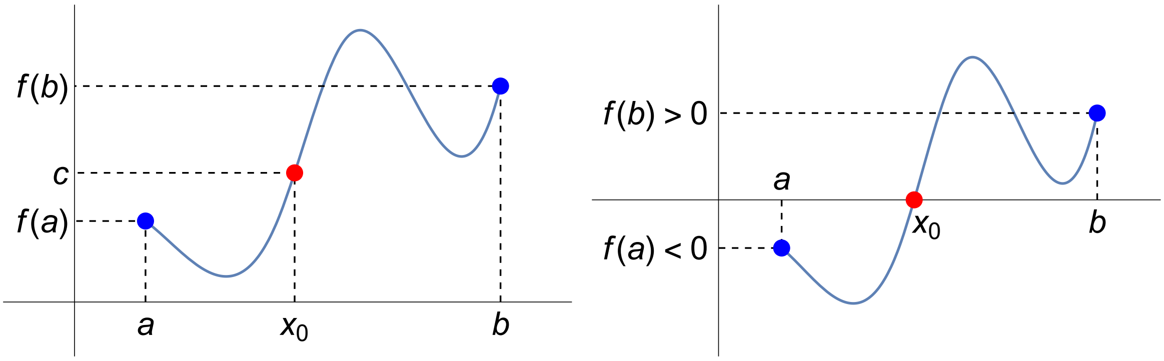
Calculus 1 - 07

Properties of continuous functions

Intermediate value theorem

Theorem (Intermediate value theorem or Bolzano's theorem).
Assume that f is continuous on $[a, b]$, $f(a) \neq f(b)$ and $f(a) < c < f(b)$ or $f(b) < c < f(a)$.
Then there exists $x_0 \in (a, b)$ such that $f(x_0) = c$.



Proof. We prove the case $f(a) < c < f(b)$. The point x_0 can be found with an interval halving method (bisection method).

1st step: Consider the midpoint $\frac{a+b}{2}$ of the interval $[a, b]$. There are three cases:

If $f\left(\frac{a+b}{2}\right) > c \implies a_1 := a,\ b_1 := \frac{a+b}{2}$

If $f\left(\frac{a+b}{2}\right) < c \implies a_1 := \frac{a+b}{2},\ b_1 := b$

If $f\left(\frac{a+b}{2}\right) = c \implies x_0 := \frac{a+b}{2}$

2nd step: Consider the midpoint $\frac{a_1+b_1}{2}$ of the interval $[a_1, b_1]$. There are again three cases:

If $f\left(\frac{a_1+b_1}{2}\right) > c \implies a_2 := a_1,\ b_2 := \frac{a_1+b_1}{2}$

If $f\left(\frac{a_1+b_1}{2}\right) < c \implies a_2 := \frac{a_1+b_1}{2},\ b_2 := b_1$

If $f\left(\frac{a_1+b_1}{2}\right) = c \implies x_0 := \frac{a_1+b_1}{2}$

Continuing the above procedure, we either reach x_0 in one of the steps, or we define the sequences (a_n) and (b_n) such that

$$[a, b] \supset [a_1, b_1] \supset [a_2, b_2] \supset \ldots \supset [a_n, b_n] \supset [a_{n+1}, b_{n+1}] \supset \ldots,$$

and

$$b_1 - a_1 = \frac{b-a}{2},\ b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b-a}{2^2},\ \ldots,\ b_n - a_n = \frac{b-a}{2^n},\ \ldots$$

From this it follows that $\lim_{n\to\infty}(b_n - a_n) = 0$, so by the Cantor axiom there exists a unique element $x_0 \in [a, b]$ such that $\bigcap_{n=1}^{\infty}[a_n, b_n] = \{x_0\}$.

Then $a_n \longrightarrow x_0$, $b_n \longrightarrow x_0$, so by the continuity of f we have that $\lim_{n\to\infty} f(a_n) = f(x_0) = \lim_{n\to\infty} f(b_n)$, and since $f(a_n) \le c \le f(b_n)$, it follows that $f(x_0) = c$.

Consequence 1. (Bolzano's theorem)

Assume that f is continuous on $[a, b]$ and $f(a)\,f(b) < 0$.
Then there exists $x_0 \in (a, b)$ such that $f(x_0) = 0$.

Remark. The above two theorems are equivalent.

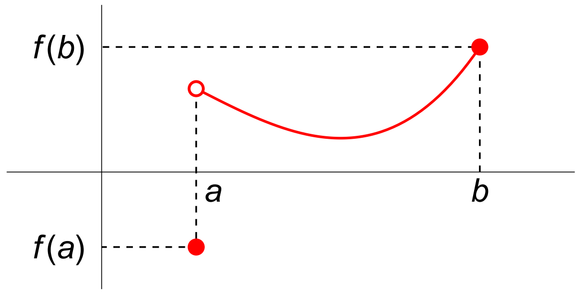
Consequence 2. Every polynomial of odd degree has at least one real root.

Proof: Let $f(x) = a_{2k+1}\,x^{2k+1} + a_{2k}\,x^{2k} + \ldots + a_1\,x + a_0$, and let $a_{2k+1} > 0$.

$\Longrightarrow$
- $\lim_{x\to\infty} f(x) = \infty$, so there exists a number b such that $f(b) > 1$, and
- $\lim_{x\to-\infty} f(x) = -\infty$, so there exists a number a such that $f(a) < -1$.

Since f is a polynomial then it is everywhere continuous, so it is also continuous on the closed interval $[a, b]$ and $f(a)\,f(b) < 0$.
Thus by Consequence 1. there exists $x \in (a, b)$, for which $f(x) = 0$.

Remark. If f is not continuous on the closed interval $[a, b]$ then the theorem is not true, as the following example shows. Here $f(a)$ and $f(b)$ have different signs but f is not continuous at a and f doesn't have a root on the interval (a, b).



Applications

Example 1. Find a real root of the polynomial $f(x) = x^3 + 4\,x^2 - 6\,x - 2$.

Solution. We apply an interval halving method. First we find two numbers a and b such that $f(a)$ and $f(b)$ have opposite signs.

1) $f(0) = -2 < 0$, $f(2) = 10 > 0 \implies f$ has a root in the interval $[0, 2]$.

Bisect the interval and examine the sign of f at $x = \frac{0+2}{2} = 1$.

2) $f(1) = -3 < 0$, $f(2) = 10 > 0 \implies f$ has a root in the interval $[1, 2]$.

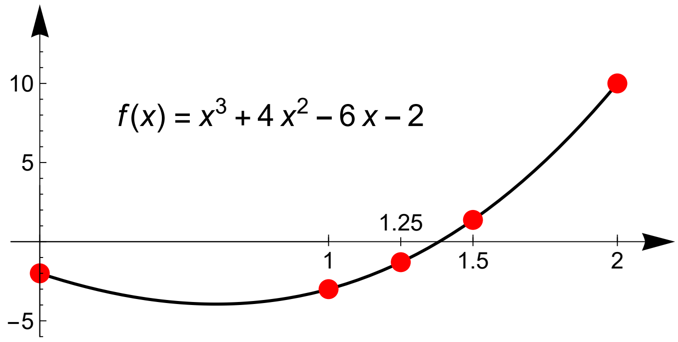
Bisect the interval again and examine the sign of f at $x = \frac{1+2}{2} = 1.5$.

3) $f(1) = -3 < 0$, $f(1.5) = 1.375 > 0 \implies f$ has a root in the interval $[1, 1.5]$.

Bisect the interval again and examine the sign of f at $x = \frac{1+1.5}{2} = 1.25$.

4) $f(1.25) \approx -1.29688 < 0$, $f(1.5) = 1.375 > 0 \implies f$ has a root in the interval $[1.25, 1.5]$.

Continuing the process, the root can be approximated as $\approx 1.38318...$.



Example 2. Show that the equation $2^x = x^2 + \lg(x)$ has a real solution.

Solution. Set the equation to zero and consider the function $f(x) = 2^x - x^2 - \lg(x)$.

We have to show that there exists a real number x such that $f(x) = 0$, that is,
we have to find two numbers a and b such that $f(a)$ and $f(b)$ have opposite signs.
For example

- $f(1) = 2 - 1 - 0 = 1 > 0$
- $f(3) = 8 - 9 - \lg(3) \approx -1.47712 < 0$

$\implies$ by Bolzano's theorem f has a root in the interval $(1, 3)$ and thus the equation has a real solution.

Weierstrass extreme value theorem

Theorem (Weierstrass boundedness theorem).
If f is continuous on $[a, b]$, then f is bounded on $[a, b]$.

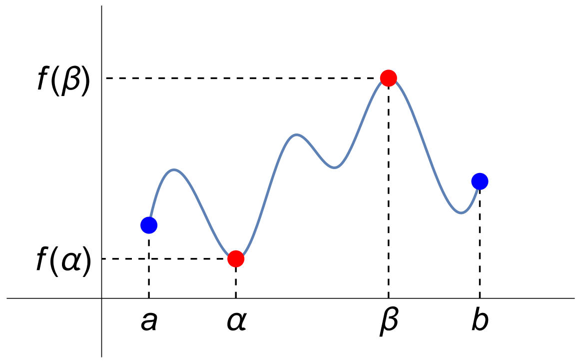
Proof. 1) Indirectly, suppose that for example f is not bounded above.
Then for all $n \in \mathbb{N}$ there exists $x_n \in [a, b]$, such that $f(x_n) > n$.

2) Obviously $x_n \in [a, b]$ for all $n \in \mathbb{N}$, so the sequence (x_n) is bounded, and thus by the Bolzano-Weierstrass theorem there exists a convergent subsequence (x_{n_k}) such that $\lim_{k\to\infty} x_{n_k} = \alpha \in [a, b]$.

3) Since f is continuous at α and $x_{n_k} \overset{k\to\infty}{\longrightarrow} \alpha$ then $\lim_{k\to\infty} f(x_{n_k}) = f(\alpha)$, so the sequence $(f(x_{n_k}))$ is bounded.

4) Since the index sequence (n_k) is strictly monotonically increasing, then $n_k \geq k$ $\Longrightarrow$ $f(x_{n_k}) > n_k \geq k$ for all $k \in \mathbb{N}$ $\Longrightarrow$ the sequence $(f(x_{n_k}))$ is not bounded above (it diverges to $+\infty$). This is a contradiction, so f is bounded above on $[a, b]$.

Theorem (Weierstrass extreme value theorem).
If f is continuous on the closed interval $[a, b]$ then
there exist numbers $\alpha \in [a, b]$ and $\beta \in [a, b]$, such that
$f(\alpha) \leq f(x) \leq f(\beta)$ for all $x \in [a, b]$,
that is, f has both a minimum and a maximum on $[a, b]$.



Proof. 1) Let $A = f([a, b]) = \{f(x) : x \in [a, b]\}$.
By the previous theorem A is bounded, so by the least-upper-bound property of the real numbers, $\exists \sup A := M \in \mathbb{R}$. We prove that $\exists\, \beta \in [a, b]$, such that $f(\beta) = M$.

2) Since M is the **least** upper bound, then for all $n \in \mathbb{N}$, $M - \frac{1}{n}$ is not an upper bound for A, so $\exists\, x_n \in [a, b]$ such that $f(x_n) > M - \frac{1}{n}$.
Since M is an upper bound for A, we have $M - \frac{1}{n} < f(x_n) \leq M$ for all $n \in \mathbb{N}$.

3) The sequence $(x_n) \subset [a, b]$ is bounded, so by the Bolzano-Weierstrass theorem there exists a convergent subsequence (x_{n_k}) such that $\lim_{k\to\infty} x_{n_k} = \beta \in [a, b]$.

4) Then $M - \frac{1}{n_k} < f(x_{n_k}) \leq M$ for all $k \in \mathbb{N}$. Since $\frac{1}{n_k} \overset{k\to\infty}{\longrightarrow} 0$, then by the sandwich theorem $f(x_{n_k}) \overset{k\to\infty}{\longrightarrow} M$.

5) Since f is continuous at β and $x_{n_k} \overset{k\to\infty}{\longrightarrow} \beta$ then $\lim_{k\to\infty} f(x_{n_k}) = f(\beta)$.
The limit is unique, so $f(\beta) = M$.

6) The existence of $\alpha \in [a, b]$ can be proved similarly.

Remark. If f is not continuous or if the interval is not compact, then the theorem is not true.

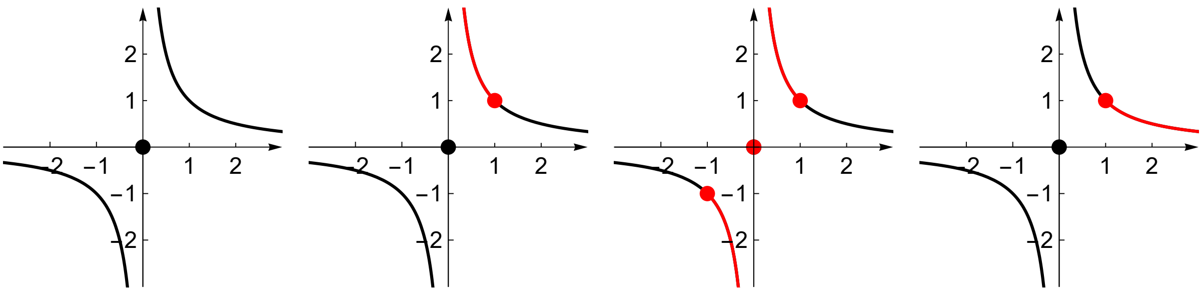
For example, let $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ and investigate f on the following intervals.

a) The interval $(0, 1]$ is bounded but **not closed**. f is continuous here but not bounded above and thus it doesn't have a maximum.

b) The interval $[-1, 1]$ is compact, but f is **not continuous** here and doesn't have a minimum and a maximum.

c) The interval $[1, \infty)$ is **not bounded**. f is continuous here, but doesn't have a minimum.

1) $f: \mathbb{R} \longrightarrow \mathbb{R}$ 2) $f: (0, 1] \longrightarrow \mathbb{R}$ 3) $f: [-1, 1] \longrightarrow \mathbb{R}$ 4) $f: [1, \infty) \longrightarrow \mathbb{R}$



Remark. It follows from the intermediate value theorem and the extreme value theorem that if f is continuous on $[a, b]$, then the range of f is a closed and bounded interval: $f([a, b]) = [c, d]$, where $c = \min\{f(x) : x \in [a, b]\}$ and $d = \max\{f(x) : x \in [a, b]\}$.

Uniform continuity

Introduction. Recall that $f : H \subset \mathbb{R} \longrightarrow \mathbb{R}$ is continuous on H if f is continuous for all $x \in H$, that is, $\forall\, x \in H \quad \forall\, \varepsilon > 0 \quad \exists\, \delta > 0$ such that $\forall\, y \in H, \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. Here $\delta = \delta(\varepsilon, x)$, that is, continuity at a point is a local property. In some cases δ can be chosen independent of x.

Definition. The function $f : E \subset \mathbb{R} \longrightarrow \mathbb{R}$ is uniformly continuous on the set $H \subset E$, if

$$\forall\; \varepsilon > 0 \quad \exists\, \delta > 0 \quad \text{such that } \forall\, x, y \in H : \quad |x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(x)| < \varepsilon.$$

Remarks. a) Here δ depends only on ε and not on x.

b) The definition implies that $\exists \inf_{x \in H} \delta(\varepsilon, x) > 0$.

c) H is usually an interval.

d) If f is uniformly continuous on the interval I (open or closed) and $J \subset I$ then f is uniformly continuous on J. The same δ is suitable for J.

e) If f is uniformly continuous on H then f is continuous for all $x \in H$.

Example. Let $f(x) = x^2$.

a) Prove that f is continuous for all $x_0 \in [1, 2]$.

b) Does there exist $\inf_{x_0 \in [1,2]} \delta(\varepsilon, x_0) > 0$, that is,

does there exist a $\delta(\varepsilon)$ that is suitable for all $x_0 \in [1, 2]$?

Is f uniformly continuous on $[1, 2]$?

c) If f uniformly continuous on $(1, 2)$?

d) Is f uniformly continuous on $(1, \infty)$?

Solution. a) $|f(x) - f(x_0)| = |x^2 - x_0^2| = |x - x_0| \cdot |x + x_0| = |x - x_0| \cdot (\boldsymbol{x} + x_0) <$

$< |x - x_0| \cdot (\boldsymbol{x_0 + 1} + x_0) < \varepsilon$ if $\left| x - x_0 \right| < \dfrac{\boldsymbol{\varepsilon}}{\boldsymbol{2 x_0 + 1}} = \boldsymbol{\delta(\varepsilon, x_0)}$

b) $\delta(\varepsilon, x_0) = \dfrac{\varepsilon}{2 x_0 + 1} \overset{x_0 \in [1,2]}{\geq} \dfrac{\varepsilon}{2 \cdot 2 + 1} = \dfrac{\varepsilon}{5} = \delta(\varepsilon, 2)$,

this is a common $\delta(\varepsilon)$ that is suitable for all $x \in [1, 2]$,

so f is uniformly continuous on $[1, 2]$.

c) Yes, $\delta(\varepsilon, 2)$ is also suitable here, see Remark d).

d) f is not uniformly continuous on $(1, \infty)$.

Let $x_n = n + \dfrac{1}{n} \longrightarrow \infty$ and $y_n = n \longrightarrow \infty$. Then $x_n - y_n = \dfrac{1}{n} \longrightarrow 0$, that is, the terms get arbitrarily close to each other if n is large enough, but

$|f(x_n) - f(y_n)| = \left| \left(n + \dfrac{1}{n}\right)^2 - n^2 \right| = 2 + \dfrac{1}{n^2} > 2$,

so if $\varepsilon < 2$ then there is no suitable δ.

Another choice: $x_n = \sqrt{n + 1}$, $y_n = \sqrt{n}$.

Example. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Solution. Let $\varepsilon > 0$. If $\delta = \varepsilon^2$ and $|x - y| < \delta$ then

$|f(x) - f(y)| = \left| \sqrt{x} - \sqrt{y} \right| = \sqrt{\left| \sqrt{x} - \sqrt{y} \right| \cdot \left| \boldsymbol{\sqrt{x} - \sqrt{y}} \right|} \leq$

$\leq \sqrt{\left| \sqrt{x} - \sqrt{y} \right| \cdot \left| \boldsymbol{\sqrt{x} + \sqrt{y}} \right|} = \sqrt{|x - y|} < \sqrt{\delta} = \varepsilon$.

Example. Let $f(x) = \dfrac{1}{x}$. Prove that

a) f is uniformly continuous on $[1, \infty)$;

b) f is not uniformly continuous on $(0, 1)$.

Solution. a) $|f(x) - f(y)| = \left| \dfrac{1}{x} - \dfrac{1}{y} \right| = \dfrac{|x - y|}{\boldsymbol{x y}} \leq \dfrac{|x - y|}{\boldsymbol{1 \cdot 1}} = |x - y| < \varepsilon = \delta$.

b) $|f(x) - f(y)| = \left| \dfrac{1}{x} - \dfrac{1}{y} \right| = \dfrac{|x - y|}{x y} < \varepsilon$ if $|x - y| < \varepsilon x y$,

but $\delta(y) = \varepsilon x y \longrightarrow 0$ if $y \longrightarrow 0$, so there is no common δ that is independent of y.

For example, if $x_n = \dfrac{1}{n}$ and $y_n = \dfrac{1}{n + 1}$ then $x_n - y_n = \dfrac{1}{n} - \dfrac{1}{n + 1} = \dfrac{1}{n(n + 1)} \longrightarrow 0$, but

$|f(x_n) - f(y_n)| = |n - (n + 1)| = 1$, so if $\varepsilon < 1$ then there is no suitable δ.

Theorem (Heine). If f is continuous on the compact set H then f is uniformly continuous on H.

Proof. 1) Indirectly assume that f is not uniformly continuous on K, that is,

$\exists\, \varepsilon > 0$ such that $\forall\, \delta > 0$ $\exists\, x, y \in H$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

2) Let $\delta = \frac{1}{n}$ for all $n \in \mathbb{N}^+$.

Then for this δ $\exists\, x_n, y_n \in H$ such that $\left| x_n - y_n \right| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon$.

3) Since H is compact, then by the Bolzano-Weierstrass theorem the sequence $(x_n) \subset H$ has a convergent subsequence whose limit belongs to H, that is, there is an index sequence (n_k) such that (x_{n_k}) is convergent and $\lim_{k\to\infty} x_{n_k} = \alpha \in H$.

4) We show that with the same index sequence (n_k), the sequence (y_{n_k}) is also convergent and $\lim_{k\to\infty} y_{n_k} = \alpha$. For all $n \in \mathbb{N}^+$ we have

$$\left| y_{n_k} - \alpha \right| \leq \left| y_{n_k} - x_{n_k} \right| + \left| x_{n_k} - \alpha \right| < \frac{1}{n_k} + \left| x_{n_k} - \alpha \right|$$

Since $\frac{1}{n_k} \xrightarrow{k\to\infty} 0$ and $\left| x_{n_k} - \alpha \right| \xrightarrow{k\to\infty} 0$ then their sum also tends to 0, so $\left| y_{n_k} - \alpha \right| \xrightarrow{k\to\infty} 0$.

5) Since $x_{n_k} \xrightarrow{k\to\infty} \alpha$ and $y_{n_k} \xrightarrow{k\to\infty} \alpha$ and f is continuous at $\alpha \in H$, then $f(x_{n_k}) \xrightarrow{k\to\infty} f(\alpha)$ and $f(y_{n_k}) \xrightarrow{k\to\infty} f(\alpha)$, from where $\lim_{k\to\infty} (f(x_{n_k}) - f(y_{n_k})) = f(\alpha) - f(\alpha) = 0$,

however, this is a contradiction, since for all $n \in \mathbb{N}^+$ $|f(x_n) - f(y_n)| \geq \varepsilon$.

It means that the indirect assumption is false, so the statement of the theorem is true.

Theorem. If f is continuous on $[a, \infty)$ and $\exists \lim_{x\to\infty} f(x) = A \in \mathbb{R}$ then f is uniformly continuous on $[a, \infty)$.

Lipschitz continuity

Definition. The function f is **Lipschitz continuous** on the set A if there exists $L \geq 0$ (Lipschitz constant), such that $|f(x) - f(y)| \leq L\, |x - y|$ for all $x, y \in A$.

Theorem. If f is Lipschitz continuous on A, then f is uniformly continuous on A.

Proof. a) If $L = 0$ then δ can be arbitrary, f is constant, so it is uniformly continuous.

b) If $L > 0$ then let $\delta = \frac{\varepsilon}{L}$. If $\left| x - y \right| < \frac{\varepsilon}{L}$ for all $x, y \in A$, then

$\left| f(x) - f(y) \right| < L \left| x - y \right| \leq L \cdot \frac{\varepsilon}{L} = \varepsilon$.

Remark. The converse of the theorem is not true.

For example $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$ but not Lipschitz continuous.

Let $x = 0$, $y > 0$ and $L > 0$. Then

$$\left|\sqrt{y} - \sqrt{x}\right| \leq L\left|y - x\right| \iff \sqrt{y} \leq L \cdot y \iff \frac{1}{L^2} \leq y$$

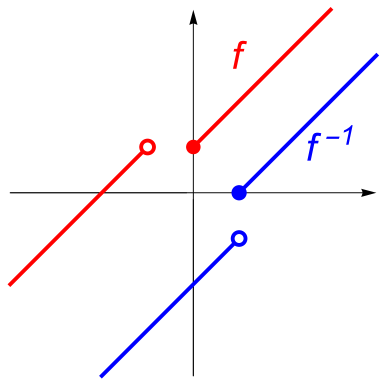
It means that there is no positive number that is less than $\frac{1}{L^2}$, which is a contradiction.

Remark. f is Lipschitz continuous on $A \implies f$ is uniformly continuous on $A \implies f$ is continuous on A.

Continuity of the inverse function

Definition. The function f is **invertible** if for all $x, y \in D_f$, $x \neq y \implies f(x) \neq f(y)$.
(Or, equivalently, for all $x, y \in D_f$: $(f(x) = f(y) \implies x = y)$).
The inverse function f^{-1} of f is defined as follows:
$D_{f^{-1}} = R_f$ and $\left(f^{-1} \circ f\right)(x) = x$ for all $x \in D_f$.

Remark. If f is invertible and continuous at x_0 then from this it doesn't follow that f^{-1} is continuous at $f(x_0)$. For example, the function $f(x) = \begin{cases} x+1 & \text{if } x \geq 0 \\ x+2 & \text{if } x < -1 \end{cases}$ is invertible.
If we express x from the equation $y = f(x)$, then we get that the inverse of f is
$f^{-1}(y) = \begin{cases} y-1 & \text{if } y \geq 1 \\ y-2 & \text{if } y < 1 \end{cases} \implies f$ is continuous but f^{-1} is not continuous.



Theorem. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and strictly monotonic.
Then f^{-1} is continuous on R_f.

Proof. 1) Since f is continuous on $[a, b]$ then it follows from the intermediate value theorem and extreme value theorem that the range of f is a closed and bounded interval.
Let $[c, d] = R_f$.
Since f is strictly monotonic then it is bijective, so it has an inverse, $f^{-1} : [c, d] \longrightarrow [a, b]$.

2) Let $v \in [c, d]$ arbitrary, $u := f^{-1}(v)$ and assume that $(y_n) \subset [c, d]$, $y_n \longrightarrow v$ is an arbitrary sequence. To prove the continuity of f^{-1} at v, it is enough to show that
$x_n := f^{-1}(y_n) \longrightarrow f^{-1}(v) = u$.

3) Assume indirectly that the sequence $(x_n) \subset [a, b]$ does not tend to u.
Then $\exists\, \delta > 0\ \ \forall\, k \in \mathbb{N}\ \ \exists\, n_k > k$, such that $\left|x_{n_k} - u\right| \geq \delta$.

4) Since the sequence $(x_{n_k}) \subset [a, b] \setminus (u - \delta, u + \delta)$ is bounded, then it has a convergent

subsequence $(x_{n_{k_l}})$. Let $\lim\limits_{l\to\infty} x_{n_{k_l}} = \alpha$. Obviously $\alpha \in [a, b]$, but $\alpha \neq u$.

5) Since f is continuous at α then $f(x_{n_{k_l}}) = y_{n_{k_l}} \longrightarrow f(\alpha)$.
Since $y_n \overset{n\to\infty}{\longrightarrow} v$ and $(y_{n_{k_l}})$ is a subsequence of (y_n), then $y_{n_{k_l}} \longrightarrow v$, so $f(\alpha) = v$.

6) We obtained that $\alpha \neq u$, but $f(\alpha) = f(u) = v$, which means that f is not bijective.
This is a contradiction, so the indirect assumption is false.
Therefore, $x_n \longrightarrow u$ and thus f^{-1} is continuous at v.

Convexity and continuity

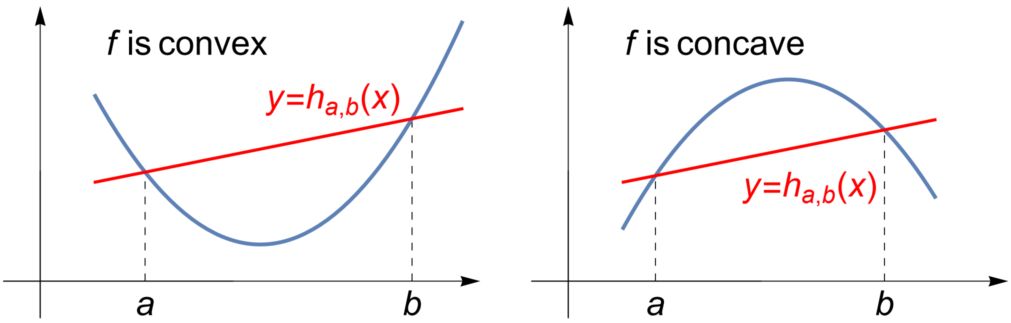
Definition. The function f is **convex** on the interval $I \subset D_f$ if for all $x, y \in I$ and $t \in [0, 1]$

$$f(t\,x + (1-t)\,y) \le t\,f(x) + (1-t)\,f(y)$$

The function f is **concave** on the interval $I \subset D_f$ if for all $x, y \in I$ and $t \in [0, 1]$

$$f(t\,x + (1-t)\,y) \ge t\,f(x) + (1-t)\,f(y).$$

f is strictly convex / strictly concave if equality doesn't hold.



Remark. Let $a, b \in I$, then the secant line passing through the points $(a, f(a))$ and $(b, f(b))$ is

$$h_{a,b}(x) = \frac{f(b) - f(a)}{b - a}\,(x - a) + f(a).$$

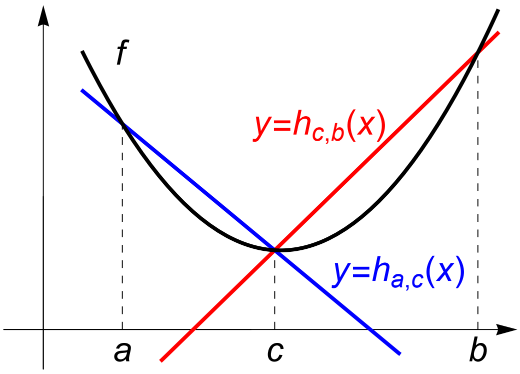
The function f is $\begin{cases} \text{convex} \\ \text{concave} \end{cases}$ on the interval $I \subset D_f$ if

$\forall\, a, b \in I,\ \ a < x < b \implies \begin{cases} f(x) \le h_{a,b}(x) \\ f(x) \ge h_{a,b}(x) \end{cases}$, that is, the secant lines of f

always lie $\begin{cases} \text{above} \\ \text{below} \end{cases}$ the graph of f.

Theorem. If f is convex on the open interval I, then f is continuous on I.

Proof. Let $a, b, c \in I$ such that $a < c < b$. If $x \in (c, b)$, then $h_{a,c} \le f(x) \le h_{c,b}(x)$.
Since $\lim\limits_{x\to c+} h_{a,c}(x) = \lim\limits_{x\to c+} h_{c,b}(x) = f(c)$, then by the sandwich theorem $\lim\limits_{x\to c+} f(x) = f(c)$,
and similarly $\lim\limits_{x\to c-} f(x) = f(c)$.



Remark. If *f* is convex on a closed interval, then *f* can be discontinuous only at the endpoints of the interval.