
Calculus 1 - 06

Limits of real functions

Definitions

A **function** $f : A \rightarrow B$ is a mapping that assigns exactly one element of B to every element from A . The set A is called the **domain** of f (notation: D_f or $\text{Dom}(f)$) and the set $f(A) = \{f(x) : x \in A\}$ is called the **range** of f (notation: R_f or $\text{Ran}(f)$).

A function $f : A \rightarrow B$ is **one-to one** or **injective** if for all $x, y \in A$: $(f(x) = f(y) \implies x = y)$.

A function $f : A \rightarrow B$ is **onto** or **surjective** if $f(A) = B$.

A function f is **bijective** if it is injective and surjective.

The function $f : D_f \subset \mathbb{R} \rightarrow \mathbb{R}$ is

- even, if $\forall x \in D_f, -x \in D_f$ and $f(x) = f(-x)$ (for example, $f(x) = x^2$ or $f(x) = \cos x$)
- odd, if $\forall x \in D_f, -x \in D_f$ and $f(-x) = -f(x)$ (for example, $f(x) = x^3$ or $f(x) = \sin x$)
- monotonically increasing if $\forall x, y \in D_f (x < y \implies f(x) \leq f(y))$
- monotonically decreasing if $\forall x, y \in D_f (x < y \implies f(x) \geq f(y))$
- strictly monotonically increasing if $\forall x, y \in D_f (x < y \implies f(x) < f(y))$ (for example, $f(x) = \sqrt{x}$, $f(x) = x^3$)
- strictly monotonically decreasing if $\forall x, y \in D_f (x < y \implies f(x) > f(y))$
- periodic with period $p > 0$ if $\forall x \in D_f, x + p \in D_f$ and $f(x) = f(x + p)$ (for example, $f(x) = \sin x$)

Limit point

Definition. Let $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. Then x is a **limit point** or **accumulation point** of A , if every interval $(x - r, x + r)$, that is, every neighbourhood of x contains a point in A that is distinct from x . (For all $r > 0$: $(x - r, x + r) \setminus \{x\} \cap A \neq \emptyset$)
Or: every neighbourhood of x contains infinitely many points of A .
The set of limit points of A is denoted by A' .

- Examples:
- $A = [0, 2) \implies A' = [0, 2]$
 - $A = (1, 2) \cup (2, 3) \implies A' = [1, 3]$
 - $A = \mathbb{R} \setminus \{0\} \implies A' = \mathbb{R}$

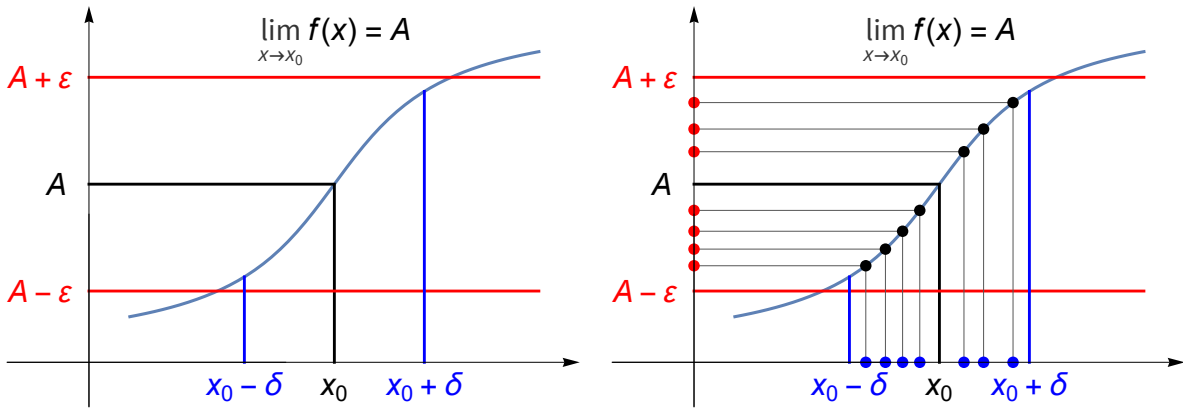
Limit at a finite point

Definition. The limit of the function $f : D_f \subset \mathbb{R} \rightarrow \mathbb{R}$ at the point $x_0 \in \mathbb{R}$ is $A \in \mathbb{R}$ if

- (1) x_0 is a limit point of D_f ($x \in D_f$)
- (2) for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that
if $x \in D_f$ and $0 < |x - x_0| < \delta(\varepsilon)$ then $|f(x) - A| < \varepsilon$

Notation: $\lim_{x \rightarrow x_0} f(x) = A$

Remark: $0 < |x - x_0| < \delta$ means that $x_0 - \delta < x < x_0$ or $x_0 < x < x_0 + \delta$.



One-sided limits:

Notation. The $\begin{cases} \text{right hand limit} \\ \text{left hand limit} \end{cases}$ of f at x_0 is denoted as $\begin{cases} \lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0+0} f(x) = f(x_0 + 0) \\ \lim_{x \rightarrow x_0^-} f(x) = \lim_{x \rightarrow x_0-0} f(x) = f(x_0 - 0) \end{cases}$

Definition. Suppose $x_0 \in \mathbb{R}$ is a limit point of $\begin{cases} D_f \cap [x_0, \infty) \\ D_f \cap (-\infty, x_0] \end{cases}$. Then

$\begin{cases} \lim_{x \rightarrow x_0^+} f(x) = A \\ \lim_{x \rightarrow x_0^-} f(x) = A \end{cases}$ if for all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $x \in D_f$ and $\begin{cases} x_0 < x < x_0 + \delta(\epsilon) \\ x_0 - \delta(\epsilon) < x < x_0 \end{cases}$ then $|f(x) - A| < \epsilon$.

Consequence. If x_0 is a limit point of D_f then $\lim_{x \rightarrow x_0} f(x)$ exists if and only if $\lim_{x \rightarrow x_0^+} f(x)$ and $\lim_{x \rightarrow x_0^-} f(x)$ exist and $\lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x)$.

Definition. Let $f : X \rightarrow Y$ be a function and $A \subset X$. The **restriction of f to A** is the function $f|_A : A \rightarrow Y, f|_A(x) = f(x)$.

Remark. $\lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0} f|_{D_f \cap [x_0, \infty)}(x)$ and $\lim_{x \rightarrow x_0^-} f(x) = \lim_{x \rightarrow x_0} f|_{D_f \cap (-\infty, x_0]}(x)$

Example 1. Using the definition, show that $\lim_{x \rightarrow -2} \frac{8 - 2x^2}{x + 2} = 8$.

Solution. We have to show that if x is “close” to x_0 , that is, $|x - x_0|$ is “small”, then $f(x)$ is “close” to A , that is, $|f(x) - A|$ is also “small”. That is, we have to show that for all $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$, then $|f(x) - A| < \epsilon$.

Here $x_0 = -2$. If $\epsilon > 0$ then

$$\begin{aligned} |f(x) - A| &= \left| \frac{8 - 2x^2}{x + 2} - 8 \right| = \left| \frac{2 \cdot (4 - x^2)}{x + 2} - 8 \right| = |2 \cdot (2 - x) - 8| = \\ &= |-2x - 4| = |(-2)(x + 2)| = 2|x + 2| = 2|x - (-2)| < \epsilon, \text{ if } \left| x + 2 \right| < \frac{\epsilon}{2} \\ \implies \text{with the choice } \delta &= \delta(\epsilon) = \frac{\epsilon}{2} \text{ the definition holds. Remark: } -2 \notin D_f. \end{aligned}$$

For example if $\epsilon = 10^{-2}$ then $\delta = 5 \cdot 10^{-3}$.

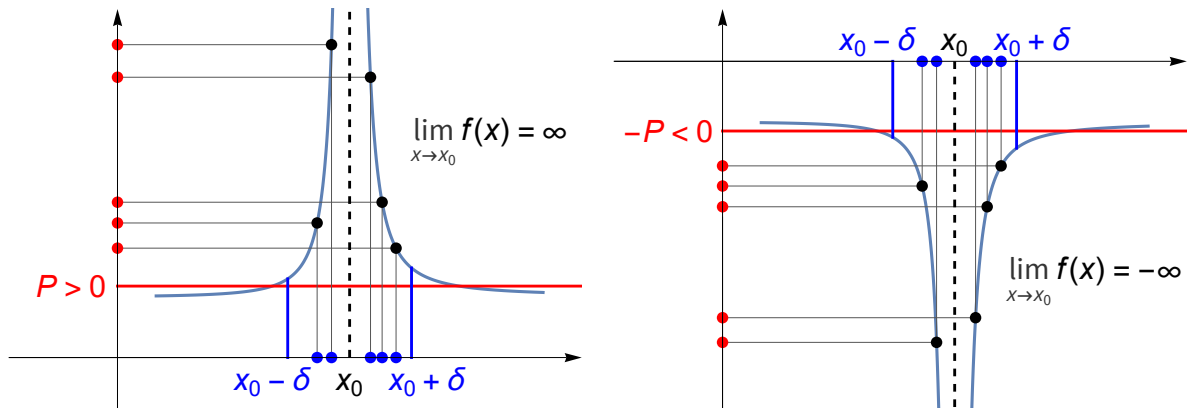
Example 2. Using the definition, show that $\lim_{x \rightarrow -3} \sqrt{1-5x} = 4$.

Solution. Let $\varepsilon > 0$. Then

$$|f(x) - A| = \left| \sqrt{1-5x} - 4 \right| = \left| \frac{1-5x-16}{\sqrt{1-5x}+4} \right| = \frac{5|x-(-3)|}{\sqrt{1-5x}+4} \leq \frac{5|x+3|}{\sqrt{1-5x}+4} < \varepsilon,$$

$$\text{if } |x+3| < \frac{4\varepsilon}{5} \implies \text{with the choice } \delta(\varepsilon) = \frac{4\varepsilon}{5} \text{ the definition holds.}$$

Definition. Suppose $f: D_f \subset \mathbb{R} \rightarrow \mathbb{R}$ is a function and $x_0 \in D_f'$. Then $\lim_{x \rightarrow x_0} f(x) = \begin{cases} \infty \\ -\infty \end{cases}$ if for all $P > 0$ there exists $\delta(P) > 0$ such that if $x \in D_f$ and $0 < |x - x_0| < \delta(P)$ then $\begin{cases} f(x) > P \\ f(x) < -P \end{cases}$.



Remark. The one-sided limits can be defined similarly:

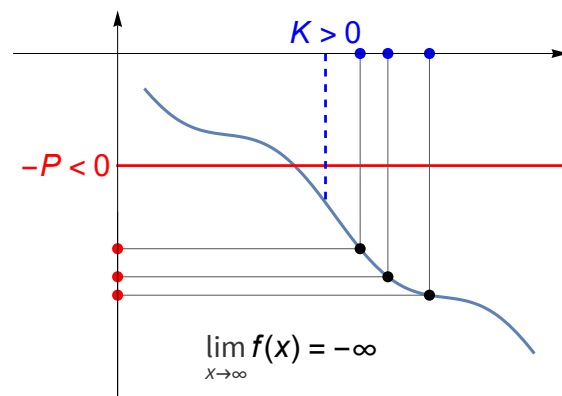
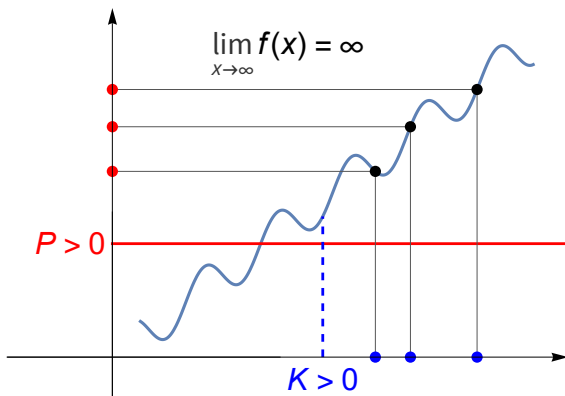
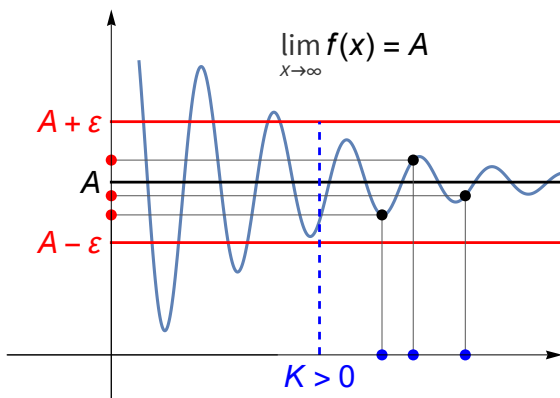
- $\lim_{x \rightarrow x_0^+} f(x) = \begin{cases} \infty \\ -\infty \end{cases}$ if $\forall P > 0 \exists \delta(P) > 0$ such that if $x \in D_f$ and $x_0 < x < x_0 + \delta(P)$ then $\begin{cases} f(x) > P \\ f(x) < -P \end{cases}$.
- $\lim_{x \rightarrow x_0^-} f(x) = \begin{cases} \infty \\ -\infty \end{cases}$ if $\forall P > 0 \exists \delta(P) > 0$ such that if $x \in D_f$ and $x_0 - \delta(P) < x < x_0$ then $\begin{cases} f(x) > P \\ f(x) < -P \end{cases}$.

Example 3. $\lim_{x \rightarrow 2} \frac{1}{(x-2)^2} = \infty$, since if $P > 0$, then $f(x) = \frac{1}{(x-2)^2} > P \iff 0 < |x-2| < \frac{1}{\sqrt{P}}$
 \implies with the choice $\delta(P) = \frac{1}{\sqrt{P}}$ the definition holds.

Limit at ∞ and $-\infty$

Definitions. Assume that D_f is not bounded above.

- (1) $\lim_{x \rightarrow \infty} f(x) = A \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists $K(\varepsilon) > 0$ such that if $x > K(\varepsilon)$ then $|f(x) - A| < \varepsilon$.
- (2) $\lim_{x \rightarrow \infty} f(x) = \infty$ if for all $P > 0$ there exists $K(P) > 0$ such that if $x > K(P)$ then $f(x) > P$.
- (3) $\lim_{x \rightarrow \infty} f(x) = -\infty$ if for all $P > 0$ there exists $K(P) > 0$ such that if $x > K(P)$ then $f(x) < -P$.



Definitions. Assume that D_f is not bounded below.

- (1) $\lim_{x \rightarrow -\infty} f(x) = A \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists $K(\varepsilon) > 0$ such that if $x < -K(\varepsilon)$ then $|f(x) - A| < \varepsilon$.
- (2) $\lim_{x \rightarrow -\infty} f(x) = \infty$ if for all $P > 0$ there exists $K(P) > 0$ such that if $x < -K(P)$ then $f(x) > P$.
- (3) $\lim_{x \rightarrow -\infty} f(x) = -\infty$ if for all $P > 0$ there exists $K(P) > 0$ such that if $x < -K(P)$ then $f(x) < -P$.

The extended set of real numbers

Definition. Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ denote the extended set of real numbers. We define $-\infty \leq x \leq \infty$ for all $x \in \bar{\mathbb{R}}$. The arithmetic operations on \mathbb{R} can be partially extended to $\bar{\mathbb{R}}$ as follows.

- | | |
|---|---|
| (1) $a + \infty = +\infty + a = \infty, \quad a \neq -\infty$ | (5) $\frac{a}{\pm\infty} = 0, \quad a \in \mathbb{R}$ |
| (2) $a - \infty = -\infty + a = -\infty, \quad a \neq +\infty$ | (6) $\frac{\pm\infty}{a} = \pm\infty, \quad a \in (0, +\infty)$ |
| (3) $a \cdot (\pm\infty) = \pm\infty \cdot a = \pm\infty, \quad a \in (0, +\infty]$ | (7) $\frac{\pm\infty}{a} = \mp\infty, \quad a \in (-\infty, 0)$ |
| (4) $a \cdot (\pm\infty) = \pm\infty \cdot a = \mp\infty, \quad a \in [-\infty, 0)$ | |

Definitions. The interval $(a - \varepsilon, a + \varepsilon)$ is called a neighbourhood of $a \in \mathbb{R}$ if $\varepsilon > 0$.
 For any $P \in \mathbb{R}$, the interval (P, ∞) is called a neighbourhood of $+\infty$ and the interval $(-\infty, P)$ is called a neighbourhood of $-\infty$.

Remark. The definition of a limit point can be extended to $\bar{\mathbb{R}}$ as follows. Let $A \subset \bar{\mathbb{R}}$ and $x \in \bar{\mathbb{R}}$. Then x is a limit point of A , if any neighbourhood of x contains a point in A that is distinct from x .

Remark. Examples for the set of limit points in $\overline{\mathbb{R}}$: $(\mathbb{N}^+)' = \{\infty\}$, $\mathbb{Z}' = \{\infty, -\infty\}$, $\mathbb{Q}' = \overline{\mathbb{R}}$, $\mathbb{R}' = \overline{\mathbb{R}}$.

Remark. If f is a sequence, that is, $D_f = \mathbb{N}^+$, then the only accumulation point of D_f is ∞ , so we can investigate the limit only here.

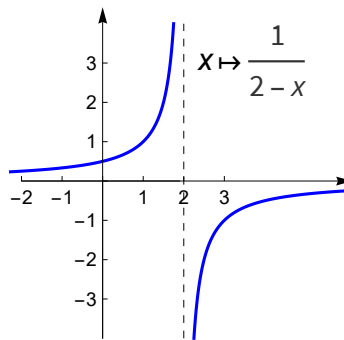
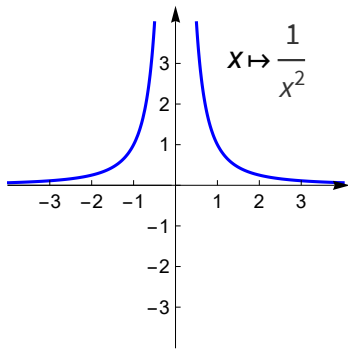
Summary

The above definitions of the limit can be summarized as follows.

Theorem. Assume that $a \in \overline{\mathbb{R}}$ is a limit point of D_f and $b \in \overline{\mathbb{R}}$. Then $\lim_{x \rightarrow a} f(x) = b$ if and only if for any neighbourhood J of b there exists a neighbourhood I of a such that if $x \in I \cap D_f$ and $x \neq a$ then $f(x) \in J$.

Examples

- $\lim_{x \rightarrow 0-0} \frac{1}{x^2} = \lim_{x \rightarrow 0+0} \frac{1}{x^2} = +\infty \implies \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty$
- $\lim_{x \rightarrow 2-0} \frac{1}{2-x} = +\infty, \lim_{x \rightarrow 2+0} \frac{1}{2-x} = -\infty \implies \lim_{x \rightarrow 2} \frac{1}{2-x}$ doesn't exist
- $\lim_{x \rightarrow \infty} \frac{1}{x^2} = \lim_{x \rightarrow -\infty} \frac{1}{x^2} = 0$
- $\lim_{x \rightarrow \infty} \frac{1}{2-x} = \lim_{x \rightarrow -\infty} \frac{1}{2-x} = 0$

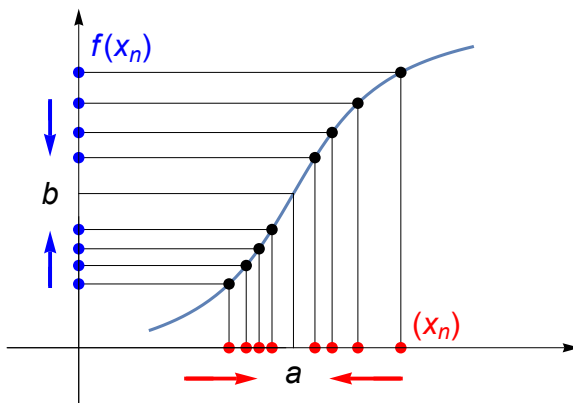


The sequential criterion for the limit of a function

Theorem. Suppose $f : D_f \subset \mathbb{R} \rightarrow \mathbb{R}$ is a function, $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, and $a \in D_f'$.

Then the following two statements are equivalent.

- (1) $\lim_{x \rightarrow a} f(x) = b$
- (2) For all sequences $(x_n) \subset D_f \setminus \{a\}$ for which $x_n \rightarrow a$, $\lim_{n \rightarrow \infty} f(x_n) = b$.



Proof. We prove it for $a, b \in \mathbb{R}$.

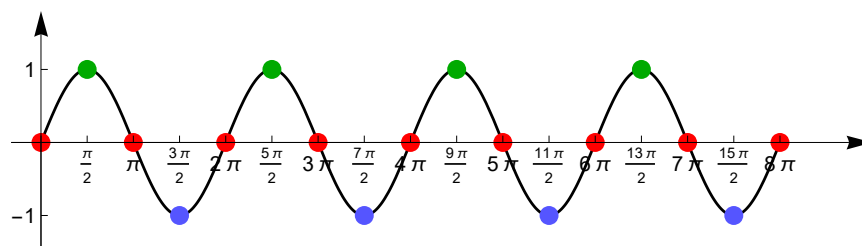
- (1) \implies (2):
- Assume that for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $0 < |x - a| < \delta(\varepsilon)$ then $|f(x) - b| < \varepsilon$.
 - Let (x_n) be a sequence for which $x_n \in D_f \setminus \{a\}$ for all $n \in \mathbb{N}$ and $x_n \rightarrow a$.
 - Then for $\delta(\varepsilon) > 0$ there exists a threshold index $N(\delta(\varepsilon)) \in \mathbb{N}$ such that if $n > N(\delta(\varepsilon))$ then $|x_n - a| < \delta(\varepsilon)$.
 - Thus for all $n > N(\delta(\varepsilon))$, $|f(x_n) - b| < \varepsilon$ also holds, so $f(x_n) \rightarrow b$.
- (2) \implies (1):
- Indirectly, assume that (2) holds but $\lim_{x \rightarrow a} f(x) \neq b$, that is, there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists $x \in D_f$ for which $0 < |x - a| < \delta$ and $|f(x) - b| \geq \varepsilon$.
 - Let $\delta_n = \frac{1}{n} > 0$ for all $n \in \mathbb{N}^+$. Then for δ_n there exists $x_n \in D_f$ such that $0 < |x_n - a| < \delta_n = \frac{1}{n}$ and $|f(x_n) - b| \geq \varepsilon$.
 - It means that $x_n \rightarrow a$, but $\lim_{n \rightarrow \infty} f(x_n) \neq b$, which is a contradiction, so $\lim_{x \rightarrow a} f(x) = b$.

Remark. The theorem is useful for problems where we prove that the limit doesn't exist.

Examples

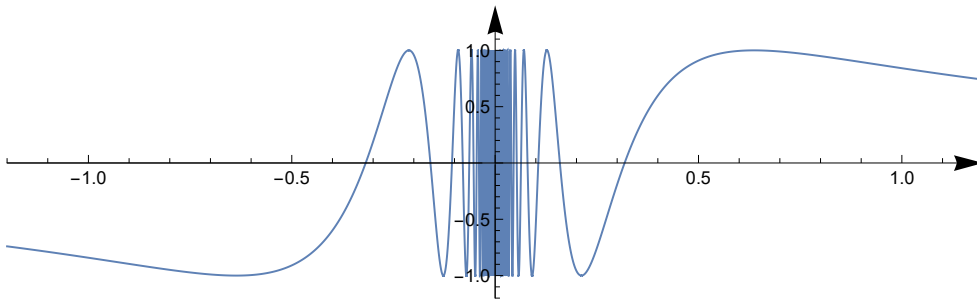
1. Show that the limit $\lim_{x \rightarrow \infty} \sin(x)$ does not exist.

Solution. We give two different sequences tending to infinity such that the sequence of the corresponding function values have different limits. For example:



- 1) If $a_n = \frac{\pi}{2} + n \cdot 2\pi$, then $a_n \rightarrow \infty$ and $\sin(a_n) = 1 \rightarrow 1$.
- 2) If $b_n = n \cdot \pi$, then $b_n \rightarrow \infty$ and $\sin(b_n) = 0 \rightarrow 0$.
- 3) If $c_n = \frac{3\pi}{2} + n \cdot 2\pi$, then $c_n \rightarrow \infty$ and $\sin(c_n) = -1 \rightarrow -1$. $\implies \lim_{x \rightarrow \infty} \sin(x)$ doesn't exist.

2. Let $f(x) = \sin\left(\frac{1}{x}\right)$, $D_f = \mathbb{R} \setminus \{0\}$. Show that f does not have a limit at 0.



Example. Let $x_n = \frac{1}{n\pi} \rightarrow 0$ and $y_n = \frac{1}{\frac{\pi}{2} + 2n\pi} \rightarrow 0$. Then

- $\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} \sin\left(\frac{1}{x_n}\right) = \lim_{x \rightarrow \infty} \sin(n\pi) = 0$ and
- $\lim_{n \rightarrow \infty} f(y_n) = \lim_{n \rightarrow \infty} \sin\left(\frac{1}{y_n}\right) = \lim_{n \rightarrow \infty} \sin\left(\frac{\pi}{2} + 2n\pi\right) = 1 \neq 0 \implies \lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$ doesn't exist.

Consequences

Theorem. Suppose $x_0 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is a limit point of $D_f \cap D_g$ and $\lim_{x \rightarrow x_0} f(x) = A \in \mathbb{R}$,

$\lim_{x \rightarrow x_0} g(x) = B \in \mathbb{R}$, $c \in \mathbb{R}$. Then

- (1) $\lim_{x \rightarrow x_0} (cf)(x) = c \cdot A$
- (2) $\lim_{x \rightarrow x_0} (f \pm g)(x) = A \pm B$
- (3) $\lim_{x \rightarrow x_0} (f \cdot g)(x) = A \cdot B$
- (4) $\lim_{x \rightarrow x_0} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$ if $B \neq 0$
- (5) If $\lim_{x \rightarrow x_0} f(x) = 0$ and g is bounded in a neighbourhood of x_0 then $\lim_{x \rightarrow x_0} (fg)(x) = 0$.

Remark. The statements (1)-(4) are also true if $A, B \in \overline{\mathbb{R}}$ and the corresponding operations are defined in $\overline{\mathbb{R}}$.

Theorem. Suppose $x_0 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is a limit point of $D_f \cap D_g$ and

$\lim_{x \rightarrow x_0} f(x) = A \in \overline{\mathbb{R}}$, $\lim_{x \rightarrow x_0} g(x) = B \in \overline{\mathbb{R}}$.

If $f(x) \leq g(x)$ for all $x \in D_f \cap D_g$ then $A \leq B$.

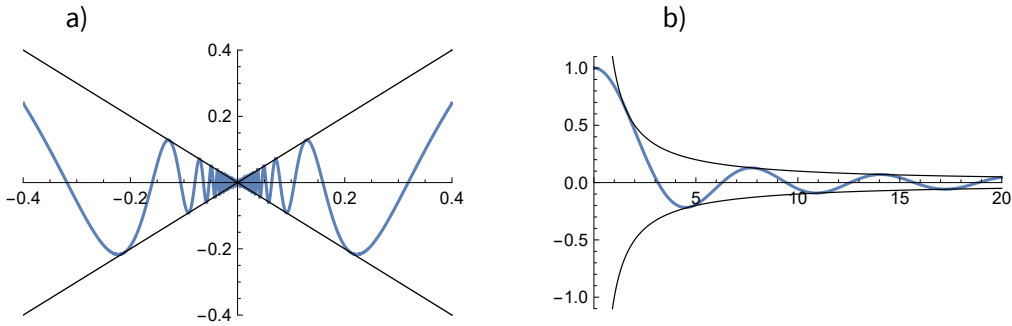
Theorem (Sandwich theorem for limits). Suppose that

- (1) $x_0 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is a limit point of $D_f \cap D_g \cap D_h$,
- (2) $f(x) \leq g(x) \leq h(x)$ for all x in a neighbourhood of x_0 and
- (3) $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} h(x) = b \in \overline{\mathbb{R}}$.

Then $\lim_{x \rightarrow x_0} g(x) = b$.

Remark. The theorem is also true for one-sided limits and if $b = \pm\infty$ then only one estimation is enough.

Example. Show that a) $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$ and b) $\lim_{x \rightarrow \infty} \frac{1}{x} \sin(x) = 0$.



Solution.

a) $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$, since $-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$, and $\lim_{x \rightarrow 0} (|x|) = \lim_{x \rightarrow 0} (-|x|) = 0$

Or: $x \rightarrow 0$ and $\sin\left(\frac{1}{x}\right)$ is bounded, so the product also tends to 0.

b) $\lim_{x \rightarrow \infty} \frac{\sin(x)}{x} = 0$, since $-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$ if $x > 0$, and $\lim_{x \rightarrow \infty} \left(-\frac{1}{x}\right) = \lim_{x \rightarrow \infty} \left(\frac{1}{x}\right) = 0$.

Or: $\frac{1}{x} \rightarrow 0$ and $\sin(x)$ is bounded, so the product also tends to 0.

Example

Theorem. $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

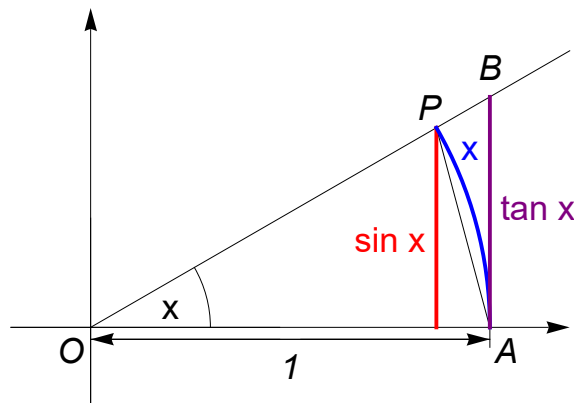
Proof. Since $f(x) = \frac{\sin x}{x}$ is even, it is enough to consider the right-hand limit $\lim_{x \rightarrow 0^+} \frac{\sin x}{x}$.

Let $0 < x < \frac{\pi}{2}$.

The area of the POA triangle is $T_1 = \frac{1 \cdot \sin x}{2}$.

The area of the POA circular sector is $T_2 = \frac{1^2 \cdot x}{2}$.

The area of the OAB triangle is $T_3 = \frac{1 \cdot \tan x}{2}$.



Obviously $T_1 < T_2 < T_3 \implies \frac{1 \cdot \sin x}{2} < \frac{1^2 \cdot x}{2} < \frac{1 \cdot \tan x}{2}$.

Multiplying both sides by $\frac{2}{\sin x} > 0$: $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$.

Since $\lim_{x \rightarrow 0^+} \frac{1}{\cos x} = 1$ then $\lim_{x \rightarrow 0^+} \frac{x}{\sin x} = 1 \implies \lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1 = \lim_{x \rightarrow 0^-} \frac{\sin x}{x}$

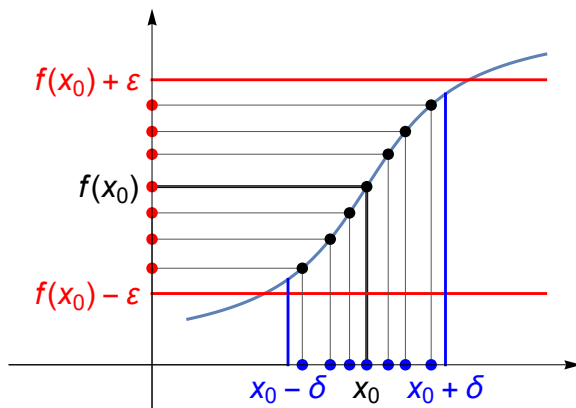
Remark. If $0 < x < \frac{\pi}{2}$, then $\sin x < x \implies |\sin x| \leq |x| \quad \forall x \in \mathbb{R}$.

Continuity

Definition. The function $f : D_f \subset \mathbb{R} \rightarrow \mathbb{R}$ is $\begin{cases} \text{continuous} \\ \text{continuous from the left} \\ \text{continuous from the right} \end{cases}$ at the point $x_0 \in D_f$ if

for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x \in D_f$ and $\begin{cases} |x - x_0| < \delta(\varepsilon) \\ x_0 - \delta(\varepsilon) < x \leq x_0 \\ x_0 \leq x < x_0 + \delta(\varepsilon) \end{cases}$

then $|f(x) - f(x_0)| < \varepsilon$.



Remarks. 1) f is continuous at $x_0 \in D_f \iff$ for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in (B(x_0, \delta) \cap D_f)$ then $f(x) \in B(f(x_0), \varepsilon)$.

2) f is $\begin{cases} \text{continuous from the right} \\ \text{continuous from the left} \end{cases}$ at $x_0 \in D_f \iff \begin{cases} f|_{D_f \cap [x_0, \infty)} \\ f|_{D_f \cap (-\infty, x_0]} \end{cases}$ is continuous at x_0 .

3) f is continuous at $x_0 \in D_f \iff f$ is continuous at x_0 from the right and from the left.

Theorem. Suppose $f : D_f \subset \mathbb{R} \rightarrow \mathbb{R}$ and $x_0 \in D_f \cap D_f'$. Then f is continuous at x_0 if and only if $\lim_{x \rightarrow x_0} f(x)$ exists and $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.

Definition. f is continuous if f is continuous for all $x \in D_f$.

Notation. If $A \subset \mathbb{R}$ then $C(A, \mathbb{R})$ or $C(A)$ denotes the set of continuous functions $f : A \rightarrow \mathbb{R}$.

For example, $f \in C([a, b])$ means that $f : [a, b] \rightarrow \mathbb{R}$ is continuous.

The sequential criterion for continuity

Theorem: The function $f : D_f \subset \mathbb{R} \rightarrow \mathbb{R}$ is continuous at $x_0 \in D_f$ if and only if for all sequences $(x_n) \subset D_f$ for which $x_n \rightarrow x_0$, $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$.

Consequences

Theorem. If f and g are continuous at $x_0 \in D_f \cap D_g$ then cf , $f \pm g$ and f/g is continuous at x_0 ($c \in \mathbb{R}$).
If $g(x_0) \neq 0$ then $\frac{f}{g}$ is also continuous at x_0 .

Theorem (Sandwich theorem for continuity): Suppose that

- (1) there exists $\delta > 0$ such that $I = (x_0 - \delta, x_0 + \delta) \subset D_f \cap D_g \cap D_h$
- (2) f and h are continuous at x_0
- (3) $f(x_0) = h(x_0)$
- (4) $f(x) \leq g(x) \leq h(x)$ for all $x \in I$

Then g is continuous at x_0 .

Definition. The composition of the functions f and g is $(f \circ g)(x) = f(g(x))$ whose domain is $D_{f \circ g} = \{x \in D_g : g(x) \in D_f\}$.

Theorem. If g is continuous at $x_0 \in D_g$ and f is continuous at $g(x_0) \in D_f$ then $f \circ g$ is continuous at x_0 .

Theorem (Limit of a composition). Let a be a limit point of $D_{f \circ g}$ for which $\lim_{x \rightarrow a} g(x) = b$.

Assume that

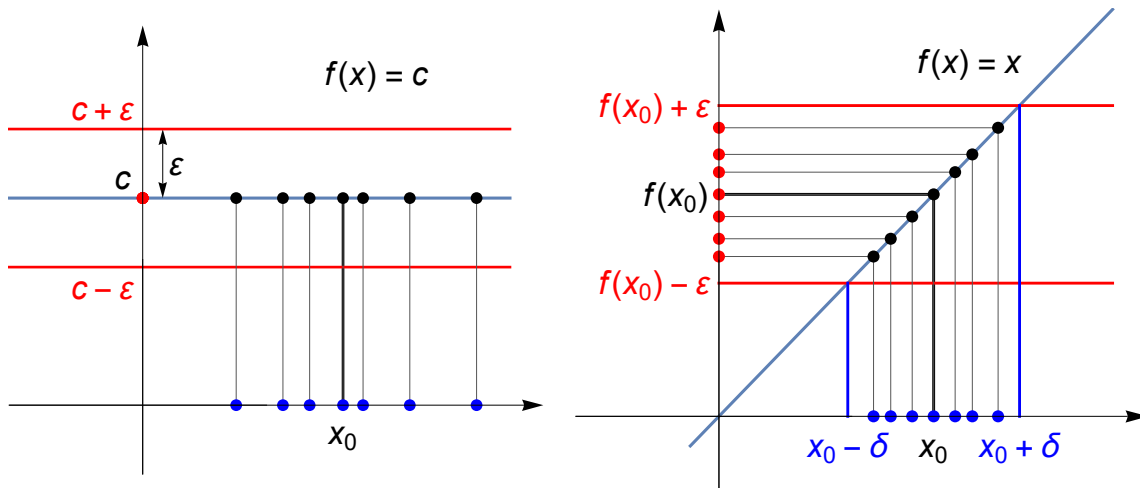
- (1) $b \in D_f$, f is continuous at b and $f(b) = c$ **or**
- (2) $b \in D_f' \setminus D_f$ and $\lim_{x \rightarrow b} f(x) = c$ **or**
- (3) g is injective, $b \in D_f'$ and $\lim_{x \rightarrow b} f(x) = c$.

Then $\lim_{x \rightarrow a} (f \circ g)(x) = c$.

Examples

1. Show that the constant function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = c$ is continuous for all $x_0 \in \mathbb{R}$.

Solution. Let $\varepsilon > 0$, then with any $\delta > 0$ if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| = |c - c| = 0 < \varepsilon$.



2. Show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x$ is continuous for all $x_0 \in \mathbb{R}$.

Solution. Let $\varepsilon > 0$, then with $\delta(\varepsilon) = \varepsilon$ if $|x - x_0| < \delta(\varepsilon) = \varepsilon$, then $|f(x) - f(x_0)| = |x - x_0| < \varepsilon$.

3. $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^n$ is continuous for all $x_0 \in \mathbb{R}$, $n \in \mathbb{N}$, since

$$f(x) = x^n = x \cdot x \cdot \dots \cdot x \rightarrow x_0 \cdot x_0 \cdot \dots \cdot x_0 = x_0^n = f(x_0)$$

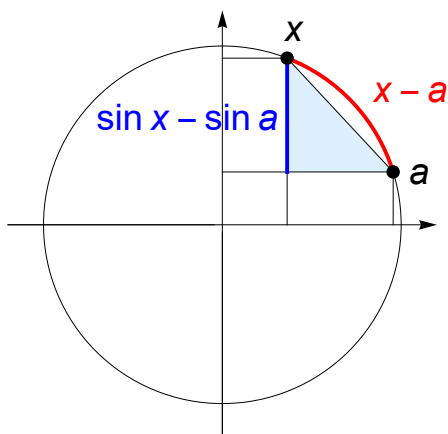
4. Polynomials ($P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, $a_i \in \mathbb{R}$) are continuous for all $x_0 \in \mathbb{R}$.

5. $f(x) = \sin x$ and $g(x) = \cos x$ are continuous for all $x \in \mathbb{R}$.

Proof. We show that $f(x) = \sin x$ is continuous at $a \in \mathbb{R}$. Let $x \in \mathbb{R}$, $x \neq a$ and consider the right-angled triangle with vertices $(\cos a, \sin a)$, $(\cos x, \sin x)$, $(\cos x, \sin a)$. Then the lengths of the legs are less than the length of the hypotenuse, which is less than the arc length $x - a$, that is,

$$|\sin x - \sin a| \leq |x - a|.$$

If $\varepsilon > 0$ and $\delta = \varepsilon$ then for all $x \in \mathbb{R}$ for which $|x - a| < \delta$ we have that $|f(x) - f(a)| = |\sin x - \sin a| \leq |x - a| < \varepsilon$, so f is continuous at a .



6. Investigate the continuity of the following functions:

a) the sign function or signum function: $\operatorname{sgn}(x) = \begin{cases} 1, & \text{ha } x > 0 \\ 0, & \text{ha } x = 0 \\ -1, & \text{ha } x < 0 \end{cases}$

b) the floor function: $f(x) = [x]$, where $[x] = \max\{k \in \mathbb{Z} : k \leq x\}$

c) the fractional part function: $f(x) = \{x\} = x - [x]$

Solution. a) $\lim_{x \rightarrow 0^+} \operatorname{sgn}(x) = 1 \neq \operatorname{sgn}(0) = 0 \implies f(x) = \operatorname{sgn}(x)$ is not continuous at 0 from the right

(and similarly not continuous at 0 from the left) $\implies f$ is not continuous at 0.

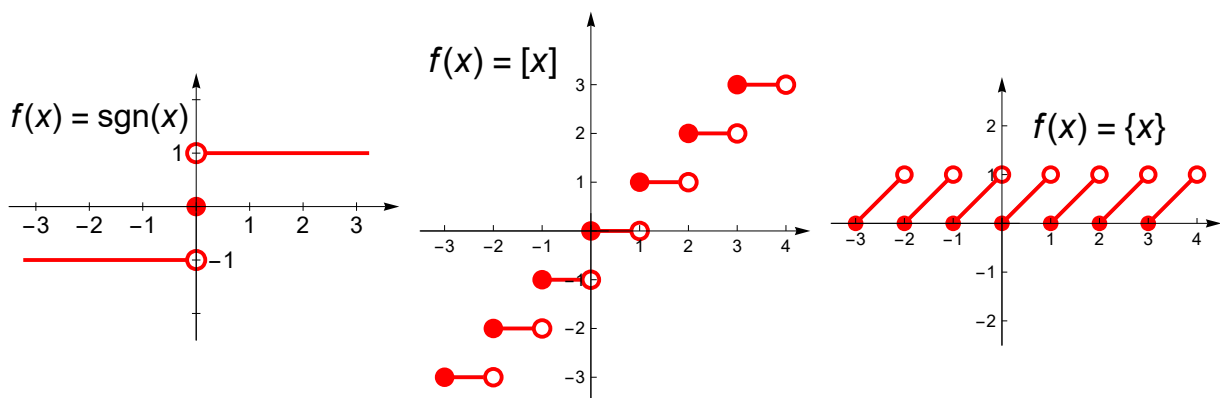
If $x \neq 0$ then f is continuous at x .

b) If $k \in \mathbb{Z}$ then $\lim_{x \rightarrow k-0} [x] = k - 1$, $\lim_{x \rightarrow k+0} [x] = k = [k]$

$\implies f(x) = [x]$ is continuous at k from the right but not from the left.

c) If $k \in \mathbb{Z}$ then $\lim_{x \rightarrow k-0} \{x\} = 1$, $\lim_{x \rightarrow k+0} \{x\} = \{k\} = 0$

$\implies f(x) = \{x\}$ is continuous at k from the right but not from the left.



7. $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is continuous for all $x \in \mathbb{R}$.

8. Show that the **Dirichlet function** $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ is not continuous at any $x \in \mathbb{R}$.

Solution. • If $x_0 \in \mathbb{Q}$, then let $x_n \in \mathbb{R} \setminus \mathbb{Q} \forall n$ such that $x_n \rightarrow x_0$. Then $f(x_n) = 0 \rightarrow 0 \neq 1 = f(x_0)$.

• If $x_0 \in \mathbb{R} \setminus \mathbb{Q}$, then let $x_n \in \mathbb{Q} \forall n$ such that $x_n \rightarrow x_0$. Then $f(x_n) = 1 \rightarrow 1 \neq 0 = f(x_0)$.

9. Show an example for a function $f: \mathbb{R} \rightarrow \mathbb{R}$ that is continuous only at one point.

Solution. Let $f(x) = \begin{cases} x, & \text{ha } x \in \mathbb{Q} \\ -x, & \text{ha } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Then f is continuous only at 0

Since $f(x) = |x|$ for all $x \in \mathbb{R}$, then

$$x_n \rightarrow 0 \iff |x_n| \rightarrow 0 \iff |f(x_n)| \rightarrow 0 \iff f(x_n) \rightarrow 0.$$

Similar examples: $f(x) = \begin{cases} x, & \text{ha } x \in \mathbb{Q} \\ 0, & \text{ha } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$, $f(x) = \begin{cases} x, & \text{ha } x \in \mathbb{Q} \\ 2x, & \text{ha } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ etc.

Types of discontinuities

Definition. We say that the function f is **discontinuous** at $x_0 \in \mathbb{R}$ or f has a discontinuity at $x_0 \in \mathbb{R}$ if x_0 is a limit point of D_f and f is not continuous at x_0 .

Classification of discontinuities:

1) Discontinuity of the first kind:

a) f has a **removable discontinuity** at x_0 if $\exists \lim_{x \rightarrow x_0} f(x) \in \mathbb{R}$ but $\lim_{x \rightarrow x_0} f(x) \neq f(x_0)$ or $f(x_0)$ is not defined.

b) f has a **jump discontinuity** at x_0 if $\exists \lim_{x \rightarrow x_0^-} f(x) \in \mathbb{R}$ and $\exists \lim_{x \rightarrow x_0^+} f(x) \in \mathbb{R}$ but $\lim_{x \rightarrow x_0^-} f(x) \neq \lim_{x \rightarrow x_0^+} f(x)$.

2) Discontinuity of the second kind:

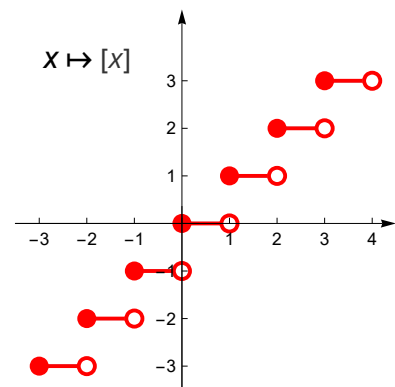
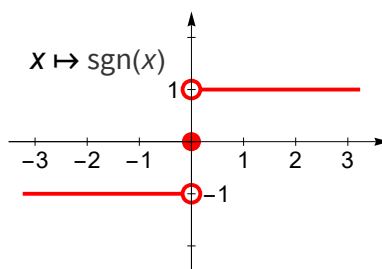
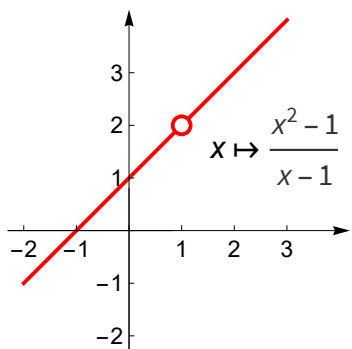
f has an **essential discontinuity** or a discontinuity of the second kind at x_0 if f has a discontinuity at x_0 but not of the first kind.

- Remarks:** 1. In the case of a discontinuity of the first kind, both one-sided limits exist and are finite.
2. In the case of an essential discontinuity, at least one of the one-sided limits doesn't exist or exists but is not finite.

Examples

1. Discontinuity of the first kind

a) $f(x) = \frac{x^2 - 1}{x - 1}$ has a removable discontinuity at $x_0 = 1$.

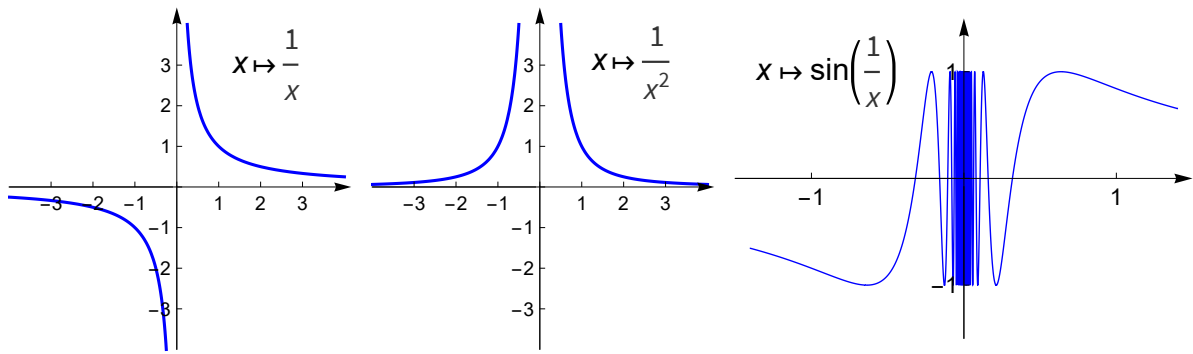


b) $f(x) = \text{sgn}(x)$ has a jump discontinuity at $x = 0$.

c) $f(x) = [x]$ has a jump discontinuity for all $x \in \mathbb{Z}$.

2. Discontinuity of the second kind

a) $f_1(x) = \frac{1}{x}$, $f_2(x) = \frac{1}{x^2}$ and $f_3 = \sin \frac{1}{x}$ have an essential discontinuity at $x = 0$.



b) The Dirichlet function has essential discontinuities for all $x \in \mathbb{R}$.

c) The function $f(x) = e^{\frac{1}{x}}$ has an essential discontinuity at $x = 0$.

- If $x \rightarrow 0+$, then $\frac{1}{x} \rightarrow \infty$, and since $\lim_{x \rightarrow \infty} e^x = \infty$, then $\lim_{x \rightarrow 0+0} e^{\frac{1}{x}} = \infty$.
- If $x \rightarrow 0-$, then $\frac{1}{x} \rightarrow -\infty$, and since $\lim_{x \rightarrow -\infty} e^x = 0$, then $\lim_{x \rightarrow 0-0} e^{\frac{1}{x}} = 0$.

