Exercises

Analyze the following functions and sketch their graphs.

1. $f(x) = x^4 - 4x^3$
2. $f(x) = -x^4 + 18x^2$
3. $f(x) = x^5 + 5x^4$
4. $f(x) = \dfrac{1}{1+x^2}$
5. $f(x) = \dfrac{1}{1+x^2}$
6. $f(x) = \dfrac{x}{1+x^2}$
7. $f(x) = \dfrac{x}{1-x^2}$
8. $f(x) = \dfrac{x}{(1-2x)^2}$
9. $f(x) = \dfrac{x^2}{x+1}$
10. $f(x) = \dfrac{x^3}{x^2-3}$
11. $f(x) = x\,e^{-x}$
12. $f(x) = (x+2)^2\,e^{-x}$
13. $f(x) = e^{-x^2}$
14. $f(x) = x\,e^{-x^2}$
15. $f(x) = x^2 \ln x$
16. $f(x) = \arctan(x^2)$

Summary of the steps:

1) finding the domain of f
2) finding the zeros of f
3) parity, periodicity
4) limits at the endpoints of the intervals constituting the domain
5) investigation of f' $\Longrightarrow$ monotonicity, extreme values
6) investigation of f'' $\Longrightarrow$ convexity/concavity, inflection points
7) linear asymptotes
8) plotting the graph of f, finding the range of f

Theorems:

1) Assume that f is differentiable at $x_0 \in \operatorname{int} D_f$.

Necessary condition for a local extremum at x_0:

$f'(x_0) = 0$

Sufficient condition for a local extremum at x_0:

a) $f'(x_0) = 0$ and f' changes sign at x_0

b) f is twice differentiable at x_0, $f'(x_0) = 0$ and $f''(x_0) \neq 0$

($f''(x_0) > 0:$ local minimum, $f''(x_0) < 0:$ local maximum)

2) Assume that f is twice differentiable at $x_0 \in \operatorname{int} D_f$.

Necessary condition for an inflection point at x_0:

$f''(x_0) = 0$

Sufficient condition for an inflection point at x_0:

a) $f''(x_0) = 0$ and f'' changes sign at x_0

b) f is three times differentiable at x_0, $f''(x_0) = 0$ and $f'''(x_0) \neq 0$

Asymptotes:

1) The straight line $x = a$ is a vertical asymptote of the function f if $\lim_{x\to a+} f(x) = \pm\infty$ or $\lim_{x\to a-} f(x) = \pm\infty$.

2) The straight line $g(x) = A\,x + B$ is a linear asymptote of the function f at ∞ or $-\infty$ if

$$\lim_{x\to\infty} (f(x) - g(x)) = 0 \text{ or } \lim_{x\to-\infty} (f(x) - g(x)) = 0. \text{ Then } A = \lim_{x\to\pm\infty} \frac{f(x)}{x} \text{ and } B = \lim_{x\to\pm\infty} (f(x) - A\,x).$$

Solutions

1. $f(x) = x^4 - 4x^3$

$D_f = \mathbb{R}$; $f(x) = 0 \iff x = 0$ or $x = 4$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0 \iff x = 0$ or $x = 3$

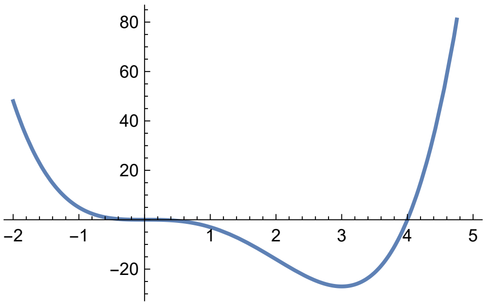
$f''(x) = 12x^2 - 24x = 12x(x-2) = 0 \iff x = 0$ or $x = 2$

$\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = +\infty$

$R_f = [-27, \infty)$

x	x<0	x=0	0<x<3	x=3	x>3
f'	-	0	-	0	+
f	↘		↘	min:-27	↗

x	x<0	x=0	0<x<2	x=2	x>2
f''	+	0	-	0	+
f	⋃	infl:0	⋂	infl:-16	⋃



2. $f(x) = -x^4 + 18x^2$

$D_f = \mathbb{R}$; $f(x) = 0 \iff x = 0$ or $x = \pm 3\sqrt{2}$; f is even;

$f'(x) = -4x^3 + 36x = 4x(-x^2 + 9) = 0 \iff x = 0$ or $x = \pm 3$

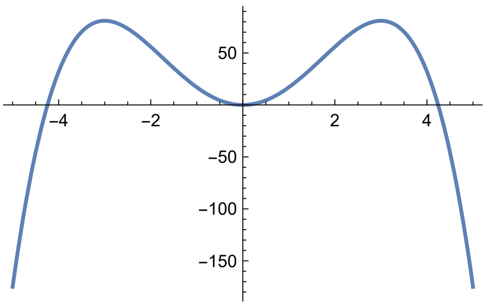
$f''(x) = -12x^2 + 36 = 0 \iff x = \pm\sqrt{3}$

$\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = -\infty$

$R_f = (-\infty, 81]$

x	x<-3	x=-3	-3<x<0	x=0	0<x<3	x=3	x>3
f'	+	0	-	0	+	0	-
f	↗	max:81	↘	min:0	↗	max:81	↘

x	$x<-\sqrt{3}$	$x=-\sqrt{3}$	$-\sqrt{3}<x<\sqrt{3}$	$x=\sqrt{3}$	$x>\sqrt{3}$
f''	-	0	+	0	-
f	⋂	infl:45	⋃	infl:45	⋂



3. $f(x) = x^5 + 5x^4$

$D_f = \mathbb{R}$; $f(x) = 0 \iff x = 0$ or $x = -5$

$f'(x) = 5x^4 + 20x^3 = 5x^3(x+4) = 0 \iff x = -4$ or $x = 0$

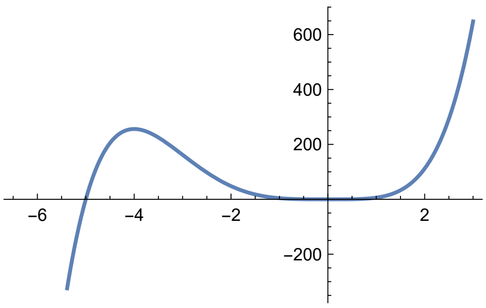
$f''(x) = 20x^3 + 60x^2 = 20x^2(x+3) = 0 \iff x = -3$ or $x = 0$

$\lim_{x\to-\infty} f(x) = -\infty, \quad \lim_{x\to+\infty} f(x) = +\infty$

$R_f = \mathbb{R}$

x	x<-4	x=-4	-4<x<0	x=0	x>0
f'	+	0	-	0	+
f	↗	max:256	↘	min:0	↗

x	x<-3	x=-3	-3<x<0	x=0	x>0
f''	-	0	+	0	+
f	∩	infl:162	∪		∪



4. $f(x) = \frac{1}{1+x^2}$

$D_f = \mathbb{R}$; $f(x) \neq 0$; f is even

$\lim_{x\to-\infty} f(x) = 0, \quad \lim_{x\to+\infty} f(x) = 0$

$f'(x) = -\frac{2x}{(1+x^2)^2} = 0 \iff x = 0$

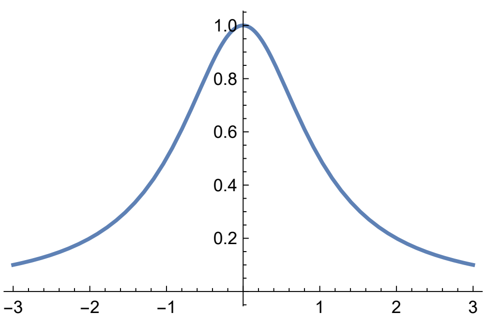
horizontal asymptote: $y = 0$

$f''(x) = \frac{2\cdot(-1+3x^2)}{(1+x^2)^3} = 0 \iff x = \pm\frac{1}{\sqrt{3}}$

$R_f = (0, 1]$

x	x<0	x=0	x>0
f'	+	0	-
f	↗	max:1	↘

x	$x<-\frac{1}{\sqrt{3}}$	$x=-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$	$x=\frac{1}{\sqrt{3}}$	$x>\frac{1}{\sqrt{3}}$
f''	+	0	-	0	+
f	∪	infl: $\frac{3}{4}$	∩	infl: $\frac{3}{4}$	∪



5. $f(x) = \frac{1}{1-x^2}$

$D_f = \mathbb{R} \setminus \{-1, 1\}$; $f(x) \neq 0$; f is even

$f'(x) = \frac{2x}{(1-x^2)^2} = 0 \iff x = 0$

$f''(x) = \frac{2 \cdot (1+3x^2)}{(1-x^2)^3} \neq 0$

$\lim_{x \to \pm\infty} f(x) = 0$, $\lim_{x \to -1 \pm 0} f(x) = \pm\infty$, $\lim_{x \to 1 \pm 0} f(x) = \mp\infty$

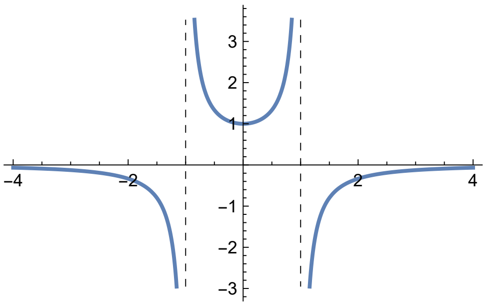
horizontal asymptote : $y = 0$

vertical asymptotes: $x = 1$, $x = -1$

$R_f = (-\infty, 0) \cup [1, +\infty)$

x	x<-1	-1<x<0	x=0	0<x<1	1<x
f'	−	−	0	+	+
f	↘	↘	min:1	↗	↗

x	x<-1	-1<x<1	x>1
f''	−	+	−
f	∩	∪	∩



6. $f(x) = \frac{x}{1+x^2}$

$D_f = \mathbb{R}$; $f(x) = 0 \iff x = 0$; f is odd

$f'(x) = \frac{1-x^2}{(1+x^2)^2} = 0 \iff x = \pm 1$

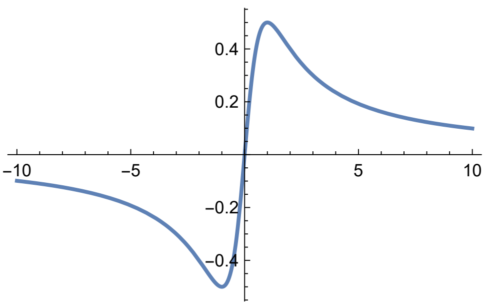
$f''(x) = \frac{2x(-3+x^2)}{(1+x^2)^3} = 0 \iff x = 0$ or $x = \pm\sqrt{3}$

$\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to +\infty} f(x) = 0$

horizontal asymptote : $y = 0$

$R_f = \left[-\frac{1}{2}, \frac{1}{2}\right]$

x	x<-1	x=-1	-1<x<1	x=1	x>1
f'	−	0	+	0	−
f	↘	min: $-\frac{1}{2}$	↗	max: $\frac{1}{2}$	↘



x	x<−$\sqrt{3}$	x=−$\sqrt{3}$	−$\sqrt{3}$<x<0	x=0	0<x<$\sqrt{3}$	x=$\sqrt{3}$	x>$\sqrt{3}$
f''	−	0	+	0	−	0	+
f	∩	infl: $-\frac{\sqrt{3}}{4}$	∪	infl:0	∩	infl: $\frac{\sqrt{3}}{4}$	∪

7. $f(x) = \frac{x}{1-x^2}$

$D_f = \mathbb{R} \setminus \{-1, 1\}$

$f(x) = 0 \iff x = 0$; f is odd

$f'(x) = \frac{1+x^2}{(1-x^2)^2} \neq 0$

$f''(x) = \frac{2x(3+x^2)}{(1-x^2)^3} = 0 \iff x = 0$

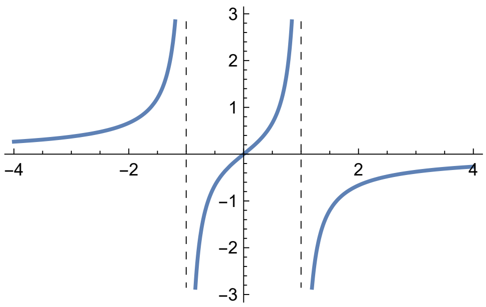
$\lim_{x\to\pm\infty} f(x) = 0,\quad \lim_{x\to -1\pm 0} f(x) = \mp\infty,\quad \lim_{x\to 1\pm 0} f(x) = \mp\infty$

horizontal asymptote : $y = 0$

vertical asymptotes: $x = 1,\ x = -1$

$R_f = \mathbb{R}$

x	x<-1	-1<x<1	x>1
f'	+	+	+
f	↗	↗	↗



x	x<-1	-1<x<0	x=0	0<x<1	1<x
f''	+	-	0	+	-
f	∪	∩	infl:0	∪	∩

8. $f(x) = \frac{x}{(1-2x)^2}$

$D_f = \mathbb{R} \setminus \left\{\frac{1}{2}\right\}$

$f(x) = 0 \iff x = 0$

$f'(x) = \frac{1+2x}{(1-2x)^3} = 0 \iff x = -\frac{1}{2}$

$f''(x) = \frac{8\cdot(1+x)}{(1-2x)^4} = 0 \iff x = -1$

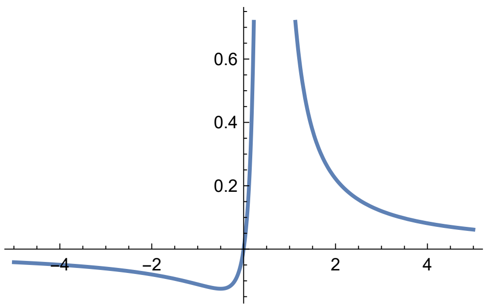
$\lim_{x\to\pm\infty} f(x) = 0,\quad \lim_{x\to -\frac{1}{2}\pm 0} f(x) = +\infty$

horizontal asymptote : $y = 0$

vertical asymptote: $x = \frac{1}{2}$

$R_f = \left[-\frac{1}{8}, +\infty\right)$

x	$x<-\frac{1}{2}$	$x=-\frac{1}{2}$	$-\frac{1}{2}<x<\frac{1}{2}$	$x>\frac{1}{2}$
f'	-	0	+	-
f	↘	min: $-\frac{1}{8}$	↗	↘



x	x<-1	x=-1	$-1<x<\frac{1}{2}$	$x>\frac{1}{2}$
f''	-	0	+	-
f	∩	infl: $-\frac{1}{9}$	∪	∩

9. $f(x) = \dfrac{x^2}{x+1}$

$D_f = \mathbb{R} \setminus \{-1\}$;

$f(x) = 0 \iff x = 0$

$f'(x) = \dfrac{x(2+x)}{(1+x)^2} = 0 \iff x = -2 \text{ or } x = 0$

$f''(x) = \dfrac{2}{(1+x)^3} \neq 0$

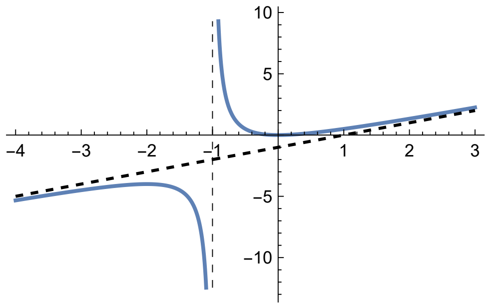
$\lim_{x\to\pm\infty} f(x) = \pm\infty, \quad \lim_{x\to -1\pm 0} f(x) = \pm\infty$

vertical asymptote : $x = -1$

linear asymptote: $y = x - 1$

$R_f = \mathbb{R}$

x	x<-2	x=-2	-2<x<-1	-1<x<0	x=0	x>0
f'	+	0	-	-	0	+
f	↗	max:-4	↘	↘	min:0	↗



x	x<-1	x>-1
f''	-	+
f	∩	∪

10. $f(x) = \dfrac{x^3}{x^2-3}$

$D_f = \mathbb{R} \setminus \left\{-\sqrt{3},\ \sqrt{3}\right\}$;

$f(x) = 0 \iff x = 0$; f is odd

$f'(x) = \dfrac{x^2\left(x^2-9\right)}{\left(x^2-3\right)^2} = 0 \iff x = 0 \text{ or } x = \pm 3$

$f''(x) = \dfrac{6x\left(9+x^2\right)}{\left(x^2-3\right)^3} = 0 \iff x = 0$

$\lim_{x\to\pm\infty} f(x) = \pm\infty, \quad \lim_{x\to -\sqrt{3}\pm 0} f(x) = \pm\infty, \quad \lim_{x\to \sqrt{3}\pm 0} f(x) = \pm\infty$

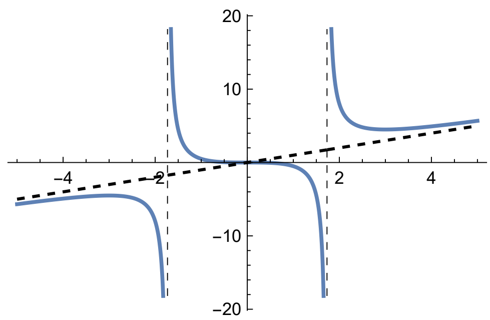
vertical asymptotes : $x = -\sqrt{3}$, $x = \sqrt{3}$

linear asymptote: $y = x$

$R_f = \mathbb{R}$

x	x<-3	x=-3	$-3<x<-\sqrt{3}$	$-\sqrt{3}<x<0$	x=0	$0<x<\sqrt{3}$	$\sqrt{3}<x<3$	x=3	x>3
f'	+	0	-	-	0	-	-	0	+
f	↗	max: $-\frac{9}{2}$	↘	↘		↘	↘	min: $\frac{9}{2}$	↗

x	$x<-\sqrt{3}$	$-\sqrt{3}<x<0$	x=0	$0<x<\sqrt{3}$	$x>\sqrt{3}$
f''	-	+	0	-	+
f	∩	∪	infl:0	∩	∪



11. $f(x) = x\,e^{-x}$

$D_f = \mathbb{R}$; $f(x) = 0 \iff x = 0$

$f'(x) = e^{-x}(1 - x) = 0 \iff x = 1$

$f''(x) = e^{-x}(x - 2) = 0 \iff x = 2$

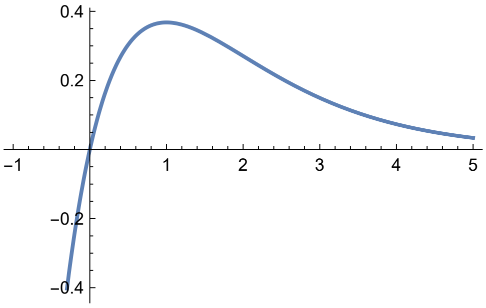
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x}{e^x} \overset{L'H}{=} \lim_{x\to+\infty} \frac{1}{e^x} = 0$$

$$\lim_{x\to-\infty} f(x) = -\infty$$

$$R_f = \left(-\infty, \frac{1}{e}\right]$$

x	x<1	x=1	x>1
f'	+	0	–
f	↗	max: $\frac{1}{e}\approx 0.37$	↘

x	x<2	x=2	x>2
f''	–	0	+
f	⋂	infl: $\frac{2}{e^2}\approx 0.27$	⋃



12. $f(x) = (x + 2)^2\,e^{-x}$

$D_f = \mathbb{R}$; $f(x) = 0 \iff x = -2$

$f'(x) = -e^{-x}\,x\,(2 + x) = 0 \iff x = 0$ or $x = -2$

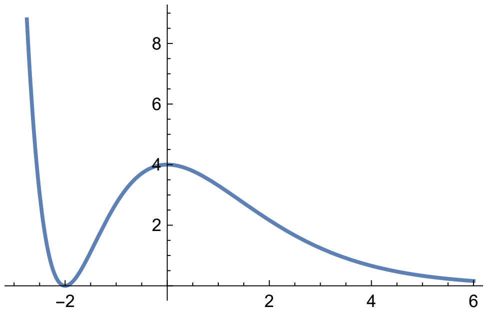
$f''(x) = e^{-x}\left(x^2 - 2\right) = 0 \iff x = \pm\sqrt{2}$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{(x+2)^2}{e^x} \overset{L'H}{=} \lim_{x\to+\infty} \frac{2\,(x+2)}{e^x} \overset{L'H}{=} \lim_{x\to+\infty} \frac{2}{e^x} = 0$$

$$\lim_{x\to-\infty} f(x) = +\infty$$

$$R_f = [0, +\infty)$$

x	x<–2	x=–2	–2<x<0	x=0	x>0
f'	–	0	+	0	–
f	↘	min:0	↗	max:4	↘



x	$x<-\sqrt{2}$	$x=-\sqrt{2}$	$-\sqrt{2}<x<\sqrt{2}$	$x=\sqrt{2}$	$x>\sqrt{2}$
f''	+	0	–	0	+
f	⋃	infl:≈1.41	⋂	infl:≈2.83	⋃

13. $f(x) = e^{-x^2}$

$D_f = \mathbb{R}$; $f(x) \neq 0$; f is even

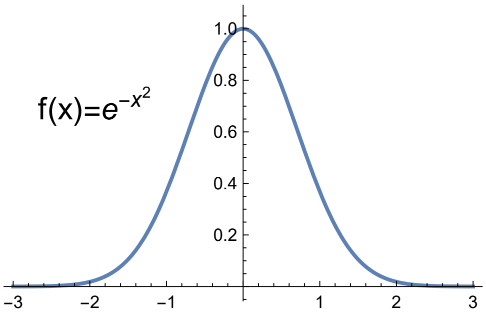
$f'(x) = e^{-x^2}(-2x) = 0 \iff x = 0$

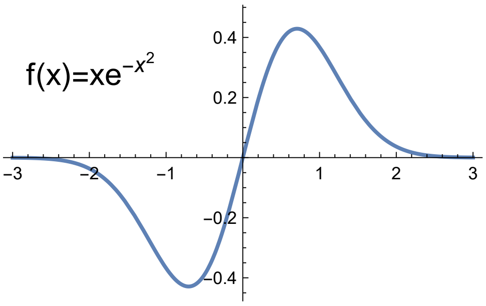
$f''(x) = 2e^{-x^2}(-1+2x^2) = 0 \iff x = \pm\frac{1}{\sqrt{2}}$

$\lim_{x\to\pm\infty} f(x) = 0$; $R_f = (0, 1]$

x	x<0	x=0	x>0
f'	+	0	−
f	↗	max:1	↘

x	$x<-\frac{1}{\sqrt{2}}$	$x=-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$	$x=\frac{1}{\sqrt{2}}$	$x>\frac{1}{\sqrt{2}}$
f''	+	0	−	0	+
f	∪	infl: $\frac{1}{\sqrt{e}}\approx 0.61$	∩	infl: $\approx\frac{1}{\sqrt{e}}\approx 0.61$	∪





14. $f(x) = x\,e^{-x^2}$

$D_f = \mathbb{R}$; $f(x) = 0 \iff x = 0$; f is odd

$f'(x) = -e^{-x^2}(-1+2x^2) = 0 \iff x = \pm\frac{1}{\sqrt{2}}$

$f''(x) = 2e^{-x^2}x(-3+2x^2) = 0 \iff x = 0$ or $x = \pm\sqrt{\frac{3}{2}}$

$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\frac{x}{e^{x^2}} \overset{L'H}{=} \lim_{x\to\pm\infty}\frac{1}{e^{x^2}\cdot 2x} = 0$

$R_f = \left[-\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right]$

x	$x<-\frac{1}{\sqrt{2}}$	$x=-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$	$x=\frac{1}{\sqrt{2}}$	$x>\frac{1}{\sqrt{2}}$
f'	−	0	+	0	−
f	↘	min: $-\frac{1}{\sqrt{2e}}\approx -0.43$	↗	max: $\frac{1}{\sqrt{2e}}\approx 0.43$	↘

x	$x<-\sqrt{\frac{3}{2}}$	$x=-\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}<x<0$	x=0	$0<x<\sqrt{\frac{3}{2}}$	$x=\sqrt{\frac{3}{2}}$	$x>\sqrt{\frac{3}{2}}$
f''	−	0	+	0	−	0	+
f	∩	infl: ≈−0.27	∪	infl:0	∩	infl: ≈0.27	∪

15. $f(x) = x^2 \ln x$

$D_f = \mathbb{R}^+$; $f(x) = 0 \iff x = 1$

$f'(x) = x(1 + 2\ln x) = 0 \iff x = \frac{1}{\sqrt{e}} \approx 0.61$

$f''(x) = 3 + 2\ln x = 0 \iff x = \frac{1}{e^{3/2}} \approx 0.22$

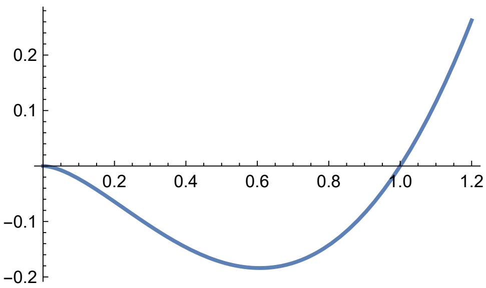
$\lim_{x\to 0+0} f(x) = \lim_{x\to 0+0} \frac{\ln x}{\frac{1}{x^2}} \overset{L'H}{=} \lim_{x\to 0+0} \frac{\frac{1}{x}}{-\frac{2}{x^3}} = \lim_{x\to 0+0} \frac{-x^2}{2} = 0$

$\lim_{x\to +\infty} f(x) = +\infty$

$R_f = \left[-\frac{1}{2e}, +\infty\right)$

x	$0<x<\frac{1}{\sqrt{e}}$	$x=\frac{1}{\sqrt{e}}$	$x>\frac{1}{\sqrt{e}}$
f'	–	0	+
f	↗	min: $-\frac{1}{2e} \approx -0.18$	↘

x	$0<x<\frac{1}{e^{3/2}}$	$x=\frac{1}{e^{3/2}}$	$x>\frac{1}{e^{3/2}}$
f''	–	0	+
f	∩	infl: $-\frac{3}{2e^3} \approx -0.07$	∪



16. $f(x) = \arctan(x^2)$

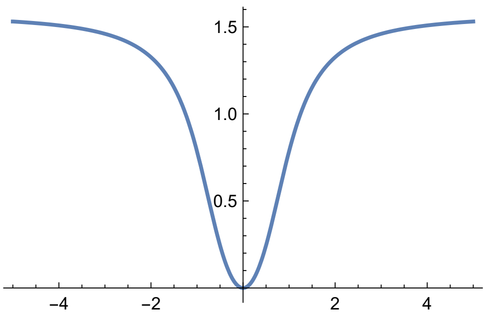
$D_f = \mathbb{R}$; $f(x) = 0 \iff x = 0$; f is even

$f'(x) = \frac{2x}{1+x^4} = 0 \iff x = 0$

$f''(x) = \frac{2\cdot(1-3x^4)}{(1+x^4)^2} = 0 \iff x = \pm\frac{1}{\sqrt[4]{3}}$

$\lim_{x\to\pm\infty} f(x) = \frac{\pi}{2}$; $R_f = [0, \frac{\pi}{2})$

x	x<0	0	x>0
f'	–	0	+
f	↘	min:0	↗



x	$x<-\frac{1}{\sqrt[4]{3}}$	$x=-\frac{1}{\sqrt[4]{3}}$	$-\frac{1}{\sqrt[4]{3}}<x<\frac{1}{\sqrt[4]{3}}$	$x=\frac{1}{\sqrt[4]{3}}$	$x>\frac{1}{\sqrt[4]{3}}$
f''	–	0	+	0	–
f	∩	infl: $\frac{\pi}{6}$	∪	infl: $\frac{\pi}{6}$	∩