

# Calculus 1 - 04

## Number sequences, part 2.

### Orders of magnitudes

**Definition:** Suppose that  $a_n \xrightarrow{n \rightarrow \infty} \infty$  and  $b_n \xrightarrow{n \rightarrow \infty} \infty$ . Then the order of magnitude of  $(a_n)$  is smaller than the order of magnitude of  $(b_n)$  if  $\frac{a_n}{b_n} \xrightarrow{n \rightarrow \infty} 0$ .

**Notation:**  $a_n \ll b_n$ .

**Theorem:**  $n^n \gg n! \gg a^n \gg n^k \gg n^{\frac{1}{k}} \gg \log n$ , where  $a > 1$  and  $k \in \mathbb{N}^+$ . That is,

$$\begin{array}{lll} \text{a) } \lim_{n \rightarrow \infty} \frac{n^n}{n!} = \infty & \text{b) } \lim_{n \rightarrow \infty} \frac{n!}{a^n} = \infty, \text{ where } a > 1 & \text{c) } \lim_{n \rightarrow \infty} \frac{a^n}{n} = \infty, \text{ where } a > 1 \\ \text{d) } \lim_{n \rightarrow \infty} \frac{a^n}{n^k} = \infty, \text{ where } a > 1 \text{ and } k \in \mathbb{N}^+ & & \text{e) } \lim_{n \rightarrow \infty} \frac{n}{\log_2 n} = \infty \end{array}$$

**Some proofs. a)**  $\frac{n^n}{n!} = \frac{n}{1} \cdot \frac{n}{2} \cdot \frac{n}{3} \cdots \frac{n}{n-2} \cdot \frac{n}{n-1} \cdot \frac{n}{n} \geq n \cdot 1 \cdot 1 \cdots 1 \cdot 1 \cdot 1 = n \rightarrow \infty \implies \frac{n^n}{n!} \rightarrow \infty$

**b)** For example, if  $a = 2$ , then  $\frac{n!}{2^n} = \frac{n}{2} \cdot \frac{n-1}{2} \cdot \frac{n-2}{2} \cdots \frac{3}{2} \cdot \frac{2}{2} \cdot \frac{1}{2} \cdot \frac{n}{2} \geq \frac{1}{2} \cdot 1 \cdot 1 \cdots 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{n}{4} \rightarrow \infty \implies \frac{n!}{2^n} \rightarrow \infty$

**Example:**  $\frac{n^2 - 3^n}{n! + n^4} = \frac{3^n \cdot \frac{n^2}{3^2} - 1}{n! \cdot (1 + \frac{n^4}{n!})} \xrightarrow{n \rightarrow \infty} 0 \cdot \frac{0-1}{1+0} = 0$ .

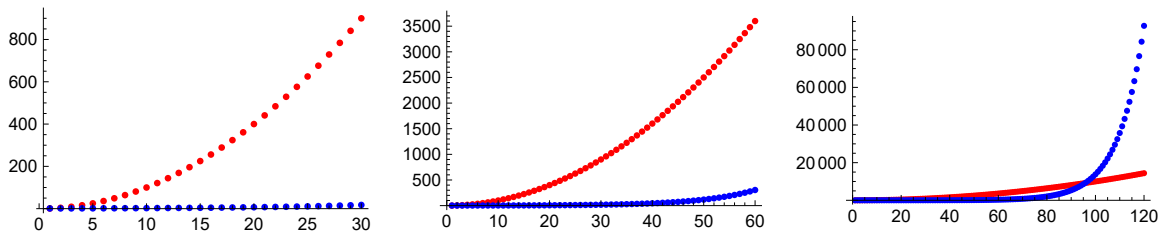
**Theorem.**  $\lim_{n \rightarrow \infty} n^k a^n = 0$ , if  $|a| < 1$  and  $k \in \mathbb{N}^+$ .

**Example:**  $\lim_{n \rightarrow \infty} \frac{n^2 + 9^{n+1}}{2n^5 + 3^{2n-1}} = ?$

**Solution:**  $\frac{n^2 + 9^{n+1}}{2n^5 + 3^{2n-1}} = \frac{9^n \cdot \frac{n^2}{9} + 9}{2n^5 \left(\frac{1}{9}\right)^n + \frac{1}{3}} \xrightarrow{n \rightarrow \infty} \frac{0+9}{0+\frac{1}{3}} = 27$ .

We used that  $\lim_{n \rightarrow \infty} n^k a^n = 0$ , if  $|a| < 1$ . Here  $a = \frac{1}{9}$ .

The graph of the sequences  $a_n = n^2$  and  $b_n = 1.1^n$



## Subsequences

**Definition.** Suppose that  $(n_k): \mathbb{N} \rightarrow \mathbb{N}$  is a strictly monotonically increasing sequence of natural numbers. Then we call the sequence  $(a_{n_k})$  a **subsequence** of  $(a_n)$ .

**Examples:** 1) The prime numbers are a subsequence of the positive integers.

2)  $b_n = \frac{1}{1+n^2}$  is a subsequence of  $a_n = \frac{1}{1+n}$  ( $b_n = a_{n^2}$ ).

3)  $c_n = \frac{1}{n!}$  is a subsequence of  $a_n = \frac{1}{n}$  ( $c_n = a_{n!}$ ).

**Remark.** A subsequence can be obtained from a given sequence by deleting some or no elements without changing the order of the remaining elements.

For example, 2, 4, 6, 8, ... is a subsequence of 1, 2, 3, 4, 5, 6, 7, 8, ...,

but 4, 2, 8, 6, ... is not a subsequence of it.

**Theorem.**  $\lim_{n \rightarrow \infty} a_n = A$  if and only for all  $(a_{n_k})$  subsequences  $\lim_{k \rightarrow \infty} a_{n_k} = A$ .

**Proof. 1)** Assume that all  $(a_{n_k})$  subsequences tend to the same limit  $A$ .

Since  $(a_{n+1})$  is also a subsequence of  $(a_n)$ , then  $\lim_{n \rightarrow \infty} a_{n+1} = A$ ,

so for all  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that if  $n > N$  then  $|a_{n+1} - A| < \varepsilon$ .

Then obviously  $|a_n - A| < \varepsilon$  if  $n > N + 1$ , so  $\lim_{n \rightarrow \infty} a_n = A$  also holds.

**2)** Assume that  $\lim_{n \rightarrow \infty} a_n = A$  and let  $(a_{n_k})$  be a subsequence of  $(a_n)$ .

Then for all  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$ , such that if  $n > N$ , then  $|a_n - A| < \varepsilon$ .

Since  $n_k \xrightarrow{k \rightarrow \infty} \infty$ , then for the number  $N \in \mathbb{N}$  above, there exists  $S \in \mathbb{N}$  such that

if  $k > S$ , then  $n_k > N$ , so  $|a_{n_k} - A| < \varepsilon$ , therefore  $a_{n_k} \xrightarrow{k \rightarrow \infty} A$ .

## The Sandwich Theorem and two applications

**Theorem (Sandwich Theorem).** If  $a_n \xrightarrow{n \rightarrow \infty} A \in \mathbb{R}$ ,  $c_n \xrightarrow{n \rightarrow \infty} A \in \mathbb{R}$  and  $a_n \leq b_n \leq c_n$  for all  $n > N$ , then  $b_n \xrightarrow{n \rightarrow \infty} A \in \mathbb{R}$

**Proof.** Let  $\varepsilon > 0$  be fixed. Then

there exists  $N_1 \in \mathbb{N}$  such that if  $n > N_1$  then  $A - \varepsilon < a_n < A + \varepsilon$  and

there exists  $N_2 \in \mathbb{N}$  such that if  $n > N_2$  then  $A - \varepsilon < c_n < A + \varepsilon$ .

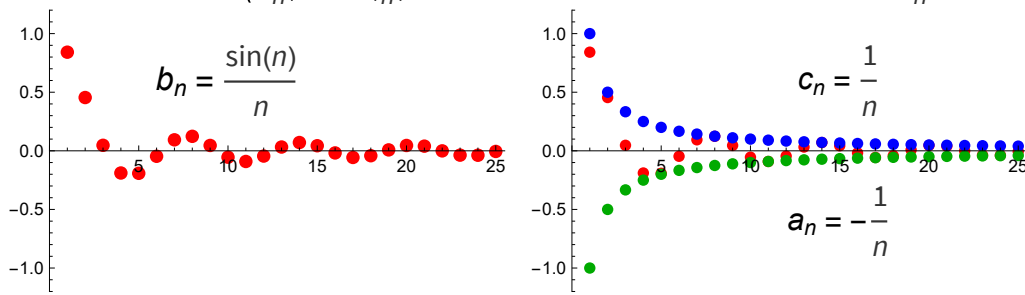
So if  $n > \max\{N, N_1, N_2\}$  then

$$A - \varepsilon < a_n \leq b_n \leq c_n < A + \varepsilon \implies |b_n - A| < \varepsilon.$$

**Example:** Calculate the limit of the sequence  $a_n = \frac{\sin(n)}{n}$ .

**Solution.** For all  $n$  we have  $-1 \leq \sin(n) \leq 1 \implies -\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$ .

Since  $\lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) = \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) = 0$ , then by the Sandwich Theorem  $\lim_{n \rightarrow \infty} \frac{\sin(n)}{n} = 0$ .



### Inequality of the arithmetic and geometric means (AM-GM inequality):

If  $a_1, a_2, \dots, a_n \geq 0$  then  $\sqrt[n]{a_1 a_2 \dots a_n} \leq \frac{a_1 + a_2 + \dots + a_n}{n}$  and equality holds if and only if  $a_1 = a_2 = \dots = a_n$ .

Proof for the special case  $n = 2$ :

$$\frac{a+b}{2} \geq \sqrt{ab} \iff (a+b)^2 \geq 4ab \iff a^2 - 2ab + b^2 \geq 0 \iff (a-b)^2 \geq 0, \text{ which always holds.}$$

**Theorem.**  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ .

**Proof.** Apply the above AM-GM inequality for  $a_1 = \dots = a_{n-2} = 1$ ,  $a_{n-1} = a_n = \sqrt{n}$ . Then

$$1 \leq \sqrt[n]{n} = \sqrt[n]{1 \cdot \dots \cdot 1 \cdot \sqrt{n} \cdot \sqrt{n}} \leq \frac{(n-2) + 2\sqrt{n}}{n} \leq 1 + \frac{2}{\sqrt{n}} \rightarrow 1 + 0 = 1,$$

so by the Sandwich Theorem,  $\sqrt[n]{n} \rightarrow 1$ .

**Theorem.** If  $p > 0$  then  $\lim_{n \rightarrow \infty} \sqrt[n]{p} = 1$ .

**Proof.** Assume that  $p \geq 1$  and apply the AM-GM inequality for  $a_1 = \dots = a_{n-2} = 1$ ,  $a_{n-1} = a_n = \sqrt{p}$ . Then

$$1 \leq \sqrt[n]{p} = \sqrt[n]{1 \cdot \dots \cdot 1 \cdot \sqrt{p} \cdot \sqrt{p}} \leq \frac{(n-2) + 2\sqrt{p}}{n} \leq 1 + \frac{2\sqrt{p}}{n} \rightarrow 1 + 0 = 1,$$

so by the Sandwich Theorem,  $\sqrt[n]{p} \rightarrow 1$ .

If  $0 < p < 1$ , then  $\frac{1}{p} > 1$ , so  $\sqrt[n]{p} = \frac{1}{\sqrt[n]{\frac{1}{p}}} \rightarrow 1$ .

## Examples

**Exercise 1.** Calculate the limit of  $a_n = \sqrt[3n]{n}$ .

**1st solution:**  $a_n = \sqrt[3n]{n} = \sqrt[3n]{\frac{3n}{3}} = \frac{\sqrt[3n]{3n}}{\sqrt[3n]{3}} \rightarrow \frac{1}{1} = 1$ . Here we use that  $\sqrt[3n]{3n} \rightarrow 1$ , since it is a subsequence of  $\sqrt[n]{n}$ , and similarly  $\sqrt[3n]{3} \rightarrow 1$ , since it is a subsequence of  $\sqrt[n]{3}$ .

**2nd solution:**  $a_n = \sqrt[3n]{n} = \sqrt[3]{\sqrt[n]{n}} \rightarrow \sqrt[3]{1} = 1$ .

**3rd solution:** Since  $1 \leq a_n = \sqrt[3n]{n} \leq \sqrt[3n]{3n}$  for all  $n \in \mathbb{N}$  and  $\sqrt[3n]{3n} \rightarrow 1$  then by the Sandwich Theorem,  $a_n \rightarrow 1$ .

**Exercise 2.** Calculate the limit of  $a_n = \sqrt[n]{\frac{2n^5 + 5n}{8n^2 - 2}}$ .

**Solution.** Estimating  $a_n$  from above and from below:

$$a_n = \sqrt[n]{\frac{2n^5 + 5n}{8n^2 - 2}} \leq \sqrt[n]{\frac{2n^5 + 5n^5}{8n^2 - 2n^2}} = \sqrt[n]{\frac{7n^5}{6n^2}} = \sqrt[n]{\frac{7}{6}} \cdot (\sqrt[n]{n})^3 \rightarrow 1 \cdot 1^3 = 1,$$

$$a_n = \sqrt[n]{\frac{2n^5 + 5n}{8n^2 - 2}} \geq \sqrt[n]{\frac{2n^5 + 0}{8n^2 + 0}} = \sqrt[n]{\frac{2n^5}{8n^2}} = \sqrt[n]{\frac{2}{8}} \cdot (\sqrt[n]{n})^3 \rightarrow 1 \cdot 1^3 = 1,$$

so by the Sandwich Theorem,  $a_n \rightarrow 1$ .

**Exercise 2.** Calculate the limit of  $a_n = \sqrt[n]{\frac{3^n + 5^n}{2^n + 4^n}}$ .

**Solution.** Estimating  $a_n$  from above and from below:

$$a_n = \sqrt[n]{\frac{3^n + 5^n}{2^n + 4^n}} \leq \sqrt[n]{\frac{5^n + 5^n}{0 + 4^n}} = \sqrt[n]{2} \cdot \frac{5}{4} \rightarrow 1 \cdot \frac{5}{4} = \frac{5}{4},$$

$$a_n = \sqrt[n]{\frac{3^n + 5^n}{2^n + 4^n}} \geq \sqrt[n]{\frac{0 + 5^n}{4^n + 4^n}} = \sqrt[n]{\frac{1}{2}} \cdot \frac{5}{4} \rightarrow 1 \cdot \frac{5}{4} = \frac{5}{4},$$

so by the Sandwich Theorem,  $a_n \rightarrow \frac{5}{4}$ .

## Monotonic sequences

**Theorem.** If  $(a_n)$  is monotonically increasing and not bounded above, then  $a_n \xrightarrow{n \rightarrow \infty} \infty$ .

**Proof.** Let  $P > 0$  be fixed. Since it is not an upper bound, there exists an  $N \in \mathbb{N}$  such that  $a_N > P$ .  
By the monotonicity, if  $n > N$  then  $a_n \geq a_N > P$ .

**Consequence.** If  $(a_n)$  is monotonically decreasing and not bounded below, then  $a_n \xrightarrow{n \rightarrow \infty} -\infty$ .

**Theorem. (1)** If  $(a_n)$  is monotonically increasing and bounded above, then  $(a_n)$  is convergent and  

$$\lim_{n \rightarrow \infty} a_n = \sup \{a_n : n \in \mathbb{N}\}.$$

**(2)** If  $(a_n)$  is monotonically decreasing and bounded below, then  $(a_n)$  is convergent and  

$$\lim_{n \rightarrow \infty} a_n = \inf \{a_n : n \in \mathbb{N}\}.$$

**Consequence.** If  $(a_n)$  is monotonic and bounded then  $(a_n)$  is convergent.

**Proof of part (1).**

1. Let  $A = \sup \{a_k : k \in \mathbb{N}\}$ , then  $a_n \leq A$  for all  $n \in \mathbb{N}$ .
2. Assume indirectly that  $\lim_{n \rightarrow \infty} a_n \neq A$ . Then there exists  $\varepsilon > 0$ , such that for all  $N \in \mathbb{N}$  there exists  $n > N$ , such that  $a_n \leq A - \varepsilon$ .
3. By the monotonicity  $a_N \leq a_n$ , so  $a_N \leq A - \varepsilon$  for all  $N \in \mathbb{N}$ .
4. However, this is a contradiction, since  $A$  is the smallest upper bound of the sequence, so  $A - \varepsilon$  is not an upper bound.  
Therefore for all  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that if  $n > N$  then  $A - \varepsilon < a_n \leq A < A + \varepsilon$ , so

$$\lim_{n \rightarrow \infty} a_n = A.$$

## Recursive sequences

In many cases, the convergence of recursively given sequences can be investigated by the application of the previous theorem.

**Exercise 1.** Let  $0 < a < 1$  and  $b_n = a^n$ . Prove that the sequence  $(b_n)$  is convergent and find its limit.

**Solution.** Since  $0 < b_{n+1} = a^{n+1} < a^n = b_n < 1$  then  $(b_n)$  is bounded and monotonically decreasing. So it is convergent, let  $A = \lim_{n \rightarrow \infty} b_n$ . Then  

$$A = \lim_{n \rightarrow \infty} b_{n+1} = \lim_{n \rightarrow \infty} a \cdot b_n = a \cdot A \iff A(1 - a) = 0, \text{ so } A = 0.$$

**Exercise 2.** Let  $a_1 = 4$  and  $a_{n+1} = 8 - \frac{15}{a_n}$ . Prove that the sequence  $(a_n)$  is convergent and find its limit.

**Solution.** The first few terms of the sequence:

$$a_1 = 4, a_2 = 4.25, a_3 = 4.47059, a_4 = 4.64474, a_5 = 4.77054, \dots$$

1) First we calculate the possible limits of  $(a_n)$ . If  $(a_n)$  is convergent then

$$A = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1} = 8 - \frac{15}{A} \implies A^2 - 8A + 15 = (A - 3)(A - 5) = 0, \text{ therefore } A = 3 \text{ or } A = 5.$$

2) Next, we investigate the boundedness and monotonicity of  $(a_n)$ . If we prove that  $(a_n)$  is bounded and monotonically increasing or decreasing, then  $(a_n)$  is convergent and its limit is the supremum or the infimum of the sequence.

(i) First we prove boundedness by induction, that is, we prove that  $3 < a_n < 5$  for all  $n \in \mathbb{N}^+$ .

I. The statement is true for  $n = 1$ :  $3 < a_1 = 4 < 5$ .

II. Assume that  $3 < a_n < 5$ . Then

$$3 < a_n < 5 \implies \frac{1}{5} < \frac{1}{a_n} < \frac{1}{3} \implies 3 < \frac{15}{a_n} < 5 \implies -3 > -\frac{15}{a_n} > -5 \implies 3 < 8 - \frac{15}{a_n} = a_{n+1} < 5.$$

(ii) Next we prove by induction that  $(a_n)$  is monotonically increasing, that is,  $a_n < a_{n+1}$  for all  $n \in \mathbb{N}^+$ .

I. The statement is true for  $n = 1$ :  $a_1 = 4 < a_2 = \frac{17}{4} = 4.25$

II. Assume that  $a_n < a_{n+1}$ . Then

$$a_n < a_{n+1} \implies \frac{1}{a_n} > \frac{1}{a_{n+1}} \text{ (since } a_n > 3 > 0) \implies -\frac{15}{a_n} < -\frac{15}{a_{n+1}} \implies a_{n+1} = 8 - \frac{15}{a_n} < 8 - \frac{15}{a_{n+1}} = a_{n+2}.$$

Since  $(a_n)$  is monotonic increasing and bounded then  $(a_n)$  is convergent. The limit of  $(a_n)$  cannot be  $A = 3$ , since  $a_1 = 4$  and the sequence is monotonic increasing. Therefore  $\lim_{n \rightarrow \infty} a_n = 5$ .

The sequence  $a_n = \left(1 + \frac{1}{n}\right)^n$

**Theorem.** The sequence  $a_n = \left(1 + \frac{1}{n}\right)^n$  is monotonically increasing and bounded, so it is convergent.

**Proof.** a) Monotonicity. We use the inequality between the arithmetic and geometric means:

$$\text{if } a_1, a_2, \dots, a_k \geq 0 \text{ then } \sqrt[k]{a_1 a_2 \dots a_k} \leq \frac{a_1 + a_2 + \dots + a_k}{k}.$$

$$\text{Let } a_1 = \dots = a_n = 1 + \frac{1}{n} \text{ and } a_{n+1} = 1. \text{ Then } \sqrt[n+1]{\left(1 + \frac{1}{n}\right)^n \cdot 1} \leq \frac{n\left(1 + \frac{1}{n}\right) + 1}{n+1} = 1 + \frac{1}{n+1},$$

$$\text{so } a_n = \left(1 + \frac{1}{n}\right)^n \leq \left(1 + \frac{1}{n+1}\right)^{n+1} = a_{n+1} \text{ for all } n \in \mathbb{N}.$$

b) Boundedness. We use the inequality between the arithmetic and geometric means

$$\text{for the numbers } a_1 = \dots = a_n = 1 + \frac{1}{n} \text{ and } a_{n+1} = a_{n+2} = \frac{1}{2}. \text{ Then}$$

$$\sqrt[n+2]{\left(1 + \frac{1}{n}\right)^n \cdot \frac{1}{4}} \leq \frac{n\left(1 + \frac{1}{n}\right) + 2 \cdot \frac{1}{2}}{n+2} = 1, \quad \text{so } a_n = \left(1 + \frac{1}{n}\right)^n \leq 4 \text{ for all } n \in \mathbb{N}.$$

**Definition:** The number  $e$  is defined as the limit of the above sequence:

$$e := \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n.$$

**Remark:**  $2 < e < 3$ .

Some terms of the sequence are:  $a_1 = 2, a_2 = 2.25, a_3 \approx 2.37, a_4 \approx 2.44, a_5 \approx 2.488$

$$a_{10} \approx 2.59, a_{20} \approx 2.65, a_{100} \approx 2.70481, a_{200} \approx 2.71152$$

$$a_{1000} \approx 2.71692, a_{10000} \approx 2.71815$$

**Theorems.** 1) The number  $e \approx 2.718281828459045235360287 \dots$  is irrational.

$$2) \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x \text{ for all } x \in \mathbb{R}$$

$$3) \text{ If } x_n \xrightarrow{n \rightarrow \infty} \infty, \text{ then } \lim_{n \rightarrow \infty} \left(1 + \frac{1}{x_n}\right)^{x_n} = e.$$

$$4) e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}\right) = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!}$$

**Remark.** The convergence of the series  $\sum_{n=0}^{\infty} \frac{1}{n!}$  is very fast, for exam-

ple

$$\sum_{n=0}^6 \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} \approx 2.718 \dots \text{ (3 digits are accurate)}$$

$$\sum_{n=0}^{10} \frac{1}{n!} \approx 2.7182818 \dots \text{ (7 digits are accurate)}$$

$$\sum_{n=0}^{15} \frac{1}{n!} \approx 2.71828182845 \dots \text{ (11 digits are accurate)}$$

$$\sum_{n=0}^{20} \frac{1}{n!} \approx 2.7182818284590452353 \dots \text{ (19 digits are accurate)}$$

## Exercises

The sequence  $a_n = \left(1 + \frac{1}{n}\right)^n$

$$1. a_n = \left(1 + \frac{1}{n^3 + n + 6}\right)^{n^3 + n + 6} \rightarrow e, \text{ since it is a subsequence of } \left(1 + \frac{1}{n}\right)^n.$$

$$2. a_n = \left(1 + \frac{1}{n-6}\right)^n = \left(1 + \frac{1}{n-6}\right)^{n-6} \cdot \left(1 + \frac{1}{n-6}\right)^6 \rightarrow e \cdot 1^6 = e$$

$$3. a_n = \left(1 + \frac{1}{6n+1}\right)^{6n-7} = \left(1 + \frac{1}{6n+1}\right)^{6n+1} \cdot \frac{1}{\left(1 + \frac{1}{6n+1}\right)^8} \rightarrow e \cdot \frac{1}{1^8} = e$$

$$4. a_n = \left(\frac{n+3}{n+4}\right)^{n-2} = \left(\frac{n+4-1}{n+4}\right)^{n+4-6} = \left(1 + \frac{-1}{n+4}\right)^{n+4} \cdot \frac{1}{\left(\frac{n+4}{n+3}\right)^6} \rightarrow e^{-1} \cdot \frac{1}{1^6} = \frac{1}{e}$$

Here we used that  $\frac{n+4}{n+3} = \frac{1 + \frac{4}{n}}{1 + \frac{3}{n}} \rightarrow \frac{1+0}{1+0} = 1.$

Another solution:  $a_n = \left(\frac{n+3}{n+4}\right)^{n-2} = \frac{\left(1 + \frac{3}{n}\right)^n}{\left(1 + \frac{4}{n}\right)^n} \cdot \left(\frac{n+4}{n+3}\right)^2 \rightarrow \frac{e^3}{e^4} \cdot 1^2 = \frac{1}{e}$

$$5. a_n = \left(\frac{n^2-2}{n^2+3}\right)^{n^2} = \frac{\left(1 + \frac{-2}{n^2}\right)^{n^2}}{\left(1 + \frac{3}{n^2}\right)^{n^2}} \rightarrow \frac{e^{-2}}{e^3} = e^{-5}$$

$$6. a_n = \left(\frac{n+1}{n+6}\right)^{2n} = \left(\frac{\left(1 + \frac{1}{n}\right)^n}{\left(1 + \frac{6}{n}\right)^n}\right)^2 \rightarrow \left(\frac{e^1}{e^6}\right)^2 = (e^{-5})^2 = e^{-10}$$

$$7. a_n = \left(\frac{2n+2}{2n+9}\right)^{2n} = \frac{\left(1 + \frac{2}{2n}\right)^{2n}}{\left(1 + \frac{9}{2n}\right)^{2n}} \rightarrow \frac{e^2}{e^9} = e^{-7}$$

8. Calculate the limit of  $a_n = \left(\frac{2n^2+5}{2n^2+3}\right)^{4n^2}$

1st solution.  $a_n = \frac{\left(\left(1 + \frac{5}{2n^2}\right)^{2n^2}\right)^2}{\left(1 + \frac{3}{2n^2}\right)^{2n^2}} \rightarrow \left(\frac{e^5}{e^3}\right)^2 = e^4$

2nd solution.  $a_n = \frac{\left(1 + \frac{5 \cdot 2}{4n^2}\right)^{4n^2}}{\left(1 + \frac{3 \cdot 2}{4n^2}\right)^{4n^2}} \rightarrow \frac{e^{10}}{e^6} = e^4$

9. Calculate the limit of the following sequences:

$$a_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{3n^2}, \quad b_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{9n^2}$$

$$c_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{3n^3}, \quad d_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{3n}$$

Solution.  $a_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{3n^2} = \frac{\left(1 + \frac{1}{3n^2}\right)^{3n^2}}{\left(1 + \frac{-2}{3n^2}\right)^{3n^2}} \rightarrow \frac{e}{e^{-2}} = e^3 = A$

$$b_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{9n^2} = (a_n)^3 \Rightarrow b_n \rightarrow A^3 = e^9$$



$$c_n = \left( \frac{3n^2 + 1}{3n^2 - 2} \right)^{3n^3} = (a_n)^n$$

In the estimation below we use that  $2 < e < 3$ , so  $e^3 > 2^3 = 8$ .

Since  $a_n \rightarrow e^3$  then  $\exists N_1$  such that if  $n > N_1$  then  $c_n = (a_n)^n > 8^n \rightarrow \infty \implies c_n \rightarrow \infty$

$$d_n = \left( \frac{3n^2 + 1}{3n^2 - 2} \right)^{3n} = \sqrt[n]{a_n}$$

Since  $a_n \rightarrow e^3$  then for  $\varepsilon = 0.1 \exists N_2$  such that if  $n > N_2$  then  $\sqrt[n]{e^3 - 0.1} \leq d_n \leq \sqrt[n]{e^3 + 0.1}$ .

Since  $\sqrt[n]{e^3 - 0.1} \rightarrow 1$  and  $\sqrt[n]{e^3 + 0.1} \rightarrow 1$  then by the Sandwich Theorem  $d_n \rightarrow 1$ .

## Recursive sequences

1. Let  $a_1 = \frac{4}{3}$  and  $a_{n+1} = \frac{3 + a_n^2}{4}$ ,  $n = 1, 2, \dots$

Prove that the sequence  $(a_n)$  is convergent and find its limit.

**Solution.**  $a_1 \approx 1.33 > a_2 = \frac{3 + \left(\frac{4}{3}\right)^2}{4} \approx 1.194 > a_3 \approx 1.1067$

Conjecture:  $(a_n)$  is monotonically decreasing, so  $a_n > a_{n+1} > 0$ .

Proof: by induction.

I.  $a_1 > a_2 > a_3 > 0$  is satisfied.

II. Assume that  $a_{n-1} > a_n$ . From the definition of the sequence it is obvious that  $a_n > 0$

$\left( a_n = \frac{3 + a_{n-1}^2}{4} \geq \frac{3}{4} > 0 \right)$ . Then

$$\begin{aligned} a_{n-1} > a_n > 0 &\implies a_{n-1}^2 > a_n^2 \implies 3 + a_{n-1}^2 > 3 + a_n^2 \implies a_n = \frac{3 + a_{n-1}^2}{4} > \frac{3 + a_n^2}{4} = a_{n+1} \\ &\implies a_n > a_{n+1}. \end{aligned}$$

Since  $(a_n)$  is monotonic decreasing and bounded below (since  $a_n > 0$ ) then  $(a_n)$  is

convergent, therefore  $A = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{3 + a_n^2}{4}$

$$\implies A = \frac{3 + A^2}{4} \implies A^2 - 4A + 3 = (A - 1)(A - 3) = 0 \implies A = 1 \text{ or } A = 3.$$

Since  $a_n < a_1 = \frac{4}{3}$  then  $A = 3$  cannot be the case, so  $A = \lim_{n \rightarrow \infty} a_n = 1$ .

2. Let  $a_1 = 1$  and  $a_{n+1} = \sqrt{6 + a_n}$ ,  $n = 1, 2, \dots$

Is the sequence convergent? If so, what is the limit?

**Solution.** The first few terms of the sequence:  $a_1 = 1$ ,  $a_2 \approx 2.646$ ,  $a_3 \approx 2.94$ , ...

Since  $\sqrt{6 + a_n} \geq 0$  then the terms of the sequence are positive.

1) First we calculate the possible limits of  $(a_n)$ . If  $(a_n)$  is convergent then

$$A = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \sqrt{6 + a_n} = \sqrt{6 + A} \implies A^2 - A - 6 = (A - 3)(A + 2) = 0, \text{ from where}$$

$A = 3$  or  $A = -2$ . Since  $a_n = \sqrt{6 + a_{n-1}} > 0$  then  $A = -2$  cannot be the case, so the only possible limit is  $A = 3$ .

2) Next, we investigate the boundedness and monotonicity of  $(a_n)$ .

(i) First we prove by induction that  $(a_n)$  is monotonically increasing and the terms are positive, that is,  $0 < a_n < a_{n+1}$  for all  $n \in \mathbb{N}^+$ .

I. The statement is true for  $n = 1$ :  $0 < a_1 = 1 < a_2 = \sqrt{7} \approx 2.646$

II. Assume that  $0 < a_n < a_{n+1}$ . Then

$$0 < a_n < a_{n+1} \implies 0 < 0 + 6 < 6 + a_n < 6 + a_{n+1} \implies 0 < \sqrt{6 + a_n} < \sqrt{6 + a_{n+1}} \implies 0 < a_{n+1} < a_{n+2}.$$

(ii) Next we prove that the sequence is bounded above.  $A = 3$  is a suitable choice for the upper bound, that is, we show that  $a_n < 3$  for all  $n \in \mathbb{N}^+$ .

I. The statement is true for  $n = 1$ :  $a_1 = 1 < 3$

II. Assume that  $a_n < 3$ . Then  $a_{n+1} = \sqrt{6 + a_n} < \sqrt{6 + 3} = 3$ .

Since  $(a_n)$  is monotonic increasing and bounded above then  $(a_n)$  is convergent, so  $\lim_{n \rightarrow \infty} a_n = 3$ .

We have seen that this is the only possible limit.

**Remark.** Monotonicity can also be proved as follows.

$$0 < a_n < a_{n+1} = \sqrt{6 + a_n} \iff a_n^2 < 6 + a_n \iff a_n^2 - a_n - 6 < 0 \iff -2 < a_n < 3.$$

Here  $-2 < a_n$  trivially holds, since  $a_n > 0$ , and  $a_n < 3$  can be proved by induction.

**3.** Let  $a_1 = -3$  and  $a_{n+1} = \frac{5 - 6a_n^2}{13}$ ,  $n = 1, 2, \dots$ . Is the sequence convergent?

**Solution.**  $a_1 = -3$ ,  $a_2 \approx -3.769$ ,  $a_3 \approx -6.1725$ , ...

Is the sequence monotonic decreasing?

$$a_{n+1} = \frac{5 - 6a_n^2}{13} < a_n \iff 6a_n^2 + 13a_n - 5 > 0 \quad \left( 6x^2 + 13x - 5 = 0 \iff x_1 = -\frac{5}{2}, x_2 = \frac{1}{3} \right)$$

It means that the sequence is monotonic decreasing if and only if  $a_n < -\frac{5}{2}$  or  $a_n > \frac{1}{3}$ .

Homework: It can be proved by induction that  $a_n \leq -3 \left( < -\frac{5}{2} \right)$ .

Therefore the sequence is monotonic decreasing with initial value  $a_1 = -3$ .

If the sequence were bounded from below then it would be convergent and for the limit

$$\text{we would have } A = \frac{5 - 6A^2}{13} \implies \text{the possible values of } A \text{ could be } A = -\frac{5}{2} \text{ or } A = \frac{1}{3}.$$

Since  $a_n \leq -3$  for all  $n$  then these numbers cannot be the limit, so  $(a_n)$  is not convergent and therefore not bounded from below. Since  $(a_n)$  is monotonic decreasing then

$$\lim_{n \rightarrow \infty} a_n = -\infty.$$

