Calculus 1 - Exercises 9

Analysing graphs of functions

1. Let $f(x) = \ln\left(x^2 + 2x + 2\right)$.
   Find those intervals on which
   a) f is monotonic increasing or decreasing
   b) f is convex or concave.
2. Let $f(x) = 2x^6 - 15x^5 + 20x^4$
   Find those intervals on which the function is convex or concave.
   Where does it have inflection points?
3. Let $f(x) = x\, e^{-x^2}$
   Find those intervals on which the function is convex or concave.
   Where does it have inflection points?

Additional exercises

https://math.bme.hu/~nagyi/calc1/calc1-functions.pdf

Practice exercises for the 2nd midterm test, pages 6-7:
https://math.bme.hu/~nagyi/calc1/calculus1-practice2.pdf

Results

1. Let $f(x) = \ln\left(x^2 + 2x + 2\right)$.
   Find those intervals on which
   a) f is monotonic increasing or decreasing
   b) f is convex or concave.

Solution.

$$f(x) = \ln\left(x^2+2x+2\right) = \ln\big(\underbrace{(x+1)^2+1}_{\geq 1}\big) \implies D_f = \mathbb{R}$$

$$f'(x) = \frac{2x+2}{x^2+2x+2}$$

x	$(-\infty,-1)$	-1	$(-1,\infty)$
f'	$-$	0	$+$
f	$\searrow$		$\nearrow$

$\implies$ f is monotonic decreasing on $(-\infty, -1)$ and monotonic increasing on $(-1, \infty)$.

$$f''(x) = \frac{2(x^2+2x+2) - (2x+2)(2x+2)}{(x^2+2x+2)^2} = \frac{-2x\,(x+2)}{(x^2+2x+2)^2}$$

(the denominator is positive, the numerator is a downward parabola)

x	$(-\infty,-2)$	-2	$(-2,0)$	0	$(0,\infty)$
f''	$-$	0	$+$	0	$-$
f	$\cap$	(infl. pont)	$\cup$	(infl. pont)	$\cap$

$\Longrightarrow$ f is convex on $(-2, 0)$ and concave on $(-\infty, -2)$ and on $(0, \infty)$.

2. Let $f(x) = 2x^6 - 15x^5 + 20x^4$

   Find those intervals on which the function is convex or concave.

   Where does it have inflection points?

Solution.

$$f'(x) = 12x^5 - 75x^4 + 80x^3$$
$$f''(x) = 60x^4 - 300x^3 + 240x^2 = \underbrace{60x^2}_{\geq 0}\underbrace{(x^2 - 5x + 4)}_{(x-1)(x-4)}$$

x	$(-\infty,0)$	0	$(0,1)$	1	$(1,4)$	4	$(4,\infty)$
f''	$+$	0	$+$	0	$-$	0	$+$
f	$\cup$		$\cup$	infl.p.	$\cap$	infl.p.	$\cup$

3. Let $f(x) = x\,e^{-x^2}$

   Find those intervals on which the function is convex or concave.

   Where does it have inflection points?

Solution.

$$f'(x) = e^{-x^2} + x\,e^{-x^2}(-2x) = e^{-x^2} - 2x^2 e^{-x^2}$$
$$f''(x) = e^{-x^2}(-2x) - 4x\,e^{-x^2} - 2x^2 e^{-x^2}(-2x) = e^{-x^2}(4x^3 - 6x) = e^{-x^2}\,2x\,(2x^2 - 3)$$

x	$\left(-\infty,-\sqrt{\frac{3}{2}}\right)$	$-\sqrt{\frac{3}{2}}$	$\left(-\sqrt{\frac{3}{2}},0\right)$	0	$\left(0,\sqrt{\frac{3}{2}}\right)$	$\sqrt{\frac{3}{2}}$	$\left(\sqrt{\frac{3}{2}},\infty\right)$
f''	$-$	0	$+$	0	$-$	0	$+$
f	$\cap$	infl.p.	$\cup$	infl.p.	$\cap$	infl.p.	$\cup$