Calculus 1 - Exercises 4

The limits $\sqrt[n]{p} \xrightarrow{n\to\infty} 1 \ (p > 0)$ and $\sqrt[n]{n} \xrightarrow{n\to\infty} 1$

1. Calculate the following limits:

a) $a_n = \sqrt[2n]{2n}$

b) $a_n = \sqrt[n]{2n}$

c) $a_n = \sqrt[2n]{n}$

d) $a_n = \sqrt[n]{n+1}$

e) $a_n = \sqrt[n]{2n^3+3}$

f) $a_n = \sqrt[n]{\dfrac{2n^2+6}{3n^2+2n}}$

g) $a_n = \sqrt[n]{\dfrac{5n^2+4n-5}{n^3+6n^2-n}}$

h) $a_n = \sqrt[n^2]{n}$

i) $a_n = \sqrt[n]{4^n+n^2+3n}$

Recursive sequences

2. Investigate the convergence of the following sequences and calculate the limit if it exists.

a) $a_1 = 4, \quad a_{n+1} = 7 - \dfrac{10}{a_n}, \quad n = 1, 2, \ldots$

c) $a_1 = 1, \quad a_{n+1} = \sqrt{2a_n+3}, \quad n = 1, 2, \ldots$ $\qquad (a_n) = (1, 2.236, 2.73, \ldots)$

d) $a_1 = 5, \quad a_{n+1} = \sqrt{2a_n+3}, \quad n = 1, 2, \ldots$ $\qquad (a_n) = (5, 3.605, 3.195, \ldots)$

e) $a_1 = 4, a_{n+1} = \sqrt{8a_n-7}, n = 1, 2, \ldots$ $\qquad (a_n) = (4, 5, 5.74, \ldots)$

f) $a_1 = 5, a_{n+1} = 8 - \dfrac{12}{a_n}, n = 1, 2, \ldots$ $\qquad (a_n) = (5, 5.6, 5.85, \ldots)$

The limit $\left(1 + \dfrac{x}{n}\right)^n \xrightarrow{n\to\infty} e^x$

4. Calculate the limits of the following sequences.

a) $a_n = \left(1 + \dfrac{1}{6n^2}\right)^{6n^2+2}, \quad b_n = \left(\dfrac{n+5}{n-4}\right)^{n+3}, \quad c_n = \left(\dfrac{n^2+2}{n^2+3}\right)^{n^2+7}$

b) $a_n = \left(\dfrac{3n+5}{3n-4}\right)^{3n}, \quad b_n = \left(\dfrac{3n+5}{3n-4}\right)^{2n}$

c) $a_n = \left(1 + \dfrac{1}{n}\right)^{n^2}, \quad b_n = \left(1 + \dfrac{1}{n^2}\right)^{n}, \quad c_n = \left(1 - \dfrac{1}{n^4}\right)^{n^3}, \quad d_n = \left(1 - \dfrac{1}{n^4}\right)^{n^5}$

d) $a_n = \left(\dfrac{4n+1}{4n+5}\right)^{n} \qquad a_n = \left(\dfrac{4n+1}{7n+5}\right)^{n}, \quad d_n = \left(\dfrac{6n+1}{4n+5}\right)^{n}$

e) $a_n = \left(\dfrac{3n^2+1}{3n^2-2}\right)^{3n^2}, \quad b_n = \left(\dfrac{3n^2+1}{3n^2-2}\right)^{3n^3}, \quad c_n = \left(\dfrac{3n^2-2}{3n^2+1}\right)^{3n^3}, \quad d_n = \left(\dfrac{3n^2-1}{3n^2+2}\right)^{3n}$

Limit superior and limit inferior

Find the limit inferior and limit superior of the following sequences.

a) $a_n = n^{((-1)^n)}$

b) $a_n = \cos\left(\dfrac{n\,\pi}{2}\right) \cdot \dfrac{2\,n^2 - 3}{n^2 + n + 8}, \quad b_n = \cos\left(\dfrac{n\,\pi}{2}\right) \cdot \dfrac{2\,n^2 - 3}{n^3 + n + 8}$

e) $a_n = \dfrac{(-3)^n + 8}{5 + 2^{2\,n+1}}, \quad b_n = \dfrac{(-4)^n + 8}{5 + 2^{2\,n+1}}$

f) $a_n = \sqrt{\dfrac{n^3 + (-1)^n\,n^3}{3\,n^3 + n + 7}} \qquad$ g) $a_n = \sqrt{\dfrac{2\,n^3 + (-1)^n\,n^3}{3\,n^3 + n + 7}}$