Calculus 1 - 02

Axioms for the real numbers.

Axioms for the real numbers

$\mathbb{R}$ is a set whose elements are called real numbers. Two operations, called addition and multiplication are defined in $\mathbb{R}$ such that $\mathbb{R}$ is closed under these operations, that is, $\forall\, a, b \in \mathbb{R}\ \ (a + b \in \mathbb{R}\ \text{ and }\ a \cdot b \in \mathbb{R})$.

Addition:

1) $\forall\, a, b \in \mathbb{R}\ (a + b = b + a)$ (commutativity),
2) $\forall\, a, b, c \in \mathbb{R}\ \ ((a + b) + c) = a + (b + c))$ (associativity),
3) $\exists\, 0 \in \mathbb{R}\ \ (\forall\, a \in \mathbb{R}\ \ (a + 0 = 0 + a = 0))$ (existence of a zero element),
4) $\forall\, a \in \mathbb{R}\ (\exists\, b \in \mathbb{R}\ (a + b = 0))$ (existence of an additive inverse, notation: $b = -a$).

Multiplication:

5) $\forall\, a, b \in \mathbb{R}\ \ (a \cdot b = b \cdot a)$ (commutativity),
6) $\forall\, a, b, c \in \mathbb{R}\ \ ((a \cdot b) \cdot c = a \cdot (b \cdot c))$ (associativity),
7) $\exists\, 1 \in \mathbb{R}\ \ (\forall\, a \in \mathbb{R}\ (a \cdot 1 = 1 \cdot a = a))$ (existence of a unit element),
8) $\forall\, a \in \mathbb{R} \setminus \{0\}\ (\exists\, b \in \mathbb{R}\ \ (a \cdot b = 1))$ (existence of a multiplicative inverse, notation: $b = a^{-1}$).

For the two operations above:
9) $\forall\, a, b, c \in \mathbb{R}\ \ (a + b) \cdot c = a \cdot c + b \cdot c$ (the multiplication is distributive with respect to the addition).

Axioms (1)–(9) are the axioms for a **field**.

Ordering:
10) Exactly one of the following is true: $a < b,\ \ b < a,\ \ a = b$ (trichotomy),
11) $\forall\, a, b, c \in \mathbb{R}\ \ ((a < b) \wedge (b < c)) \implies (a < c)$ (transitivity),
12) $\forall\, a, b, c \in \mathbb{R}\ \ ((a < b) \wedge c > 0) \implies a \cdot c < b \cdot c$
13) $\forall\, a, b, c \in \mathbb{R}\ \ (a < b) \implies a + c < b + c$ (monotonicity)

Axioms (1)–(13) are the axioms for an **ordered field**.

Archimedian axiom:

14) $\forall\, a \in \mathbb{R}\ \ (\exists\, n \in \mathbb{N}\ \ (a < n))$.

Axioms 1) – 14) are true both for $\mathbb{R}$ and $\mathbb{Q}$.

Cantor axiom:

15) Let $a_1, b_1, a_2, b_2, \ldots \in \mathbb{R}$.
$(\forall\, n \in \mathbb{N}\ \ (a_n \le a_{n+1} \le b_{n+1} \le b_n)) \implies (\exists\, x \in \mathbb{R}\ \ (\forall\, n \in \mathbb{N}\ \ (x \in [a_n, b_n])))$
$\left(\text{so } \bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset\right)$.
It states that any nested sequence of closed intervals has a non-empty intersection.

Example: Let $a_1 = 1.4$ and $b_1 = 1.5$

$a_2 = 1.41$ $b_2 = 1.42$

$a_3 = 1.414$ $b_3 = 1.415$

$a_4 = 1.4142$ $b_4 = 1.4143$

... ...

$a_n = \left[10^n \cdot \sqrt{2}\right] \cdot 10^{-n}$ $b_n = \left(\left[10^n \cdot \sqrt{2}\right] + 1\right) \cdot 10^{-n}$

where [.] denotes the floor function.

Then $a_1 < a_2 < a_3 < a_4 < ... < \sqrt{2} < ... < b_4 < b_3 < b_2 < b_1$, so $\bigcap_{n=1}^{\infty} [a_n, b_n] = \left\{\sqrt{2}\right\} \in \mathbb{R} \setminus \mathbb{Q}$.

Remark. Closeness is important, for example if $I_n = \left(0, \frac{1}{n}\right]$ then $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Consequences

Some elementary laws of algebra and inequalities follow from the axioms. For example:

1) For all $a \in \mathbb{R}$, exactly one of the following properties hold: $a > 0$, $a = 0$, $a < 0$.
   $(a > 0 \iff -a < 0)$

2) $(a < b) \wedge (c < d) \implies a + c < b + d$
   Specifically: $(a > 0) \wedge (b > 0) \implies a + b > 0$

3) $(0 \leq a < b) \wedge (0 \leq c < d) \implies a\, c < b\, d$
   Specifically: $(a > 0) \wedge (b > 0) \implies a\, b > 0$

4) $(a < b) \wedge (c < 0) \implies a\, c > b\, c$
   Specifically: $a < b \implies -a > -b$

5) (i) $0 < a < b \implies \frac{1}{a} > \frac{1}{b}$ (ii) $a < b < 0 \implies \frac{1}{a} > \frac{1}{b}$ (iii) $a < 0 < b \implies \frac{1}{a} < \frac{1}{b}$

6) For all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$ and $||a| - |b|| \leq |a - b|$.

7) If n is a positive integer and $0 < a < b$ then $a^n < b^n$.

8) $\forall x \in \mathbb{R}$ $(x \cdot 0 = 0)$

9) $\forall x \in \mathbb{R}$ $(x \cdot y = 0 \implies x = 0$ or $y = 0)$

Proof of 8):
$x \cdot 0 = x \cdot 0 + 0 = x \cdot 0 + (x \cdot 0 - x \cdot 0) = (x \cdot 0 + x \cdot 0) - x \cdot 0 = x \cdot (0 + 0) - x \cdot 0 = x \cdot 0 - x \cdot 0 = 0.$

Proof of 9):
$x \neq 0 \implies y = 1 \cdot y = ((1/x) \cdot x) \cdot y = (1/x) \cdot (x \cdot y) = (1/x) \cdot 0 = 0.$

Bounded subsets of real numbers

Definition. $A \subset \mathbb{R}$ **is bounded above** if there exists a $K \in \mathbb{R}$ such that $a \leq K$ for all $a \in A$.
In this case K is an **upper bound** of A.

Definition. $A \subset \mathbb{R}$ is **bounded below** if there exists a $k \in \mathbb{R}$ such that $a \geq k$ for all $a \in A$.
In this case k is a **lower bound** of A.

Definition. $A \subset \mathbb{R}$ is **bounded** if it is has an upper bound and a lower bound.
It means that there exists a $K > 0$ such that $|a| < K$ for all $a \in A$.

Remark: A bounded set has infinitely many lower and upper bounds.

Examples: 1) $\mathbb{N}$ is bounded below

2) $(0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$ is bounded (for example, upper bounds are $1, 2, \pi, \ldots$, lower bounds are $0, -3, -100, \ldots$)

3) $\mathbb{Q}$ has no upper bound or lower bound

Definition. If a set A is bounded above, then the **supremum** of A is the **least upper bound** of A.
Notation: $\sup A$. If A is not bounded above, then $\sup A = \infty$.

Definition. If a set A is bounded below then the **infimum** of A is the **greatest lower bound** of A.
Notation: $\inf A$. If A is not bounded below, then $\inf A = -\infty$.

Examples: 1) $\inf \mathbb{N} = 1$, $\sup \mathbb{N} = \infty$; 2) $\inf(0, 1] = 0$, $\sup(0, 1] = 1$; 3) $\inf \mathbb{Q} = -\infty$, $\sup \mathbb{Q} = \infty$

Definition. The **minimum** of the set A is a if $a \in A$ and $a = \inf A$.
The **maximum** of the set A is b if $b \in A$ and $b = \sup A$.

Examples: 1) The minimum of $\mathbb{N}$ is 1 and it has no maximum.

2) The maximum of $(0, 1]$ is 1 and it has no minimum.

3) $\mathbb{Q}$ has no minimum and no maximum.

Least-upper-bound property

Theorem (Least-upper-bound property, Dedekind):
If a non-empty subset of $\mathbb{R}$ is bounded above then it has a least upper bound in $\mathbb{R}$.

Consequence. If a non-empty subset of $\mathbb{R}$ is bounded below then it has a greatest lower bound in $\mathbb{R}$.

Remarks. 1) In the above system of axioms, the axioms of Cantor and Archimedes can be replaced by this statement.

2) The set of rational numbers does not have the least-upper-bound property under the usual order.

For example, $\{x \in \mathbb{Q} : x^2 \leq 2\} = \mathbb{Q} \cap \left(-\sqrt{2}, \sqrt{2}\right)$ has an upper bound in $\mathbb{Q}$ but does not have a least upper bound in $\mathbb{Q}$ since $\sqrt{2}$ is irrational.