Calculus 1 - 10

Indefinite integrals

Antiderivative

Definition. If F is differentiable on the interval I and $F'(x) = f(x)$ for all $x \in I$
F is an **antiderivative** or **primitive function** of f.

Example. If $f(x) = \sin x \cos x = \frac{1}{2} \sin 2x$ then

$F(x) = \frac{\sin^2 x}{2}$ and $G(x) = -\frac{\cos 2x}{4}$ are primitive functions of f on $\mathbb{R}$

since $F'(x) = G'(x) = f(x)$.

Theorem. If F and G are antiderivatives of f on the interval I then there exists
$c \in \mathbb{R}$ such that $F(x) = G(x) + c$.

Proof. F and G are antiderivatives of $f \iff F' = G' \iff (F - G)' = 0 \iff$
$\exists\, c \in \mathbb{R}: F - G = c \iff \exists\, c \in \mathbb{R}: F(x) = G(x) + c \;\; \forall\, x \in I.$

Remark. This theorem holds only on an interval.

Definition. If f has an antiderivative then the set of antiderivatives of f is called
the **indefinite integral** of f:

$$\int f(x)\,\mathrm{dx} = \{H : H'(x) = f(x)\; \forall\, x \in I\} = F(x) + c$$

Remark. Let $F(x) = \begin{cases} \ln x & \text{if } x > 0 \\ 4 + \ln(-x) & \text{if } x < 0 \end{cases}$ and $G(x) = \begin{cases} 3 + \ln x & \text{if } x > 0 \\ -2 + \ln(-x) & \text{if } x < 0 \end{cases}$.

$\implies F'(x) = G'(x) = \frac{1}{x} \quad \forall\, x \in \mathbb{R} \setminus \{0\} = H$

$\implies F$ and G are antiderivatives of $f(x) = \frac{1}{x}$ on the set H, however,

$F(x) - G(x) = \begin{cases} -3 & \text{if } x > 0 \\ 6 & \text{if } x < 0 \end{cases}$, so their difference is not a constant.

It is important, that the above theorem holds only on an interval.

On the contrary, we use the following notation: $\int \frac{1}{x}\,\mathrm{dx} = \ln |x| + c$

It means that $\int \frac{1}{x}\,\mathrm{dx} = \ln x + c$ if $I \subset (0\ \infty)$ and $\int \frac{1}{x}\,\mathrm{dx} = \ln(-x) + c$ if $I \subset (-\infty, 0)$.

Theorem. If f and g have antiderivatives on I and $c \in \mathbb{R}$ then
$f + g,\ f - g$ and $c\,f$ also have antiderivatives on I and

1) $\int (f \pm g) = \int f \pm \int g$
2) $\int c\,f = c \int f$

Basic integrals

$$\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + c \ (\alpha \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int e^x\,dx = e^x + c$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + c \ (0 < a \neq 1)$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x + c$$

$$\int \frac{1}{\sin^2 x}\,dx = -\cot x + c$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c = -\arccos x + c$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + c = -\operatorname{arccot} x + c$$

$$\int \sinh x\,dx = \cosh x + c$$

$$\int \cosh x\,dx = \sinh x + c$$

$$\int \frac{1}{\cosh^2 x}\,dx = \tanh x + c$$

$$\int \frac{1}{\sinh^2 x}\,dx = -\coth x + c$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \operatorname{arsinh} x + c$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \operatorname{arcosh} x + c$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c$$

Integration methods

Theorem. $\int f(a\,x+b)\,dx = \frac{F(a\,x+b)}{a} + c$, where $F' = f$ and $a \neq 0$

Proof. $\left(\frac{F(a\,x+b)}{a} + c\right)' = \frac{1}{a} F'(a\,x+b)\cdot a = f(a\,x+b)$

Example 1. $\int \sqrt{5\,x-8}\,dx = \int (5\,x-8)^{\frac{1}{2}}\,dx = \frac{(5\,x-8)^{\frac{3}{2}}}{\frac{3}{2}\cdot 5} + c$

Example 2. $\int (e^{-x} + \cos 2\,x)\,dx = \frac{e^{-x}}{-1} + \frac{\sin 2\,x}{2} + c$

Example 3. $\int \frac{5}{4\,x^2+1}\,dx = \int 5\cdot\frac{1}{(2\,x)^2+1}\,dx = 5\cdot\frac{\operatorname{arctg}(2\,x)}{2} + c$

Example 4. $\int \frac{5}{4\,x^2+2}\,dx = \int \frac{5}{2}\cdot\frac{1}{2\,x^2+1}\,dx = \int \frac{5}{2}\cdot\frac{1}{\left(\sqrt{2}\,x\right)^2+1}\,dx = \frac{5}{2}\cdot\frac{\operatorname{arctg}\left(\sqrt{2}\,x\right)}{\sqrt{2}} + c$

Theorem. 1) $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

2) $\int f'(x)\,(f(x))^\alpha\,dx = \frac{(f(x))^{\alpha+1}}{\alpha+1} + c, \quad \alpha \neq -1$

Proof. These are consequences of the differentiation rules.

Example 1. $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2\,x}{x^2+1}\,dx = \frac{1}{2}\ln\left(x^2+1\right) + c$

Example 2. $\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} dx = -\ln|\cos x| + c$

Example 3. $\int \cos x \sin^3 x \, dx = \frac{\sin^4 x}{4} + c \quad (f(x) = \sin x, \; f'(x) = \cos x, \; \alpha = 3)$

Example 4. $\int x \sqrt{1+x^2} \, dx = \frac{1}{2} \cdot \int 2x(1+x^2)^{\frac{1}{2}} dx = \frac{1}{2} \cdot \frac{(1+x^2)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{1}{3} \cdot \sqrt{(1+x^2)^3} + c$

Integration by parts

Theorem. Assume that f and g are differentiable on the interval I and $f \cdot g'$ has an antiderivative on I. Then $f' \cdot g$ also has an antiderivative here and

$$\int f'(x) \, g(x) \, dx = f(x) \, g(x) - \int f(x) \, g'(x) \, dx$$

Proof. The right-hand side is differentiable and its derivative is

$$\left(f(x) \, g(x) - \int f(x) \, g'(x) \, dx\right)' = f'(x) \, g(x) + f(x) \, g'(x) - f(x) \, g'(x) = f'(x) \, g(x).$$

Applications

1. $g(x)$ is a polynomial of degree n and
$f'(x) = e^{a x + b}, \; \sin(a x + b), \; \cos(a x + b), \; \sinh(a x + b), \; \cosh(a x + b)$
$\Longrightarrow$ the method has to be applied n times

Example 1.

$$\int x \cos 2x \, dx = \frac{\sin 2x}{2} \cdot x - \int \frac{\sin 2x}{2} \cdot 1 \, dx = x \cdot \frac{\sin 2x}{2} + \frac{\cos 2x}{4} + c$$

$f'(x) = \cos 2x \implies f(x) = \frac{\sin 2x}{2}$

$g(x) = x \implies g'(x) = 1$

Example 2.

$$\int x^2 e^x \, dx = e^x \cdot x^2 - \int e^x \cdot 2x \, dx = e^x \cdot x^2 - \left(e^x \cdot 2x - \int e^x \cdot 2 \, dx\right) =$$
$$= e^x \cdot x^2 - e^x \cdot 2x + 2e^x + c$$

$f'(x) = e^x \implies f(x) = e^x$ $\qquad$ $u'(x) = e^x \implies u(x) = e^x$

$g(x) = x^2 \implies g'(x) = 2x$ $\qquad$ $v(x) = 2x \implies v'(x) = 2$

2. $f'(x)$ is a polynomial and
$g(x) = \ln x, \; \arcsin x, \; \arccos x, \; \arctan x, \; \text{arccot}\, x, \; \text{arsinh}\, x, \; \ldots$

Example 1.

$$\int \ln x \, dx = \int 1 \cdot \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int 1 \, dx = x \ln x - x + c$$

$$f'(x) = 1 \Longrightarrow f(x) = x$$

$$g(x) = \ln x \Longrightarrow g'(x) = \frac{1}{x}$$

Example 2.

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$$

$$f'(x) = x \Longrightarrow f(x) = \frac{x^2}{2}$$

$$g(x) = \ln x \Longrightarrow g'(x) = \frac{1}{x}$$

Example 3.

$$\int \operatorname{arctg} x \, dx = \int 1 \cdot \operatorname{arctg} x \, dx = x \cdot \operatorname{arctg} x - \int x \cdot \frac{1}{1 + x^2} \, dx$$

$$= x \cdot \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{1 + x^2} \, dx = x \cdot \operatorname{arctg} x - \frac{1}{2} \ln (1 + x^2) + c$$

$$f'(x) = 1 \Longrightarrow f(x) = x$$

$$g(x) = \operatorname{arctg} x \Longrightarrow g'(x) = \frac{1}{1 + x^2}$$

$$\int \frac{f'(x)}{f(x)} \, dx = \ln | f(x) | + c$$

Example 4.

$$\int \arcsin x \, dx = \int 1 \cdot \arcsin x \, dx = x \cdot \arcsin x - \int x \cdot \frac{1}{\sqrt{1 - x^2}} \, dx$$

$$= x \cdot \arcsin x - \int x \cdot (1 - x^2)^{-\frac{1}{2}} \, dx = x \cdot \arcsin x + \frac{1}{2} \cdot \int (-2x) (1 - x^2)^{-\frac{1}{2}} \, dx =$$

$$= x \cdot \arcsin x + \frac{1}{2} \frac{(1 - x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c = x \arcsin x + \sqrt{1 - x^2} + c$$

$$f'(x) = 1 \Longrightarrow f(x) = x$$

$$g(x) = \arcsin x \Longrightarrow g'(x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\int f'(x) \cdot (f(x))^\alpha \, dx = \frac{(f(x))^{\alpha+1}}{\alpha + 1} + c, \ \alpha \neq 0$$

3. $f'(x)$ and $g(x)$ are both one of the following functions:
$e^{a x + b}$, $\sin(a x + b)$, $\cos(a x + b)$, $\sinh(a x + b)$, $\cosh(a x + b)$
$\Longrightarrow$ the method has to be applied twice

Example. $I = \int e^x \sin x \, dx = ?$

$$I = \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

$f'(x) = e^x \implies f(x) = e^x$
$g(x) = \sin x \implies g'(x) = \cos x$

$I = \int e^x \sin x \, dx = e^x(-\cos x) - \int e^x(-\cos x) \, dx$

$f'(x) = \sin x \implies f(x) = -\cos x$
$g(x) = e^x \implies g'(x) = e^x$

(1) $I = e^x \sin x - \int e^x \cos x \, dx$
(2) $I = e^x(-\cos x) + \int e^x \cos x \, dx$

$$(1) + (2) \implies 2I = e^x \sin x - e^x \cos x \implies I = \frac{1}{2} e^x(\sin x - \cos x) + c$$

Powers of sin *x* and cos *x*

Odd powers of sin *x* and cos *x*:
$\sin^{2n+1} x = \sin x \cdot (\sin^2 x)^n = \sin x \cdot (1 - \cos^2 x)^n = \ldots$
$\cos^{2n+1} x = \cos x \cdot (\cos^2 x)^n = \cos x \cdot (1 - \sin^2 x)^n = \ldots$

Even powers of sin *x* and cos *x*:
$\sin^{2n} x = (\sin^2 x)^n = \left(\frac{1 - \cos 2x}{2}\right)^n + \ldots$
$\cos^{2n} x = (\cos^2 x)^n = \left(\frac{1 + \cos 2x}{2}\right)^n + \ldots$

Example 1.

$$I = \int \sin^3 x \, dx = \int \sin x \sin^2 x \, dx = \int \sin x \left(1 - \cos^2 x\right) dx =$$
$$= \int \left(\sin x - \sin x \cos^2 x\right) dx = -\cos x + \frac{\cos^3 x}{3} + c$$
$$\int f'(x) \cdot (f(x))^\alpha \, dx = \frac{(f(x))^{\alpha+1}}{\alpha+1} + c, \quad \alpha \neq 0$$

Example 2.

$$I = \int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int \left(1 - \sin^2 x\right)^2 \cos x \, dx =$$
$$= \int \left(1 - 2\sin^2 x + \sin^4 x\right) \cos x \, dx = \int \left(\cos x - 2\sin^2 x \cos x + \sin^4 x \cos x\right) dx$$
$$= \sin x - \frac{2}{3} \sin^3 x + \frac{\sin^5 x}{5} + c$$
$$\int f'(x) \cdot (f(x))^\alpha \, dx = \frac{(f(x))^{\alpha+1}}{\alpha+1} + c, \quad \alpha \neq 0$$

Example 3.

$$I = \int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$$

Example 4.

$$I = \int \cos^4 x \, dx = \int \left(\cos^2 x\right)^2 dx = \int \left(\frac{1 + \cos 2x}{2}\right)^2 dx =$$

$$= \int \frac{1}{4} \cdot \left(1 + 2\cos 2x + \cos^2 2x\right) dx =$$

$$\cos^2 2x = \frac{1 + \cos 4x}{2}$$

$$= \int \left(\frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x\right) dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$$

Products of powers of sin x and cos x:

$$\int \sin^n x \cos^m x \, dx = ? \quad n, m \in \mathbb{N}^+$$

1. If n or m is odd:

$$\int \sin^3 x \cos^4 x \, dx = \int \sin x \sin^2 x \cos^4 x \, dx = \int \sin x \left(1 - \cos^2 x\right) \cos^4 x \, dx =$$

$$= \int \sin x \left(\cos^4 x - \cos^6 x\right) dx = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + c$$

2. If n and m are even:

$$\int \sin^2 x \cos^4 x \, dx = \int \left(1 - \cos^2 x\right) \cos^4 x \, dx = \int \left(\cos^4 x - \cos^6 x\right) dx = \ldots$$

Integrals of rational functions

Polynomial division

Definition: If $p_1(x)$ and $p_2(x)$ are polynomials then the function $R(x) = \frac{p_1(x)}{p_2(x)}$ is called a rational function.

Statement. Polynomials can be divided in the following sense:

$p_1(x) = q(x)\, p_2(x) + r(x)$

where $\deg r < \deg p_2$. If $\deg p_1 < \deg p_2$ then $q = 0$ and $r = p_1$.

The polynomial q is the quotient and r is the remainder.

Example 1. Divide $p_1(x) = x^3 - 4x^2 + 5x - 8$ by $p_2(x) = x^2 + 3$

$(x^3 - 4x^2 + 5x - 8) \; : (x^2 + 3) = x - 4$ ($x^3 : x^2 = x$, then multiply $p_2(x)$ by x^2)

$-(x^3 \quad + 3x)$

$0 - 4x^2 + 2x - 8$ ($-4x^2 : x^2 = -4$, then multiply $p_2(x)$ by -4)

$-(-4x^2 \quad - 12)$

$0 + 2x + 4$ $\Longrightarrow$ the remainder is $2x + 4$, since $\deg(2x + 4) < \deg\left(x^2 + 3\right)$

$$\Longrightarrow x^3 - 4x^2 + 5x - 8 = \left(x^2 + 3\right)(x - 4) + (2x + 4)$$

$$\Longrightarrow \frac{x^3 - 4x^2 + 5x - 8}{x^2 + 3} = (x - 4) + \frac{2x + 4}{x^2 + 3}$$

Example 2. Divide $p_1(x) = x^4$ by $p_2(x) = x^2 + x - 2$

$x^4 \quad : \left(x^2 + x - 2\right) = x^2 - x + 3$ ($x^4 : x^2 = x^2$, then multiply $p_2(x)$ by x^2)

$-(x^4 + x^3 - 2x^2)$ (the product is subtracted from the line above)

$0 - x^3 + 2x^2$ ($-x^3 : x^2 = -x$, then multiply $p_2(x)$ by $-x$)

$-(-x^3 - x^2 + 2x)$

$0 + 3x^2 - 2x$ ($3x^2 : x^2 = 3$, then multiply $p_2(x)$ by 3)

$-(3x^2 + 3x - 6)$

$0 - 5x + 6$ $\Longrightarrow$ the remainder is $-5x + 6$, since $\deg(-5x+6) < \deg(x^2 - x + 3)$

$$\Longrightarrow x^4 = (x^2 + x - 2)(x^2 - x + 3) + (-5x + 6)$$

$$\Longrightarrow \frac{x^4}{x^2 + x - 2} = (x^2 - x + 3) + \frac{-5x + 6}{x^2 + x - 2}$$

Integration of rational functions

1st step. If $R(x) = \frac{T(x)}{Q(x)}$ and $\deg T(x) \geq \deg Q(x)$ then with polynomial division we bring it to the form $R(x) = E(x) + \frac{P(x)}{Q(x)}$, where $E(x)$ is a polynomial and $\deg P(x) < \deg Q(x)$.

2nd step. The denominator can be written as

$$Q(x) = (x - a_1)^{\alpha_1} \ldots (x - a_r)^{\alpha_r} (x^2 + b_1 x + c_1)^{\beta_1} \ldots (x^2 + b_s x + c_s)^{\beta_s}$$

where $b_i^2 - 4c_i < 0$ and $\alpha_1, \ldots, \alpha_r$ are the multiplicities of the real roots and $\beta_1, \ldots, \beta_s$ are the multiplicities of the complex roots.

3rd step. Partial fraction decomposition. It means that to each term in the above form of $Q(x)$ we assign an elementary fraction (or partial fraction) such that the sum of these fractions is equal to $\frac{P(x)}{Q(x)}$. This decomposition is unique.

	Factor in the denominator	Term in the partial fraction decomposition
Single real root:	$x - a$	$\frac{A}{x-a}$
Multiple real root:	$(x-a)^k$	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_k}{(x-a)^k}$
Single complex roots $(b^2 - 4c < 0)$:	$x^2 + bx + c$	$\frac{Bx + C}{x^2 + bx + c}$
Multiple complex roots $(b^2 - 4c < 0)$:	$(x^2 + bx + c)^k$	$\frac{B_1 x + C_1}{x^2 + bx + c} + \frac{B_2 x + C_2}{(x^2 + bx + c)^2} + \ldots + \frac{B_k x + C_k}{(x^2 + bx + c)^k}$

4th step. We integrate the polynomial $E(x)$ and the partial fractions term by term.

Integration of the elementary fractions

(1) $\int \frac{A}{x-1} \, dx = A \ln | x - a | + c$

(2) $k \in \mathbb{N}$, $k > 1$: $\int \frac{A}{(x-1)^k}\,dx = \frac{-k\,A}{(x-1)^{k-1}} + c$

(3) $b^2 - 4c < 0$: $\int \frac{B\,x + C}{x^2 + b\,x + c}\,dx = \int \frac{B}{2}\frac{2\,x + b}{x^2 + b\,x + c}\,dx + \int \frac{-\frac{B\,b}{2} + C}{x^2 + b\,x + c}\,dx$

(i) $\int \frac{2\,x + b}{x^2 + b\,x + c}\,dx = \ln(x^2 + bx + c) + \text{constant}$

(ii) $\int \frac{1}{x^2 + b\,x + c}\,dx = \int \frac{1}{\left(x + \frac{b}{2}\right)^2 + d^2}\,dx = \frac{1}{d^2}\int \frac{1}{\left(\frac{x+\frac{b}{2}}{d}\right)^2 + 1}\,dx = \frac{1}{d^2}\,\frac{\arctan\left(\frac{x+\frac{b}{2}}{d}\right)}{\frac{1}{d}} + \text{constant}$

where $d = \sqrt{c - \frac{b^2}{4}}$.

(4) $b^2 - 4c < 0$, $k \in \mathbb{N}$, $k > 1$: $\int \frac{B\,x + C}{(x^2 + b\,x + c)^k}\,dx = \int \frac{B}{2}\frac{2\,x + b}{(x^2 + b\,x + c)^k}\,dx + \int \frac{-\frac{B\,b}{2} + C}{(x^2 + b\,x + c)^k}\,dx$

(i) $\int \frac{2\,x + b}{(x^2 + b\,x + c)^k}\,dx = \frac{1}{1-k}\,\frac{1}{(x^2 + b\,x + c)^{k-1}} + \text{constant}$

(ii)* $\int \frac{1}{(x^2 + b\,x + c)^k}\,dx = \int \frac{1}{\left(\left(x + \frac{b}{2}\right)^2 + d^2\right)^k}\,dx$, where $d = \sqrt{c - \frac{b^2}{4}}$

If $\int \frac{1}{(x^2 + 1)^n}\,dx = F_n(x) + c$ then $\int \frac{1}{\left(\left(x + \frac{b}{2}\right)^2 + d^2\right)^k}\,dx = \frac{1}{d^{2n-1}}\,F_n\left(\frac{x}{d} + \frac{b}{2\,d}\right) + c.$

Remark. If $\int \frac{1}{(x^2 + 1)^n}\,dx = F_n(x) + c$ then for $n \geq 1$: $F_{n+1}(x) = \frac{1}{2\,n}\,\frac{x}{(x^2 + 1)^n} + \frac{2\,n - 1}{2\,n}\,F_n(x) + c.$

For example, $F_1(x) = \arctan x + c$

$$F_2(x) = \frac{1}{2}\,\frac{x}{x^2 + 1} + \frac{1}{2}\arctan x + c$$

$$F_3(x) = \frac{1}{4}\,\frac{x}{x^2 + 1} + \frac{3}{8}\,\frac{x}{x^2 + 1} + \frac{3}{8}\arctan x + c$$

Examples

Example 1. $I = \int \frac{x + 1}{x^2 + 3\,x}\,dx = ?$

Solution. The denominator has two distinct real roots: $x_1 = 0$, $x_2 = -3$.

Partial fraction decomposition:

$$\frac{x + 1}{x^2 + 3\,x} = \frac{x + 1}{x(x + 3)} = \frac{A}{x} + \frac{B}{x + 3} = \frac{A(x + 3) + B\,x}{x(x + 3)}$$

$\Longrightarrow x + 1 = A(x + 3) + B\,x$

We want to find A and B such that this equation holds **for all** $x \in \mathbb{R}$.

1st method (comparison of the coefficients)

$x+1=(A+B)\,x+3A$ holds **for all** $x\in\mathbb{R}$ if and only if

$$\begin{matrix} A+B=1 \\ 3A=1 \end{matrix} \implies A=\frac{1}{3},\ B=\frac{2}{3}$$

2nd method (substitution) (it is worth substituting the roots of the denominator)

$$x+1=A(x+3)+B\,x$$

if $x=0 \implies 1=A\cdot 3+B\cdot 0 \implies A=\frac{1}{3}$

if $x=-3 \implies -2=A\cdot 0+B\cdot(-3) \implies B=\frac{2}{3}$

$$\implies I=\int\frac{x+1}{x^2+3x}\,\mathrm{dx}=\int\left(\frac{1}{3}\cdot\frac{1}{x}+\frac{2}{3}\cdot\frac{1}{x+3}\right)\mathrm{dx}=\frac{1}{3}\ln|x|+\frac{2}{3}\ln|x+3|+c$$

Example 2. $I=\int\frac{x+5}{x^2+6x+9}\,\mathrm{dx}=?$

Solution. The denominator has multiple real roots: $x_{1,2}=-3$.

Partial fraction decomposition:

$$\frac{x+5}{x^2+6x+9}=\frac{x+5}{(x+3)^2}=\frac{A}{x+3}+\frac{B}{(x+3)^2}=\frac{A(x+3)+B}{(x+3)^2}$$

$$\implies x+5=A(x+3)+B$$

1st method: $x+5=A\,x+(3A+B) \implies \begin{matrix} A=1 \\ 3A+B=5 \end{matrix} \implies A=1,\ B=2$

2nd method: $x+5=A(x+3)+B$

$x=-3 \implies 2=0+B \implies B=2$

$x=0 \implies 5=3A+B \implies A=1$

$$\implies I=\int\frac{x+5}{x^2+6x+9}\,\mathrm{dx}=\int\left(\frac{1}{x+3}+\frac{2}{(x+3)^2}\right)\mathrm{dx}=\int\left(\frac{1}{x+3}+2\,(x+3)^{-2}\right)\mathrm{dx}=$$

$$=\ln|x+3|+2\,\frac{(x+3)^{-1}}{-1}+c=\ln|x+3|-\frac{2}{x+3}+c$$

Example 3. $I=\int\frac{1}{(x-1)^2\,(x^2+1)}\,\mathrm{dx}=?$

Solution: The roots of the denominator are:

$x_{1,2}=1$ (multiple real roots), $x_{3,4}=\pm i$ (simple complex roots)

Partial fraction decomposition:

$$\frac{1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} = \frac{A(x-1)(x^2+1)+B(x^2+1)+(Cx+D)(x-1)^2}{(x-1)^2(x^2+1)}$$

$\Longrightarrow\ 1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$

Substitutions:

$x = 1 \Longrightarrow\ 1 = 0 + B + 0 \Longrightarrow\ B = 1$

$x = 0 \Longrightarrow\ 1 = -A + B + D$

$x = -1 \Longrightarrow\ 1 = -4A + 2B - 4C + 4D$

$x = 2 \Longrightarrow\ 1 = 5A + 5B + 2C + D$

Homework: the solution of this equation system is

$C = \frac{1}{2},\ D = 0,\ A = -\frac{1}{2},\ B = \frac{1}{2}$

$$\Longrightarrow I = \int \frac{1}{(x-1)^2(x^2+1)}\,dx = \int\left(-\frac{1}{2}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{(x-1)^2} + \frac{1}{2}\cdot\frac{x}{x^2+1}\right)dx =$$

$$= \int\left(-\frac{1}{2}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot(x-1)^{-2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2x}{x^2+1}\right)dx = \left(\text{the last term has the form } \frac{f'}{f}\right)$$

$$= -\frac{1}{2}\ln\left|x-1\right| + \frac{1}{2}\cdot\frac{(x-1)^{-1}}{-1} + \frac{1}{4}\ln(x^2+1) + c$$

Integration by change of variables

The substitution formula

Theorem. Assume that g is differentiable on the interval I, f is defined on $J = g(I)$ and f has a primitive function on J. Then $(f \circ g)\, g'$ also has a primitive function on I and

$$\int f(g(x))\cdot g'(x)\,dx = F(g(x)) + c \quad \text{where} \quad \int f(x)\,dx = F(x) + c$$

Proof. $(F(g(x)) + c)' = F'(g(x))\cdot g'(x) = f(g(x))\cdot g'(x)$

Remark. If g is invertible then the above formula can be written in the form

$$\int f(g(t))\cdot g'(t)\,dt\,\big|_{t=g^{-1}(x)} = \int f(x)\,dx$$

Examples

Example 1. $I = \int \sin x\, e^{\cos x}\,dx = ?$

Substitution: $t = \cos x \Longrightarrow\ x = x(t) = \arccos t$

$$x'(t) = \frac{dx}{dt} = -\frac{1}{\sqrt{1-t^2}} \Longrightarrow\ dx = -\frac{1}{\sqrt{1-t^2}}\,dt$$

$$\sin x = \sqrt{\sin^2 x} = \sqrt{1-\cos^2 x} = \sqrt{1-t^2}$$

$$I = \int \sin x\, e^{\cos x}\, dx = \int \sqrt{1-t^2}\, e^t \cdot \left(-\frac{1}{\sqrt{1-t^2}}\right) dt = \int -e^t\, dt = -e^t + c = -e^{\cos x} + c$$

Remark. $\int e^{f(x)} \cdot f'(x)\, dx = e^{f(x)} + c$

Example 2. $I = \int x^2 \sin x^3\, dx = ?$

Substitution: $t = x^3 \implies x = x(t) = \sqrt[3]{t} = t^{\frac{1}{3}} \implies x^2 = t^{\frac{2}{3}}$

$$x'(t) = \frac{dx}{dt} = \frac{1}{3} t^{-\frac{2}{3}} \implies dx = \frac{1}{3} t^{-\frac{2}{3}}\, dt$$

$$I = \int x^2 \sin(x^3)\, dx = \int t^{\frac{2}{3}} \sin t \cdot \frac{1}{3} t^{-\frac{2}{3}}\, dt = \int \frac{1}{3} \sin t\, dt = -\frac{1}{3}\cos t + c = -\frac{1}{3}\cos x^3 + c$$

Example 3. $I = \int \sin^4 x \cos dx = ?$

1st solution. $\int f'(x) f^{\alpha}(x)\, dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + c \quad (\alpha \neq -1)$

Here: $f(x) = \sin x,\ f'(x) = \cos x,\ \alpha = 4 \implies$

$$I = \int \sin^4 x \cos x\, dx = \frac{\sin^5 x}{5} + c$$

2nd solution.

Substitution: $t = \sin x \implies x = x(t) = \arcsin t$

$$x'(t) = \frac{dx}{dt} = \frac{1}{\sqrt{1-t^2}} \implies dx = \frac{1}{\sqrt{1-t^2}}\, dt$$

$$\cos x = \sqrt{\cos^2 x} = \sqrt{1-\sin^2 x} = \sqrt{1-t^2}$$

$$I = \int \sin^4 x \cos x\, dx = \int t^4 \sqrt{1-t^2} \cdot \frac{1}{\sqrt{1-t^2}}\, dt = \int t^4\, dt = \frac{t^5}{5} + c = \frac{\sin^5 x}{5} + c$$

Rational functions of e^x

Statement. The integral $\int R(e^x)\, dx$ (where R is a rational function) can be transformed to the integral of a rational function $t = e^x$.

Example 1. $I = \int \frac{e^{2x}}{e^x + 1}\, dx = ?$

Substitution: $t = e^x \implies x = x(t) = \ln t$

$$x'(t) = \frac{dx}{dt} = \frac{1}{t} \implies dx = \frac{1}{t}\, dt$$

$$I = \int \frac{e^{2x}}{e^x + 1}\,dx = \int \frac{t^2}{t+1} \cdot \frac{1}{t}\,dt = \int \frac{t}{t+1}\,dt = \int \frac{(t+1)-1}{t+1}\,dt = \int \left(1 - \frac{1}{t+1}\right) dt =$$
$$= t - \ln|t+1| + c = e^x - \ln(e^x + 1) + c$$

Example 2. $I = \int \frac{4}{e^{2x} - 4}\,dx = ?$

Substitution: $t = e^x \implies x = x(t) = \ln t$

$$x'(t) = \frac{dx}{dt} = \frac{1}{t} \implies dx = \frac{1}{t}\,dt$$

$$I = \int \frac{4}{e^{2x} - 4}\,dx = \int \frac{4}{t^2 - 4} \cdot \frac{1}{t}\,dt = \int \frac{4}{t(t-2)(t+2)}\,dt$$

Partial fraction decomposition:

$$\frac{4}{t(t-2)(t+2)} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-2}$$

$\implies$ $4 = A(t+2)(t-2) + B\,t(t-2) + C\,t(t+2)$

$t = 0$: $4 = -4A + 0 + 0 \implies A = -1$

$t = -2$: $4 = 0 + 8B + 0 \implies B = \frac{1}{2}$

$t = 2$: $4 = 0 + 0 + 8C \implies C = \frac{1}{2}$

$$\implies I = \int \left(-\frac{1}{t} + \frac{1}{2}\frac{1}{t+2} + \frac{1}{2}\frac{1}{t-2}\right) dt = -\ln|t| + \frac{1}{2}\ln|t+2| + \frac{1}{2}\ln|t-2| + c =$$
$$= -\ln e^x + \frac{1}{2}\ln(e^x + 2) + \frac{1}{2}\ln|e^x - 2| + c$$

Some integrals with roots

Remark. In the following cases $R(u, v)$ denotes a two-variable rational function,

that is, $R(u, v) = \frac{P(u, v)}{Q(u, v)}$, $P(u, v) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\,u^i\,v^j$, $Q(u, v) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\,u^i\,v^j$, $n \in \mathbb{N}$.

1. The integral $\int R\left(x, \sqrt[n]{\frac{a\,x + b}{c\,x + d}}\right) dx$ (where $c^2 + d^2 \neq 0$, $a\,d \neq b\,c$) can be transformed to the integral of a rational function with the substitution $t = \sqrt[n]{\frac{a\,x + b}{c\,x + d}}$.

Example 1. $I = \int \frac{1}{x^2} \sqrt[3]{\frac{x+1}{x}}\,dx = ?$

Substitution: $t = \sqrt[3]{\frac{x+1}{x}} \implies x + 1 = t^3\,x$

$$x = x(t) = \frac{1}{t^3 - 1}$$

$$x'(t) = \frac{dx}{dt} = -\frac{3t^2}{(t^3-1)^2} \Longrightarrow dx = -\frac{3t^2}{(t^3-1)^2}\,dt$$

$$I = \int \frac{1}{x^2}\sqrt[3]{\frac{x+1}{x}}\,dx = \int (t^3-1)^2 \cdot t \cdot \frac{-3t^2}{(t^3-1)^2}\,dt = \int -3t^3\,dt = -\frac{3}{4}t^4 + c = -\frac{3}{4}\left(\frac{x+1}{x}\right)^{\frac{4}{3}} + c$$

Example 2. $I = \int \sqrt{\frac{x-3}{x-1}}\,dx = ?$

Substitution: $t = \sqrt{\frac{x-3}{x-1}} \Longrightarrow t^2(x-1) = x-3 \Longrightarrow x(t^2-1) = t^2-3$

$$x = x(t) = \frac{t^2-3}{t^2-1} = \frac{t^2-1-2}{t^2-1} = 1 - \frac{2}{t^2-1}$$

$$x'(t) = \frac{dx}{dt} = \frac{4t}{(t^2-1)^2} \Longrightarrow dx = \frac{4t}{(t^2-1)^2}\,dt$$

$$I = \int \sqrt{\frac{x-3}{x-1}}\,dx = \int t \cdot \frac{4t}{(t^2-1)^2}\,dt = \int \frac{4t^2}{(t-1)^2(t+1)^2}\,dt$$

Partial fraction decomposition: $\frac{4t^2}{(t-1)^2(t+1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$

$\Longrightarrow A = 1,\ B = 1,\ C = -1,\ D = 1$

$$\Longrightarrow I = \int\left(\frac{1}{t-1} + \frac{1}{(t-1)^2} - \frac{1}{t+1} + \frac{1}{(t+1)^2}\right)dt = \ln|t-1| - \frac{1}{t-1} - \ln|t+1| - \frac{1}{t+1} + c$$

$$= -\frac{2t}{t^2-1} + \ln\left|\frac{t-1}{t+1}\right| + c = -\frac{2\sqrt{\frac{x-3}{x-1}}}{\frac{x-3}{x-1}-1} + \ln\left|\frac{\sqrt{\frac{x-3}{x-1}}-1}{\sqrt{\frac{x-3}{x-1}}+1}\right| + c$$

2. The integral $\int R\left(x, \sqrt[n]{ax+b}\right)dx$ can be transformed to the integral of a rational function with the substitution $t = \sqrt[n]{ax+b}$.

Example 1. $I = \int x\sqrt{5x+3}\,dx = ?$

Substitution: $t = \sqrt{5x+3} \Longrightarrow x = x(t) = \frac{t^2-3}{5}$

$$x'(t) = \frac{dx}{dt} = \frac{2}{5}t \Longrightarrow dx = \frac{2}{5}t\,dt$$

$$I = \int x\sqrt{5x+3}\,dx = \int \frac{t^2-3}{5} \cdot t \cdot \frac{2}{5}t\,dt = \int \frac{2}{25}(t^4-3t^2)\,dt = \frac{2}{25}\left(\frac{t^5}{5} - t^3\right) + c =$$

$$= \frac{2}{25}\left(\frac{\sqrt{(5x+3)^5}}{5} - \sqrt{(5x+3)^3}\right) + c$$

Example 2. $I = \int \frac{1}{\sqrt{x}+1}\,dx = ?$

Substitution: $t = \sqrt{x} \Longrightarrow x = x(t) = t^2$

$$x'(t) = \frac{dx}{dt} = 2t \Longrightarrow dx = 2t\,dt$$

$$I = \int \frac{1}{\sqrt{x}+1}\,dx = \int \frac{2t}{t+1}\,dt = \int \frac{2(t+1)-1}{t+1}\,dt = \int 2\cdot\left(1-\frac{1}{t+1}\right)dt =$$
$$= 2t - 2\ln|1+t| + c = 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + c$$

3. In the integral $\int R\left(x, \sqrt{a x^2 + b x + c}\right) dx$ $(a \neq 0)$ after completing the square under the root sign, either of the following substitutions can be used:

$\sqrt{1-A^2}$ $\quad A = \sin t,\ t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (or $A = \cos t,\ t \in [0, \pi]$)

$\sqrt{B^2+1}$ $\quad B = \sinh t$

$\sqrt{C^2-1}$ $\quad C = \cosh t$

Identities for taking the square root: $\cos^2 t + \sin^2 t = 1$, $\cosh^2 t - \sinh^2 t = 1$.

Example 1. $I = \int \sqrt{4-x^2}\,dx = ?$

Substitution: $x = x(t) = 2\sin t \implies t = \arcsin\left(\frac{x}{2}\right)$

$$x'(t) = \frac{dx}{dt} = 2\cos t \implies dx = 2\cos t\,dt$$

$$I = \int \sqrt{4-x^2}\,dx = \int \sqrt{4-4\sin^2 t}\cdot 2\cos t\,dt = \int 2\sqrt{1-\sin^2 t}\cdot 2\cos t\,dt = \int 4\sqrt{\cos^2 t}\cdot\cos t\,dt$$

$$= \int 4\cos^2 t\,dt = \int 4\cdot\frac{1+\cos 2t}{2}\,dt = \int 2\cdot(1+\cos 2t)\,dt = 2\left(t + \frac{\sin 2t}{2}\right) + c =$$
$$= 2t + \sin 2t + c = 2\cdot\arcsin\left(\frac{x}{2}\right) + 2\cdot\frac{x}{2}\sqrt{1-\left(\frac{x}{2}\right)^2} + c$$

Identities:

$\cos^2 x + \sin^2 x = 1 \implies \cos^2 x = \frac{1+\cos 2x}{2},\ \sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x - \sin^2 x = \cos 2x$

$\sin 2t = 2\sin t\cos t = 2\sin t\sqrt{\cos^2 t} = 2\sin t\sqrt{1-\sin^2 t} = 2\cdot\frac{x}{2}\cdot\sqrt{1-\left(\frac{x}{2}\right)^2}$

Example 2. $I = \int \frac{x^2}{\sqrt{9-x^2}}\,dx = ?$

Substitution: $x = 3\sin t \implies t = \arcsin\left(\frac{x}{3}\right)$

$$x'(t) = \frac{dx}{dt} = 3\cos t \implies dx = 3\cos t\,dt$$

$$I = \int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{(3\sin t)^2}{\sqrt{9-(3\sin t)^2}}\cdot 3\cos t\,dt = \int \frac{9\sin^2 t}{\sqrt{9\cdot(1-\sin^2 t)}}\cdot 3\cos t\,dt =$$
$$= \int \frac{9\sin^2 t}{\sqrt{\cos^2 t}}\cos t\,dt = \int \frac{9\sin^2 t}{\cos t}\cos t\,dt = \int 9\sin^2 t\,dt = \int 9\cdot\frac{1-\cos 2t}{2}\,dt =$$

$$= \frac{9}{2}\cdot\left(t-\frac{\sin 2t}{2}\right)+c=\frac{9}{2}\,t-\frac{9}{4}\,\sin 2t+c=\frac{9}{2}\,\arcsin\left(\frac{x}{3}\right)-\frac{9}{4}\cdot 2\cdot\frac{x}{3}\,\sqrt{1-\left(\frac{x}{3}\right)^2}+c=$$

Identities:

$$\sin^2 x=\frac{1-\cos 2x}{2}$$

$$\sin 2t=2\sin t\cos t=2\sin t\,\sqrt{\cos^2 t}=2\sin t\,\sqrt{1-\sin^2 t}=2\cdot\frac{x}{3}\cdot\sqrt{1-\left(\frac{x}{3}\right)^2}$$

Rational functions of sin x and cos x

Statement. The integral $\int R(\sin x, \cos x)\,\mathrm{dx}$ where R is a rational function can be transformed to the integral of a rational function with the substitution

$$t=\tan\frac{x}{2}$$

$$\Longrightarrow\ x=2\arctan t,\ \mathrm{dx}=\frac{2}{1+t^2}\,\mathrm{dt},\ \sin x=\frac{2t}{1+t^2},\ \cos x=\frac{1-t^2}{1+t^2}$$

Remark. $t=\tan\frac{x}{2}\ \Longrightarrow\ x=x(t)=2\arctan t\Longrightarrow\frac{\mathrm{dx}}{\mathrm{dt}}=x'(t)=\frac{2}{1+t^2}$

$$\tan^2 x=\frac{\sin^2 x}{\cos^2 x}=\frac{1-\cos^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}-1\ \Longrightarrow\ \cos^2 x=\frac{1}{1+\tan^2 x}$$

$$\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}=2\tan\frac{x}{2}\cos^2\frac{x}{2}=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2t}{1+t^2}$$

$$\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}=\cos^2\frac{x}{2}\left(1-\tan^2\frac{x}{2}\right)=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$$

Example 1. $I=\int\frac{1}{\sin x}\,\mathrm{dx}=?$

Substitution: $t=\tan\frac{x}{2}\ \Longrightarrow\ \sin x=\frac{2t}{1+t^2},\ \mathrm{dx}=\frac{2}{1+t^2}\,\mathrm{dt}$

$$I=\int\frac{1}{\sin x}\,\mathrm{dx}=\int\frac{1+t^2}{2t}\,\frac{2}{1+t^2}\,\mathrm{dt}=\int\frac{1}{t}\,\mathrm{dt}=\ln|t|+c=\ln\left|\tan\frac{x}{2}\right|+c$$

Example 2. $I=\int\frac{1}{\cos x}\,\mathrm{dx}=?$

Substitution: $t=\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)$

$$I=\int\frac{1}{\cos x}\,\mathrm{dx}=\int\frac{1}{\sin\left(\frac{\pi}{2}-x\right)}\,\mathrm{dx}=-\int\frac{1}{\sin\left(x-\frac{\pi}{2}\right)}\,\mathrm{dx}=-\int\frac{1}{t}\,\mathrm{dt}=-\ln\left|\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)\right|+c$$

Example 3. $I=\int\frac{1}{1+\cos x}\,\mathrm{dx}=?$

Substitution: $t=\tan\frac{x}{2}\ \Longrightarrow\ \cos x=\frac{1-t^2}{1+t^2},\ \mathrm{dx}=\frac{2}{1+t^2}\,\mathrm{dt}$

$$I = \int \frac{1}{1+\cos x}\,dx = \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2}\,dt = \int 1\,dt = t + c = \tan\frac{x}{2} + c$$

Example 4. $I = \int \frac{1}{\sin x(1+\cos x)}\,dx = ?$

Substitution: $t = \tan\frac{x}{2} \Longrightarrow \sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ dx = \frac{2}{1+t^2}\,dt$

$$I = \int \frac{1}{\sin x(1+\cos x)}\,dx = \int \frac{1}{\frac{2t}{1+t^2}\left(1+\frac{1-t^2}{1+t^2}\right)} \frac{2}{1+t^2}\,dt = \int \frac{1+t^2}{2t}\,dt$$

$$= \int\left(\frac{1}{2t} + \frac{t}{2}\right)dt = \frac{1}{2}\ln|t| + \frac{t^2}{4} + c = \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + \frac{1}{4}\tan^2\frac{x}{2} + c$$

Example 5*. $I = \int \frac{1+\sin x}{1-\cos x}\,dx = ?$

Substitution: $t = \tan\frac{x}{2} \Longrightarrow \sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ dx = \frac{2}{1+t^2}\,dt$

$$I = \int \frac{1+\sin x}{1-\cos x}\,dx = \int \frac{1+\frac{2t}{1+t^2}}{1-\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2}\,dt = \int \frac{t^2+2t+1}{t^2(t^2+1)}\,dt =$$

Partial fraction decomposition:

$$\frac{t^2+2t+1}{t^2(t^2+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+1} \Longrightarrow A = 2,\ B = 1,\ C = -2,\ D = 0$$

$$I = \int\left(\frac{2}{t} + \frac{1}{t^2} - \frac{2t}{t^2+1}\right)dt = 2\ln\left|\tan\frac{x}{2}\right| - \frac{1}{\tan\frac{x}{2}} - \ln\left(1+\tan^2\frac{x}{2}\right) + c$$

Remark. For even powers of $\sin x$ and $\cos x$ the following transformations are better:

$$1+\cot^2 x = \frac{\sin^2 x+\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \Longrightarrow \sin^2 x = \frac{1}{1+\cot^2 x}$$

$$1+\tan^2 x = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Longrightarrow \cos^2 x = \frac{1}{1+\tan^2 x}$$

Example 1*. $I = \int \frac{1}{\sin^6 x}\,dx = ?$

$$I = \int \frac{1}{\sin^6 x}\,dx = \int\left(\frac{1}{\sin^2 x}\right)^2 \frac{1}{\sin^2 x}\,dx = \int\left(1+\cot^2 x\right)^2 \frac{1}{\sin^2 x}\,dx$$

Substitution: $y = \cot x \Longrightarrow \frac{dy}{dx} = -\frac{1}{\sin^2 x}$

$$I = -\int(1+y^2)^2\,dy = -\int(1+2y^2+y^4)\,dy = -\left(y + \frac{2}{3}y^3 + \frac{y^5}{5}\right) + c$$

$$= -\left(\cot x + \frac{2}{3}(\cot x)^3 + \frac{(\cot x)^5}{5}\right) + c$$

Example 2*. $I = \int \frac{1}{2 + \sin^2 x}\,\mathrm{dx} = ?$

$$I = \int \frac{1}{2 + \sin^2 x}\,\mathrm{dx} = \int \frac{1}{2 + \frac{1}{1+\cot^2 x}}\,\mathrm{dx} = \int \frac{1 + \cot^2 x}{3 + 2\cot^2 x}\,\mathrm{dx}$$

Substitution: $y = \cot x$

$$x = \operatorname{arccot} y \Longrightarrow \mathrm{dx} = -\frac{1}{1 + y^2}\,\mathrm{dy}$$

$$I = -\int \frac{1 + y^2}{3 + 2y^2}\,\frac{1}{1 + y^2}\,\mathrm{dy} = -\int \frac{1}{3 + 2y^2}\,\mathrm{dy} = -\frac{1}{3}\int \frac{1}{1 + \left(\sqrt{\frac{2}{3}}\,y\right)^2}\,\mathrm{dy}$$

$$= -\frac{1}{3}\,\frac{\arctan\left(\sqrt{\frac{2}{3}}\,y\right)}{\sqrt{\frac{2}{3}}} + c = -\frac{1}{\sqrt{6}}\arctan\left(\sqrt{\frac{2}{3}}\cot x\right) + c$$

Additional examples

Substitution

Example 1. $I = \int \tan^6 x\,\mathrm{dx} = ?$

Substitution: $y = \tan x \Longrightarrow x = x(t) = \arctan y$

$$x'(y) = \frac{\mathrm{dx}}{\mathrm{dy}} = \frac{1}{1 + y^2} \Longrightarrow \mathrm{dx} = \frac{1}{1 + y^2}\,\mathrm{dy}$$

$$I = \int \tan^6 x\,\mathrm{dx} = \int \frac{y^6}{y^2 + 1}\,\mathrm{dy}$$

Here y^6 has to be divided by $y^2 + 1$. This can be done in several ways, for example:

$$\frac{y^6}{y^2 + 1} = \frac{y^4(y^2 + 1) - y^4}{y^2 + 1} = y^4 - \frac{y^4}{y^2 + 1} = y^4 - \frac{y^2(y^2 + 1) - y^2}{y^2 + 1} = y^4 - y^2 + \frac{y^2}{y^2 + 1} =$$

$$= y^4 - y^2 + \frac{y^2 + 1 - 1}{y^2 + 1} = y^4 - y^2 + 1 - \frac{1}{y^2 + 1}$$

or: $\frac{y^6}{y^2 + 1} = \frac{(y^6 + 1) - 1}{y^2 + 1} = \frac{(y^2 + 1)(y^4 - y^2 + 1) - 1}{y^2 + 1} = y^4 - y^2 + 1 - \frac{1}{y^2 + 1}$

or polynomial division can also be applied.

So $I = \int \frac{y^6}{y^2 + 1}\,\mathrm{dy} = \int\left(y^4 - y^2 + 1 - \frac{1}{y^2 + 1}\right)\mathrm{dy} = \frac{y^5}{5} - \frac{y^3}{3} + y - \arctan y + c =$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c$$

Example 2*. $I = \int \frac{1}{(x^2+1)^2}\,\mathrm{dx} = ?$

Substitution: $x = x(t) = \tan t \quad \Longrightarrow \; t = \arctan x$

$$x'(t) = \frac{\mathrm{dx}}{\mathrm{dt}} = \frac{1}{\cos^2 t} \Longrightarrow \mathrm{dx} = \frac{1}{\cos^2 t}\,\mathrm{dt}$$

$$I = \int \frac{1}{(x^2+1)^2}\,\mathrm{dx} = \int \frac{1}{(\tan^2 t+1)^2}\,\frac{1}{\cos^2 t}\,\mathrm{dt} = \int \cos^4 t\,\frac{1}{\cos^2 t}\,\mathrm{dt} = \int \cos^2 t\,\mathrm{dt} =$$

$$\tan^2 t + 1 = \frac{1}{\cos^2 t}$$

$$= \int \frac{1+\cos 2t}{2}\,\mathrm{dt} = \frac{1}{2}\,t + \frac{1}{4}\,\sin 2\,t + c = \frac{1}{2}\,\arctan x + \frac{1}{2}\,\frac{x}{1+x^2} + c$$

$$\sin 2\,t = 2\sin t\cos t = 2\tan t\cos^2 t = \frac{2\tan t}{1+\tan^2 t} = \frac{2\,x}{1+x^2}$$

Example 3. $I = \int \frac{2\,x}{\sqrt{1-x^4}}\,\mathrm{dx} = ?$

Substitution: $t = x^2 \Longrightarrow x = x(t) = \sqrt{t} = t^{\frac{1}{2}}$

$$x'(t) = \frac{\mathrm{dx}}{\mathrm{dt}} = \frac{1}{2}\,t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}} \Longrightarrow \mathrm{dx} = \frac{1}{2\sqrt{t}}\,\mathrm{dt}$$

$$I = \int \frac{2\,x}{\sqrt{1-x^4}}\,\mathrm{dx} = \int \frac{2\sqrt{t}}{\sqrt{1-t^2}}\cdot\frac{1}{2\sqrt{t}}\,\mathrm{dt} = \int \frac{1}{\sqrt{1-t^2}}\,\mathrm{dt} = \arcsin t + c = \arcsin(x^2) + c$$

Example 4. $I = \int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{dx} = ?$

Substitution: $t = \sqrt{x^2-1} \Longrightarrow t^2 = x^2 - 1 \Longrightarrow t^2 + 1 = x^2$

$$x = x(t) = \sqrt{t^2+1} = (t^2+1)^{\frac{1}{2}}$$

$$x'(t) = \frac{\mathrm{dx}}{\mathrm{dt}} = \frac{1}{2}\,(t^2+1)^{-\frac{1}{2}}\cdot 2\,t = \frac{t}{\sqrt{t^2+1}} \Longrightarrow \mathrm{dx} = \frac{t}{\sqrt{t^2+1}}\,\mathrm{dt}$$

$$I = \int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{dx} = \int \frac{1}{\sqrt{t^2+1}\cdot t}\cdot\frac{t}{\sqrt{t^2+1}}\,\mathrm{dt} = \int \frac{1}{t^2+1}\,\mathrm{dt} =$$

$$= \operatorname{arctg} t + c = \operatorname{arctg}\left(\sqrt{x^2-1}\right) + c$$