Calculus 1 - 01

Complex numbers

Algebraic form

The algebraic form of complex numbers is

$$z = a + b\,i$$

where $a,\ b \in \mathbb{R}$, i is called the **imaginary unit** and $i^2 = -1$.

$\Longrightarrow$ Addition: $(a + b\,i) + (c + d\,i) = (a + b) + (c + d)\,i$

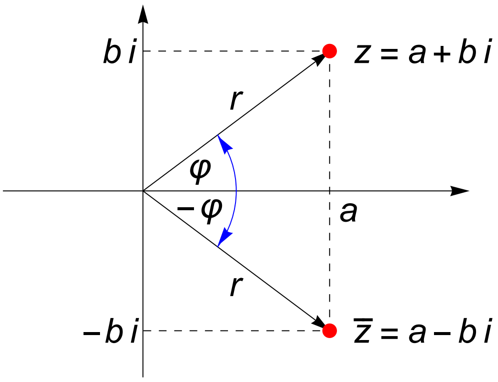
Multiplication: $(a + b\,i)\,(c + d\,i) = a\,c + b\,d\,i^2 + a\,d\,i + b\,c\,i = (a\,c - b\,d) + (a\,d + b\,c)\,i$

The set of complex numbers is denoted by $\mathbb{C}$.

Remark: $i^2 = (-i)^2 = -1$.

The complex plane

To each complex number $z = a + b\,i$ we associate the point $(a,\ b)$ in the Cartesian plane.
Real numbers are thus associated with points on the x-axis, called the **real axis** and
the purely imaginary numbers $b\,i$ correspond to points on the y-axis, called the **imaginary axis**.



Definitions

If $z = a + b\,i$ then

- the **real part** of z is $\operatorname{Re}(z) = a \in \mathbb{R}$
- the **imaginary part** of z is $\operatorname{Im}(z) = b \in \mathbb{R}$
- the **conjugate** of z is $\bar{z} = a - b\,i$
- the **absolute value** or **modulus** of z is $r = |\,z\,| = \sqrt{a^2 + b^2} \geq 0$ (the length of the vector z)
- the **argument** of z, defined for $z \neq 0$, is the angle which the vector originating from 0 to z makes with the positive x-axis. Thus $\arg(z) = \varphi$ (modulo $2\,\pi$) for which

$$\cos\varphi = \frac{\operatorname{Re}(z)}{|z|} = \frac{a}{r} \quad \text{and} \quad \sin\varphi = \frac{\operatorname{Im}(z)}{|z|} = \frac{b}{r}$$

Some identities

- $z\,\bar{z} = (a + b\,i)\,(a - b\,i) = a^2 - b^2\,i^2 = a^2 + b^2 = |z|^2$
- $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_1, \qquad \overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_1, \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \qquad \bar{\bar{z}} = z$
- $\operatorname{Re}(z) = \frac{z + \bar{z}}{2}, \qquad \operatorname{Im}(z) = \frac{z - \bar{z}}{2\,i}$

The trigonometric form (or polar form) of complex numbers

Let $z = a + b\,i \neq 0$, $r = |z|$ and $\varphi = \arg(z)$. Then $a = r\cos\varphi$ and $b = r\sin\varphi$ and

$$z = r(\cos\varphi + i\sin\varphi)$$

where r and φ are called the polar coordinates of z.

Multiplication and division

Let $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$. Then

- $z_1\,z_2 = r_1\,r_2(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)) \quad (\text{if } r_2 \neq 0)$

Remark: The above formulas can be proved by the trigonometric identities:

- $z_1\,z_2 = r_1(\cos\varphi_1 + i\sin\varphi_1) \cdot r_2(\cos\varphi_2 + i\sin\varphi_2) =$
  $= r_1\,r_2((\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2) + i(\sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2)) =$
  $= r_1\,r_2(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))$

- $\frac{z_1}{z_2} = \frac{r_1(\cos\varphi_1 + i\sin\varphi_1)}{r_2(\cos\varphi_2 + i\sin\varphi_2)} \cdot \frac{\cos\varphi_2 - i\sin\varphi_2}{\cos\varphi_2 - i\sin\varphi_2} =$

$$= \frac{r_1}{r_2} \cdot \frac{(\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2) + i(\sin\varphi_1\cos\varphi_2 - \cos\varphi_1\sin\varphi_2)}{\cos^2\varphi_2 + \sin^2\varphi_2} =$$

$$= \frac{r_1}{r_2} \cdot \frac{\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)}{1}$$

Reciprocal, conjugation, *n*th power

Let $z = r(\cos\varphi + i\sin\varphi)$. Then

- $\frac{1}{z} = \frac{1}{r}(\cos(-\varphi) + i \sin(-\varphi))$ (if $r \neq 0$)
- $\overline{z} = r(\cos(-\varphi) + i \sin(-\varphi))$
- $z^n = r^n(\cos(n\,\varphi) + i \sin(n\,\varphi))$ ($n \in \mathbb{N}^+$) If $r \neq 0$ then it holds for $n \in \mathbb{Z}$.

The *n*th root

If $z \neq 0$ and $n \in \mathbb{N}^+$ then $w \in \mathbb{C}$ is an nth root of z if $w^n = z$. Then

$$w = \sqrt[n]{r(\cos\varphi + i \sin\varphi)} = \sqrt[n]{r}\left(\cos\frac{\varphi + k \cdot 2\pi}{n} + i \sin\frac{\varphi + k \cdot 2\pi}{n}\right) \text{ where } k = 0, 1, \ldots, n-1.$$

Some identities

$$|z_1 z_2| = |z_1| \cdot |z_2|, \quad \left|\frac{1}{z}\right| = \frac{1}{|z|}, \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad |z^n| = |z|^n, \quad |\overline{z}| = |z|$$

Fundamental theorem of algebra

Every degree n polynomial with complex coefficients has exactly n complex roots, if counted with multiplicity.