

(2) $-y = 2$

From here $y = -2$, and from the first equation $\sqrt{x^2+4} - x = 1 \implies \sqrt{x^2+4} = x+1$.
Since $\sqrt{x^2+4} \geq 0$, then $x+1 \geq 0$, that is, $x \geq -1$. Taking the squares of both sides we have $x^2+4 = x^2+2x+1 \implies 2x = 3 \implies x = \frac{3}{2}$, which satisfies the previous condition.

The solution of the equation is: $z = \frac{3}{2} - 2i$.

6. c) $z^2 = \overline{z}$

Solution. c) Let $z = x + y\,i$, where $x, y \in \mathbb{R}$. Then

- $z^2 = (x+y\,i)^2 = x^2 + 2\,x\,y\,i + y^2\,i^2 = (x^2 - y^2) + 2\,x\,y\,i$
- $\overline{z} = x - y\,i$

The equation can be written as

$$z^2 = \overline{z} \iff (x^2-y^2) + 2\,x\,y\,i = x - y\,i$$
$$\iff (x^2-y^2) + 2\,x\,y\,i - x + y\,i = 0$$

A complex number is zero if and only if both the real and imaginary parts are zero.
Therefore, the equation above is equivalent with the following equation system:

(1) $(x^2-y^2) - x = 0$

(2) $2\,x\,y + y = 0$

Writing equation (2) as a product: $y(2\,x+1) = 0$
A product is equal to zero if and only at least one of the factors is zero, so we consider the following two cases.
Case 1: if $y = 0$, then substituting this into equation (1) we have:

$$(x^2-y^2) - x = 0 \implies x^2 - x = x(x-1) = 0$$

From here $x_1 = 0$ and $x_2 = 1$, so in this case we obtain two complex solutions:
$z_1 = 0 + 0 \cdot i = 0$ and $z_2 = 1 + 0 \cdot i = 1$.

Case 2: if $2\,x+1 = 0$, then $x = -\frac{1}{2}$. Substituting this into equation (1) we have:

$$(x^2-y^2) - x = 0 \implies \frac{1}{4} - y^2 + \frac{1}{2} = 0 \implies y^2 = \frac{3}{4}$$

From here $y_{1,2} = \pm\frac{\sqrt{3}}{2}$, so in this case we obtain two complex solutions:
$z_3 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $z_4 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

The solution of the equation is: $z_1 = 0, z_2 = 1, z_3 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, z_4 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

7. Solve the following equations on the set of complex numbers. Give the result in algebraic form.
a) $z^2 + (1+i)\,\overline{z} + 4\,i = 0$ b) $2\,i\,z^3 = (1+i)^8$

Solution. a) Let $z = x + y\,i$, where $x, y \in \mathbb{R}$. Then

- $z^2 = (x + y\,i)^2 = x^2 + 2\,x\,y\,i + y^2\,i^2 = \left(x^2 - y^2\right) + 2\,x\,y\,i$
- $(1+i)\,\overline{z} = (1+i)\,(x - y\,i) = x - y\,i^2 + x\,i - y\,i = (x+y) + (x-y)\,i$

The equation can be written as

$$z^2 + (1+i)\,\overline{z} + 4\,i = 0 \iff \left(x^2 - y^2\right) + 2\,x\,y\,i + (x+y) + (x-y)\,i + 4\,i = 0$$

A complex number is zero if and only if both the real and imaginary parts are zero.
Therefore, the equation above is equivalent with the following equation system:

(1) $\left(x^2 - y^2\right) + (x+y) = 0$
(2) $2\,x\,y + (x-y) + 4 = 0$

Writing equation (1) as a product, we have: $(x-y)\,(x+y) + (x+y) = 0$
$(x+y)\,(x-y+1) = 0$

A product is equal to zero if and only at least one of the factors is zero, so we consider the following two cases.

Case 1: if $x + y = 0$, then $y = -x$. Substituting this into equation (2) we have:

$$\begin{aligned} 2\,x\,y + (x-y) + 4 = 0 \implies\ & 2\,x\,(-x) + (x+x) + 4 = 0 \\ & -2\,x^2 + 2\,x + 4 = 0 \\ & x^2 - x - 2 = 0 \\ & (x-2)\,(x+1) = 0 \end{aligned}$$

The roots of this quadratic equation are $x_1 = 2$ and $x_2 = -1$, from where $y_1 = -2$ and $y_2 = 1$, so in this case we obtain two complex solutions: $z_1 = 2 - 2\,i$ and $z_2 = -1 + i$.

Case 2: if $x - y + 1 = 0$, then $y = x + 1$. Substituting this into equation (2) we have:

$$\begin{aligned} 2\,x\,y + (x-y) + 4 = 0 \implies\ & 2\,x\,(x+1) + (x - (x+1)) + 4 = 0 \\ & 2\,x^2 + 2\,x - 1 + 4 = 0 \\ & 2\,x^2 + 2\,x + 3 = 0 \end{aligned}$$

The discriminant of this quadratic equation is negative: $D = 2^2 - 4 \cdot 2 \cdot 3 = -20 < 0$, so the equation doesn't have a real root. Since x is a real number, then in this case there is no solution.

The solution of the equation is: $z_1 = 2 - 2\,i$ és $z_2 = -1 + i$.

7. b) $2\,i\,z^3 = (1+i)^8$

Solution. b) Let us express the value of z^3:

- $(1+i)^2 = 1 + 2\,i + i^2 = 1 + 2\,i - 1 = 2\,i \implies$
- $(1+i)^8 = \left((1+i)^2\right)^4 = (2\,i)^4 = 2^4\,i^4 = 16\,\left(i^2\right)^2 = 16\,(-1)^2 = 16 \implies$

- $z^3 = \frac{(1+i)^8}{2i} = \frac{16}{2i} = \frac{8}{i} \cdot \frac{i}{i} = \frac{8i}{i^2} = \frac{8i}{-1} = -8i$

The trigonometric form of $-8i$ is: $-8i = 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

$\Longrightarrow z_k = \sqrt[3]{-8i} = 2\left(\cos\frac{\frac{3\pi}{2}+k\cdot 2\pi}{3} + i\sin\frac{\frac{3\pi}{2}+k\cdot 2\pi}{3}\right)$, where $k = 0, 1, 2$.

The arguments are: $k = 0 \Longrightarrow \arg(z) = \frac{\pi}{2}$

$k = 1 \Longrightarrow \arg(z) = \frac{\pi}{2} + \frac{2\pi}{3} = \frac{7\pi}{6}$

$k = 2 \Longrightarrow \arg(z) = \frac{\pi}{2} + \frac{4\pi}{3} = \frac{11\pi}{6}$

The roots are:

If $k = 0$: $z_0 = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 2\,(0 + i\cdot 1) = 2i$

If $k = 1$: $z_1 = 2\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right) = 2\left(-\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right) = -\sqrt{3} - i$

If $k = 1$: $z_2 = 2\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right) = \sqrt{3} - i$

8. Give the algebraic form of all complex solutions of the following equation whose real part is positive and whose imaginary part is negative.

$\frac{7i+3}{7-3i} z^4 + 8\left(\sqrt{3} + i\right) = 0$

Solution. $\frac{7i+3}{7-3i} = \frac{7i+3}{7-3i} \cdot \frac{7+3i}{7+3i} = \frac{58i}{58} = i$

$\Longrightarrow i z^4 + 8\left(\sqrt{3} + i\right) = 0$

$\Longrightarrow z^4 = \frac{-8\left(\sqrt{3}+i\right)}{i} \cdot \frac{i}{i} = \frac{-8\left(i\sqrt{3} - 1\right)}{-1} = 8\left(-1 + \sqrt{3}\, i\right) = 8\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = 16\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$

$\Longrightarrow z_k = 2\left(\cos\left(\frac{\frac{2\pi}{3}+k\cdot 2\pi}{4}\right) + i\sin\left(\frac{\frac{2\pi}{3}+k\cdot 2\pi}{4}\right)\right)$, where $k = 0, 1, 2, 3$.

The arguments are: $\frac{\frac{2\pi}{3}+k\cdot 2\pi}{4} = \frac{\pi}{6} + k\cdot\frac{\pi}{2}$, where k is an integer.

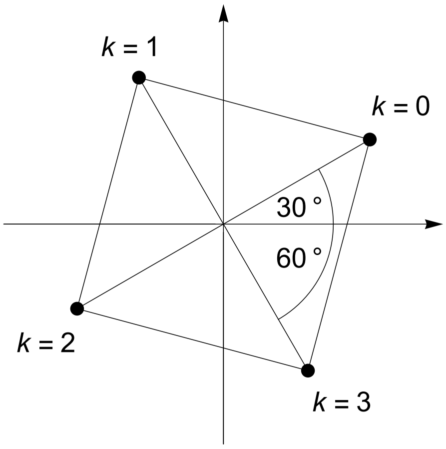
We have to find those roots whose real part is positive and whose imaginary part is negative, that is, the argument is in the 4th quadrant. The value of k can be determined algebraically:

$\frac{3\pi}{2} < \frac{\pi}{6} + k\cdot\frac{\pi}{2} < 2\pi \Longleftrightarrow \frac{3}{2} < \frac{1}{6} + k\cdot\frac{1}{2} < 2 \Longleftrightarrow \frac{4}{3} < \frac{k}{2} < \frac{11}{6} \Longleftrightarrow \frac{8}{3} \approx 2.67 < k < \frac{11}{3} \approx 3.67$

Since k is an integer, then from here $k = 3$, and the argument is $\frac{\pi}{6} + 3\cdot\frac{\pi}{2} = \frac{5\pi}{3}$.

The solution is: $z_3 = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = 1 - \sqrt{3}\, i$.

Remark: the value of k can also be determined geometrically. The roots z_k are the vertices of a square in which the argument of one vertex is $\frac{\pi}{6}$ (if $k = 0$). The following figure shows that for the vertex in the 4th quadrant we have $k = 3$.



9. Assume that the imaginary part of the complex number z is not zero, but the imaginary part of the complex number $z + \frac{1}{z}$ is zero. Find $|z|$, the absolute value of z.

Solution. Let $z = x + y\,i$, where $x, y \in \mathbb{R}$, $y \neq 0$.

$$z + \frac{1}{z} = (x + y\,i) + \frac{1}{x + y\,i} = (x + y\,i) + \frac{1}{x + y\,i} \cdot \frac{x - y\,i}{x - y\,i} =$$

$$= (x + y\,i) + \frac{x - y\,i}{x^2 + y^2} = \left(x + \frac{x}{x^2 + y^2}\right) + i\left(y - \frac{y}{x^2 + y^2}\right)$$

From here $\text{Re}\left(z + \frac{1}{z}\right) = x + \frac{x}{x^2 + y^2}$, $\text{Im}\left(z + \frac{1}{z}\right) = y - \frac{y}{x^2 + y^2}$.

From the conditions $\text{Im}\left(z + \frac{1}{z}\right) = y - \frac{y}{x^2 + y^2} = y\left(1 - \frac{1}{x^2 + y^2}\right) = 0$. Since $y \neq 0$, then from here we have $1 - \frac{1}{x^2 + y^2} = 0$, and thus the absolute value of z is: $|z| = x^2 + y^2 = 1$.

10. Give the algebraic form of all complex solutions of the following equation whose real and imaginary parts are both negative.

$$i\,z^6 = (7 + i)^2 + \frac{2 - 30\,i}{1 - i}$$

Solution:

Let us express z^6:

- $\frac{2 - 30\,i}{1 - i} = \frac{2 - 30\,i}{1 - i} \cdot \frac{1 + i}{1 + i} = \frac{2 - 30\,i^2 + 2\,i - 30\,i}{1 - i^2} = \frac{32 - 28\,i}{2} = 16 - 14\,i$
- $(7 + i)^2 = 49 + 14\,i + i^2 = 48 + 14\,i$

$\Longrightarrow i\,z^6 = 48 + 14\,i + 16 - 14\,i = 64$

$\Longrightarrow z^6 = \frac{64}{i} = \frac{64\,i}{i^2} = -64\,i$

The trigonometric form of $-64\,i$ is: $-64\,i = 64\left(\cos\frac{3\,\pi}{2} + i\sin\frac{3\,\pi}{2}\right)$

$\Longrightarrow z_k = \sqrt[6]{-64\,i} = 2\left(\cos\frac{\frac{3\,\pi}{2}+k\cdot 2\,\pi}{6} + i\sin\frac{\frac{3\,\pi}{2}+k\cdot 2\,\pi}{6}\right)$, where $k = 0, 1, 2, 3, 4, 5$.

The arguments are: $\frac{\frac{3\,\pi}{2}+k\cdot 2\,\pi}{6} = \frac{\pi}{4} + k\cdot\frac{\pi}{3}$, where k is an integer.

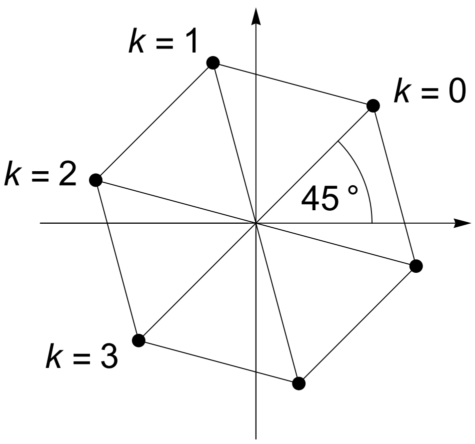
We have to find those roots whose real and imaginary parts are both negative, that is, the argument is in the 3rd quadrant. The value of k can be determined algebraically:

$\pi < \frac{\pi}{4} + k\cdot\frac{\pi}{3} < \frac{3\,\pi}{2} \Longleftrightarrow 1 < \frac{1}{4} + k\cdot\frac{1}{3} < \frac{3}{2} \Longleftrightarrow \frac{3}{4} < \frac{k}{3} < \frac{5}{4} \Longleftrightarrow \frac{9}{4} = 2.25 < k < \frac{15}{4} = 3.75$

Since k is an integer, then from here $k = 3$, and the argument is $\frac{\pi}{4} + 3\cdot\frac{\pi}{3} = \frac{5\,\pi}{4}$.

The solution is: $z_3 = 2\left(\cos\frac{5\,\pi}{4} + i\sin\frac{5\,\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2} - i\cdot\frac{\sqrt{2}}{2}\right) = -\sqrt{2} - i\,\sqrt{2}$.

Remark: the value of k can also be determined geometrically. The roots z_k are the vertices of a regular hexagon in which the argument of one vertex is $\frac{\pi}{4}$ (if $k = 0$). The following figure shows that the vertex opposite to it is in the 3rd quadrant, from where $k = 3$.



11. Solve the following equation on the set of complex numbers:

$z^2 = z + 3\,\bar{z}$

Solution. Let $z = x + y\,i$ $(x, y \in \mathbb{R})$. Then $z^2 = (x^2 - y^2) + 2\,x\,y\,i$ and $\bar{z} = x - y\,i$.
We obtain the following equation system:

(1) $x^2 - y^2 = 4\,x$

(2) $2\,x\,y = -2\,y$

From the second equation $2\,y(x+1) = 0 \Longrightarrow y = 0$ or $x = -1$

If $y = 0$ then from the first equation $x = 0$ or $x = 4$
If $x = -1$ then from the first equation $y = \pm\sqrt{5}$
The solutions are $z_1 = 0$, $z_2 = 4$, $z_3 = -1 + i\sqrt{5}$, $z_4 = -1 - i\sqrt{5}$

12. Find those solutions z of the following equation for which $\text{Re}(z) > 0$ and $\text{Im}(z) < 0$.
Give these solutions in algebraic form.
$z^6 + 7z^3 - 8 = 0$

Solution. $z^6 + 7z^3 - 8 = (z^3 + 8)(z^3 - 1) = 0 \iff z^3 = -8$ or $z^3 = 1$.

a) If $z^3 = -8 = 8(\cos\pi + i\sin\pi)$ then $z_k = 2\left(\cos\frac{\pi + k\cdot 2\pi}{3} + i\sin\frac{\pi + k\cdot 2\pi}{3}\right)$, where $k = 0, 1, 2$.

$z_0 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + \sqrt{3}\,i$

$z_1 = 2(\cos\pi + i\sin\pi) = -2$

$z_3 = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 1 - \sqrt{3}\,i$

From here the condition $\text{Re}(z) > 0$, $\text{Im}(z) < 0$ holds for $1 - \sqrt{3}\,i$.

b) If $z^3 = 1 = (\cos 0 + i\sin 0)$ then $z_k = \cos\frac{k\cdot 2\pi}{3} + i\sin\frac{k\cdot 2\pi}{3}$, where $k = 0, 1, 2$.

$z_0 = \cos 0 + i\sin 0 = 1$

$z_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$

$z_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}\,i$

From here no solutions are suitable.

13. Find the algebraic form of $\frac{z^2 - |z^2|}{z - \bar{z}}$ if $z = \sqrt{3} + i$.

Solution. $z = \sqrt{3} + i \implies$ • $z^2 = 2 + 2\sqrt{3}\,i$

• $|z^2| = \sqrt{2^2 + (2\sqrt{3})^2} = 4$

• $z - \bar{z} = (\sqrt{3} + i) - (\sqrt{3} - i) = 2i$

Then $\frac{z^2 - |z^2|}{z - \bar{z}} = \frac{2 + 2\sqrt{3}\,i - 4}{2i} = \frac{-2 + 2\sqrt{3}\,i}{2i}\cdot\frac{-i}{-i} = \frac{2\sqrt{3} + 2i}{2} = \sqrt{3} + i$.

14. Give all the solutions of the following equation in algebraic form:
$i z^3 = \frac{1}{2}(1 - i)^8$.

Solution. First we simplify the right-hand side:
$1 - i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \implies$
$(1 - i)^8 = \left(\sqrt{2}\right)^8\left(\cos\frac{8\pi}{4} + i\sin\frac{8\pi}{4}\right) = 16(\cos 2\pi + i\sin 2\pi) = 16(1 + i\cdot 0) = 16$

or in another way:
$(1 - i)^2 = 1 - 2i + i^2 = 1 - 2i - 1 = -2i \implies$
$(1 - i)^8 = ((1 - i)^2)^4 = (-2i)^4 = 16i^4 = 16(i^2)^2 = 16(-1)^2 = 16$

$z^3 = \frac{1}{2i} \cdot 16 = 8 \cdot \frac{1}{i} \cdot \frac{-i}{-i} = -\frac{8i}{-(-1)} = -8i$

In order to take the 3rd root, we find the trigonometric form of $-8i$: $-8i = 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$,

from where $z_k = 2\left(\cos\frac{\frac{3\pi}{2} + 2k\pi}{3} + i\sin\frac{\frac{3\pi}{2} + 2k\pi}{3}\right)$, $k = 0, 1, 2.$

The algebraic form of the solutions are:

$k = 0: \; z_0 = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2 \cdot (0 + i) = 2i$

$k = 1: \; z_1 = 2\left(\cos\frac{\frac{3\pi}{2} + 2\pi}{3} + i\sin\frac{\frac{3\pi}{2} + 2\pi}{3}\right) = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i$

$k = 2: \; z_2 = 2\left(\cos\frac{\frac{3\pi}{2} + 4\pi}{3} + i\sin\frac{\frac{3\pi}{2} + 4\pi}{3}\right) = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i$