Calculus 1 - Exercises 10

Analysing graphs of functions

1. Perform a complete analysis of the following functions and sketch their graphs.
   a) $f(x) = x^3 e^{-x}$
   b) $f(x) = \dfrac{x^2 + x - 2}{x}$

Global extremum

2. Let $f(x) = x^3 + \dfrac{48}{x^2}$
   a) Perform a complete analysis of the function and sketch its graph.
   b) Does the function have a minimum or maximum on the interval [1, 3]?
   If yes, what are their values?

3. Let $f(x) = x^2 e^{-3x}$
   Does the function f have a minimum or maximum on the interval [0, 1]?

Optimization problems

1. From a rectangular sheet of cardboard measuring 1×2 meters we fold an open-top rectangular box by cutting out four congruent squares from the four corners, then folding up the sides of the box. What size squares should we cut out so that the volume of the box is maximized?

2. The sum of the lengths of the edges of a square prism is 1m. What are the edge lengths of the prism with maximal volume? What is this maximal volume?

3.* The radius of the base circle of a right circular cone is 2 meters, and its height is 5 meters. Determine the dimensions of the cylinder with the maximum volume that can be inscribed in the cone.

4.* The widths of two perpendicular corridors are 2.4 m and 1.6 m, respectively. What is the longest ladder that can be moved (in a horizontal position) from one corridor to another?

Solutions

> **1.** From a rectangular sheet of cardboard measuring 1×2 meters we fold an open-top rectangular box by cutting out four congruent squares from the four corners, then folding up the sides of the box. What size squares should we cut out so that the volume of the box is maximized?

Solution. Let x denote the side length of the squares cut out. Then the volume of the box is
$V(x) = x(1 - 2x)(2 - 2x) = 4x^3 - 6x^2 + 2x$, where $0 < x < \frac{1}{2}$. We want to find the maximum of

the function V.

$V'(x) = 12x^2 - 12x + 2 = 0 \iff x_1 = \frac{3-\sqrt{3}}{6} \approx 0.211$ or $x_2 = \frac{3+\sqrt{3}}{6} \approx 0.789$.

The function V can have a maximum at x_1. V cannot have a maximum at x_2, since $x_2 > \frac{1}{2}$.

$V''(x) = 24x - 12$ and $V''(x_1) = -4\sqrt{3} < 0$, so so the function indeed has local a maximum at x_1.

2. The sum of the lengths of the edges of a square prism is 1m. What are the edge lengths of the prism with maximal volume? What is this maximal volume?

Solution. Let x denote the side of the square base and let y denote the height of the prism.
Then the volume of the prism is $V = x^2 y$ and the sum of the lengths of the edges is $8x + 4y = 1$.
From here $y = \frac{1-8x}{4}$. Substituting this into the volume, we get that

$V(x) = x^2 \cdot \frac{1-8x}{4} = \frac{1}{4}(x^2 - 8x^3)$.

We want to find the maximum of this function if $0 < x < \frac{1}{8}$.

$V'(x) = \frac{1}{4}(2x - 24x^2) = \frac{1}{4} \cdot 2x(1 - 12x) = 0 \iff x_1 = 0, \quad x_2 = \frac{1}{12}$

Because of the conditions, $x = 0$ cannot be the case.

$V''(x) = \frac{1}{4}(2 - 48x)$ and $V''\left(\frac{1}{12}\right) = \frac{1}{4}\left(2 - 48 \cdot \frac{1}{12}\right) = -\frac{1}{2} < 0$, so V has a maximum at $x = \frac{1}{12}$.

The sides of the prism with maximal volume are $\frac{1}{12}$, $\frac{1}{12}$ and $\frac{1}{12}$ m (so it is a cube) and the maximum of the
volume is $\frac{1}{12^3}$ m^3.

3.* The radius of the base circle of a right circular cone is 2 meters, and its height is 5 meters. Determine the dimensions of the cylinder with the maximum volume that can be inscribed in the cone.

Solution. The height of the cone is $m = 5$ and the radius of the base circle is $r = 2$.
Let x and y respectively denote the radius of the base circle and the height of the cylinder and let α denote the angle formed by the slant height of the cone with the plane of its base.
Then $\tan\alpha = \frac{m}{r} = \frac{y}{r-x}$.
The volume of the cylinder that we want to maximize is

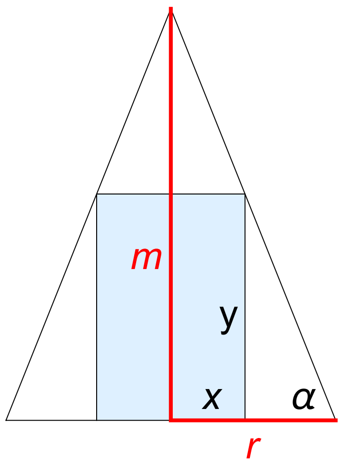
$V(x) = \pi x^2 y = \pi\left(\frac{5}{2}x^2(2-x)\right) = \pi\left(-\frac{5}{2}x^3 + 5x^2\right)$, where $0 < x < 2$.

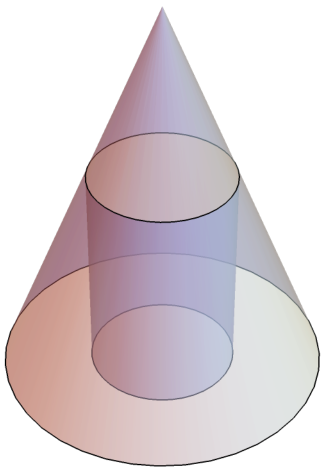
Then $V'(x) = \pi\left(-\frac{15}{2}x^2 + 10x\right) = 0$, from where $x = \frac{4}{3}$ (since $x > 0$).

At this point the function has a maximum, since

$V''(x) = \pi(-15x + 10)$, so $V''\left(\frac{4}{3}\right) = -10\pi < 0$.

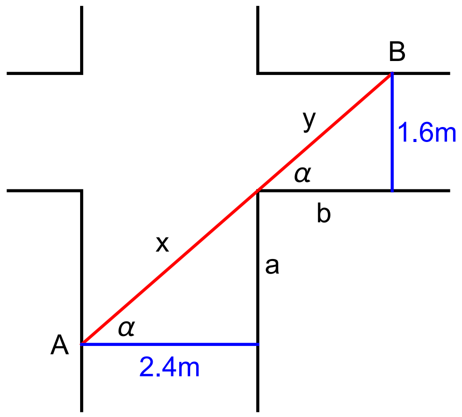
The radius of the base circle and the height of the cylinder with maximum volume are
$x = \frac{4}{3}$ and $y = \frac{5}{3}$ meters, respectively.





4.* The widths of two perpendicular corridors are 2.4 m and 1.6 m, respectively.
What is the longest ladder that can be moved (in a horizontal position) from one corridor to another?

1st solution.



The ladder is denoted by the line segment AB in the figure. The lengths x and y can be expressed with α, so the length of the ladder is $AB = f(\alpha) = \dfrac{2.4}{\cos\alpha} + \dfrac{1.6}{\sin\alpha} \implies$

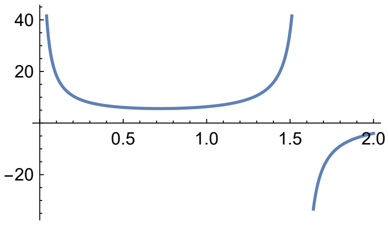
$$f'(\alpha) = \frac{2.4\sin\alpha}{\cos^2\alpha} - \frac{1.6\cos\alpha}{\sin^2\alpha} = \frac{0.8}{\sin^2\alpha\cos^2\alpha}\left(3\sin^3\alpha - 2\cos^3\alpha\right)$$

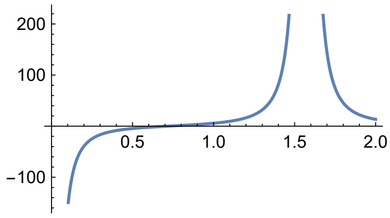
$$f'(\alpha) = 0 \text{ if } 3\sin^3\alpha = 2\cos^3\alpha \implies \tan\alpha = \sqrt[3]{\frac{2}{3}} \implies \alpha = \arctan\left(\sqrt[3]{\frac{2}{3}}\right) \approx 0.718025$$

$\implies \alpha \approx 41\,°$. The length of the ladder is $AB \approx 3.2 + 2.4 = 5.6$ m

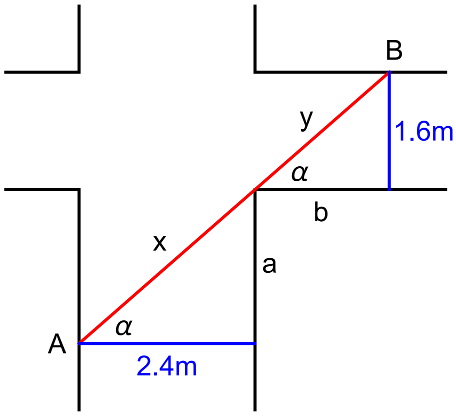
Remark. f has a local minimum at $\alpha \approx 0.718$. f' changes sign from negative to positive.

The graph of f: The graph of f':





2nd solution.



According to the figure, $\frac{a}{2.4} = \frac{1.6}{b}$, so $a = \frac{1.6 \cdot 2.4}{b}$. The length of the ladder is

$$f(b) = \sqrt{2.4^2 + a^2} + \sqrt{1.6^2 + b^2} = \sqrt{2.4^2 + \frac{1.6^2 \times 2.4^2}{b^2}} + \sqrt{1.6^2 + b^2} =$$

$$= 2.4 \sqrt{1 + \frac{1.6^2}{b^2}} + \sqrt{1.6^2 + b^2} = \frac{2.4}{b} \sqrt{1.6^2 + b^2} + \sqrt{1.6^2 + b^2} = \left(\frac{2.4}{b} + 1\right) \sqrt{1.6^2 + b^2}$$

$$f'(b) = -\frac{2.4}{b^2} \sqrt{1.6^2 + b^2} + \left(\frac{2.4}{b} + 1\right) \frac{b}{\sqrt{1.6^2 + b^2}} = 0$$

$$\Longleftrightarrow\ 1.6^2 + b^2 = \left(\frac{2.4}{b} + 1\right) \frac{b^3}{2.4} \ \Longleftrightarrow\ 1.6^2 + b^2 = b^2 + \frac{b^3}{2.4} \ \Longleftrightarrow\ b = \sqrt[3]{1.6^2 \cdot 2.4} \approx 1.83154$$

Substituting this value into $f(b) = \left(\frac{2.4}{b} + 1\right) \sqrt{1.6^2 + b^2}$, the length of the ladder is approximately 5.61879 m.

Remark. f has a local minimum at $b = \sqrt[3]{1.6^2 \cdot 2.4}$, since $f'(b) = \frac{-6.144 + b^3}{b^2 \sqrt{2.56 + b^2}}$ changes sign from negative to positive at this point.