Calculus 1

Practice exercises for the first midterm test

Complex numbers

1. Solve the following equation on the set of complex numbers: $|z|^2 + 2iz + \bar{z} = 0$

Solution. Let $z = x + yi$, where $x, y \in \mathbb{R}$. Then
$|z|^2 = x^2 + y^2$, $\bar{z} = x - yi$.

$$
\begin{aligned}
|z|^2 + 2iz + \bar{z} = 0 &\iff x^2 + y^2 + 2i(x + yi) + (x - yi) = 0 \\
&\iff x^2 + y^2 + 2ix - 2y + x - yi = 0 \\
&\iff (x^2 + y^2 + x - 2y) + i(2x - y) = 0
\end{aligned}
$$

A complex number is equal to zero if and only if its real and imaginary parts are equal to zero, so the above equation is equivalent to the following equation system:

(1) $x^2 + y^2 + x - 2y = 0$
(2) $2x - y = 0 \qquad \Longrightarrow y = 2x$

Substituting into (1): $\quad x^2 + 4x^2 + x - 4x = 0 \Longrightarrow 5x^2 - 3x = x(5x - 3) = 0$

From here $x_1 = 0$, $x_2 = \frac{3}{5}$ and $y_1 = 0$, $y_2 = \frac{6}{5}$

The solutions are: $z_1 = 0$, $z_2 = \frac{3}{5} + \frac{6}{5}i$

2. Find those solutions z of the following equation for which $\operatorname{Re}(z) > 0$ and $\operatorname{Im}(z) < 0$.
Give these solutions in algebraic form.

$$z^6 + 7z^3 - 8 = 0$$

Solution. $z^6 + 7z^3 - 8 = (z^3 + 8)(z^3 - 1) = 0 \iff z^3 = -8$ or $z^3 = 1$.

a) If $z^3 = -8 = 8(\cos\pi + i\sin\pi)$ then $z_k = 2\left(\cos\frac{\pi + k\cdot 2\pi}{3} + i\sin\frac{\pi + k\cdot 2\pi}{3}\right)$, where $k = 0, 1, 2$.

$z_0 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + \sqrt{3}\,i$

$z_1 = 2(\cos\pi + i\sin\pi) = -2$

$z_3 = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 1 - \sqrt{3}\,i$

From here the condition $\operatorname{Re}(z) > 0$, $\operatorname{Im}(z) < 0$ holds for $1 - \sqrt{3}\,i$.

b) If $z^3 = 1 = (\cos 0 + i\sin 0)$ then $z_k = \cos\frac{k\cdot 2\pi}{3} + i\sin\frac{k\cdot 2\pi}{3}$, where $k = 0, 1, 2$.

$z_0 = \cos 0 + i\sin 0 = 1$

$$z_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$$
$$z_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}\,i$$

From here no solutions are suitable.

Definition of the limit

1. Let $a_n = \frac{4n^2+3n-1}{2n^2-n+17}$. Find the limit of a_n and provide a threshold index N for $\varepsilon = 0.01$.

Solution. $a_n = \frac{4n^2+3n-1}{2n^2-n+17} = \frac{4+\frac{3}{n}-\frac{1}{n^2}}{2-\frac{1}{n}+\frac{17}{n^2}} \longrightarrow \frac{4+0-0}{2-0+0} = 2$

Let $\varepsilon > 0$. We have to find $N(\varepsilon) \in \mathbb{N}$ such that if $n > N$ then $|\,a_n - A\,| < \varepsilon$. $(A = 2)$

$$|\,a_n - A\,| = \left|\frac{4n^2+3n-1}{2n^2-n+17} - 2\right| = \left|\frac{4n^2+3n-1-2\cdot(2n^2-n+17)}{2n^2-n+17}\right| = \left|\frac{5n-35}{2n^2-n+17}\right| \overset{\text{if } n\geq 7}{=} \frac{5n-35}{2n^2-n+17}$$

$$\frac{5n-35}{2n^2-n+17} \leq \frac{5n+0}{2n^2-n^2+0} = \frac{5n}{n^2} = \frac{5}{n} < \varepsilon \iff n > \frac{5}{\varepsilon}$$

so with the choice $N(\varepsilon) \geq \max\left\{7, \left[\frac{5}{\varepsilon}\right]\right\}$ the definition holds.

If $\varepsilon = 0.01$ then $N \geq \left[\frac{5}{0.01}\right] = 500$.

2. Let $a_n = \frac{3n^4-5n}{n^4+n+2}$. Find the limit of a_n and provide a threshold index N for $\varepsilon = 0.001$.

Solution. $a_n = \frac{3n^4-5n}{n^4+n+2} = \frac{3-\frac{5}{n^3}}{1+\frac{1}{n^3}+\frac{2}{n^4}} \longrightarrow \frac{3-0}{1+0+0} = 3$

Let $\varepsilon > 0$. We have to find $N(\varepsilon) \in \mathbb{N}$ such that if $n > N$ then $|\,a_n - A\,| < \varepsilon$. $(A = 3)$

$$|\,a_n - A\,| = \left|\frac{3n^4-5n}{n^4+n+2} - 3\right| = \left|\frac{3n^4-5n-3\cdot(n^4+n+2)}{n^4+n+2}\right| =$$

$$= \left|\frac{-8n-6}{n^4+n+2}\right| = \frac{8n+6}{n^4+n+2} \leq \frac{8n+6n}{n^4+0+0} = \frac{14}{n^3} < \varepsilon \iff n > \sqrt[3]{\frac{14}{\varepsilon}},$$

so with the choice $N(\varepsilon) \geq \left[\frac{14}{\varepsilon}\right]$ the definition holds.

If $\varepsilon = 0.001$ then $N \geq 14\,000$.

Calculating the limit

1. Find the limit of the following sequence: $a_n = \frac{1}{3n+1-\sqrt{9n^2+5n}}$

Solution. $a_n = \frac{1}{3n+1-\sqrt{9n^2+5n}} \cdot \frac{3n+1+\sqrt{9n^2+5n}}{3n+1+\sqrt{9n^2+5n}} =$

$$= \frac{3n+1+\sqrt{9n^2+5n}}{(3n+1)^2-(9n^2+5n)} = \frac{3n+1+\sqrt{9n^2+5n}}{9n^2+6n+1-(9n^2+5n)} = \frac{3n+1+\sqrt{9n^2+5n}}{n+1}$$

$$= \frac{n}{n}\cdot\frac{3+\frac{1}{n}+\sqrt{9+\frac{5}{n}}}{1+\frac{1}{n}} \longrightarrow \frac{3+0+\sqrt{9+0}}{1+0} = 6$$

2. Find the limit of the following sequence: $a_n = n\left(\sqrt{n^4+8n} - \sqrt{n^4-1}\right)$.

Solution. $a_n = n\left(\sqrt{n^4+8n} - \sqrt{n^4-1}\right)\cdot\frac{\sqrt{n^4+8n}+\sqrt{n^4-1}}{\sqrt{n^4+8n}+\sqrt{n^4-1}} =$

$$= n\cdot\frac{(n^4+8n)-(n^4-1)}{\sqrt{n^4+8n}+\sqrt{n^4-1}} = n\cdot\frac{8n+1}{\sqrt{n^4+8n}+\sqrt{n^4-1}} =$$

$$= \frac{n^2}{n^2}\cdot\frac{8+\frac{1}{n}}{\sqrt{1+\frac{8}{n^3}}+\sqrt{1-\frac{1}{n^4}}} \longrightarrow \frac{8+0}{\sqrt{1+0}+\sqrt{1-0}} = 4$$

3. Find the limit of the following sequence: $a_n = \frac{1}{\sqrt{4n^2+3n}-2n}$

Solution. $a_n = \frac{1}{\sqrt{4n^2+3n}-2n}\cdot\frac{\sqrt{4n^2+3n}+2n}{\sqrt{4n^2+3n}+2n} =$

$$= \frac{\sqrt{4n^2+3n}+2n}{(4n^2+3n)-4n^2} = \frac{n}{n}\cdot\frac{\sqrt{4+\frac{3}{n}}+2}{3} \longrightarrow \frac{\sqrt{4+0}+2}{3} = \frac{4}{3}$$

4. Find the limit of the following sequence: $a_n = \sqrt[n]{\frac{7^n+5^n}{n^3+2}}$

Solution. Because of the Sandwich Theorem, $\lim_{n\to\infty}\sqrt[n]{\frac{7^n+5^n}{n^3+2}} = 7$ **(2p)**, since

$$7 \overset{(1p)}{\longleftarrow} \frac{7}{\sqrt[n]{3}\,(\sqrt[n]{n})^3} \overset{(1p)}{=} \sqrt[n]{\frac{7^n}{3n^3}} \overset{(2p)}{\leq} \sqrt[n]{\frac{7^n+5^n}{n^3+2}} \overset{(2p)}{\leq} \sqrt[n]{\frac{2\cdot 7^n}{n^3}} \overset{(1p)}{=} \frac{7\sqrt[n]{2}}{(\sqrt[n]{n})^3} \overset{(1p)}{\longrightarrow} 7$$

5. Find the limit of the following sequence: $a_n = \sqrt[n]{\frac{7n^2-n}{n^3+3}}$

Solution.

Upper estimation:

$$a_n = \sqrt[n]{\frac{7n^2-n}{n^3+3}} \leq \sqrt[n]{\frac{7n^2+0}{n^3+3n^3}} = \sqrt[n]{\frac{7}{4n}} = \sqrt[n]{\frac{7}{4}}\cdot\frac{1}{\sqrt[n]{n}} \longrightarrow 1\cdot\frac{1}{1} = 1$$

Lower estimation:

$$a_n = \sqrt[n]{\frac{7n^2 - n}{n^3 + 3}} \geq \sqrt[n]{\frac{7n^2 - n^2}{n^3 + 3n^3}} = \sqrt[n]{\frac{6}{4n}} = \sqrt[n]{\frac{3}{2}} \cdot \frac{1}{\sqrt[n]{n}} \longrightarrow 1 \cdot \frac{1}{1} = 1$$

So by the Sandwich theorem, $a_n \longrightarrow 1$.

The limit $(1 + \frac{x}{n})^n \longrightarrow e^x$

1. Find the limit of the following sequences:

a) $a_n = \left(\frac{3n^2 + 4}{3n^2 + 1}\right)^n$ b) $b_n = \left(\frac{3n + 2}{4n + 3}\right)^{n+3}$

Solution. a) $a_n^n = \left(\frac{3n^2 + 4}{3n^2 + 1}\right)^{n^2} = \frac{\left(1 + \frac{4}{3n^2}\right)^{n^2}}{\left(1 + \frac{1}{3n^2}\right)^{n^2}} \longrightarrow \frac{e^{\frac{4}{3}}}{e^{\frac{1}{3}}} = e.$

Since $2 < e < 3$ then $2 < a_n^n < 3$ if n is large enough.
Then $\sqrt[n]{2} < a_n < \sqrt[n]{3}$, and since $\sqrt[n]{2} \longrightarrow 1$ and $\sqrt[n]{3} \longrightarrow 1$,
then by the sandwich theorem $a_n \longrightarrow 1$.

b) $b_n = \left(\frac{3n + 2}{4n + 3}\right)^{n+3} = \frac{\left(3n\left(1 + \frac{2}{3n}\right)\right)^{n+3}}{\left(4n\left(1 + \frac{3}{4n}\right)\right)^{n+3}} = \left(\frac{3}{4}\right)^{n+3} \cdot \frac{\left(1 + \frac{2}{3n}\right)^n}{\left(1 + \frac{3}{4n}\right)^n} \cdot \frac{\left(1 + \frac{2}{3n}\right)^3}{\left(1 + \frac{3}{4n}\right)^3} \longrightarrow 0 \cdot \frac{e^{\frac{2}{3}}}{e^{\frac{3}{4}}} \cdot \frac{1}{1} = 0$

2. Find the limit of the following sequences:
a) $a_n = \left(\frac{n^2 + 1}{n^2 + 4}\right)^{n^2}$ b) $b_n = \left(\frac{n^2 + 1}{n^2 + 4}\right)^n$ c) $c_n = \left(\frac{n^2 + 1}{n^2 + 4}\right)^{n^3}$

Solution. a) $a_n = \left(\frac{n^2 + 1}{n^2 + 4}\right)^{n^2} = \frac{\left(1 + \frac{1}{n^2}\right)^{n^2}}{\left(1 + \frac{4}{n^2}\right)^{n^2}} \longrightarrow \frac{e}{e^4} = \frac{1}{e^3}$

b) $b_n = \sqrt[n]{a_n}$. Since $a_n \longrightarrow \frac{1}{e^3}$ and $0 < \frac{1}{e^3} < 1$ then there exists $N \in \mathbb{N}$ such that if $n > N$ then
$\frac{1}{2e^3} < a_n < 1 \implies \sqrt[n]{\frac{1}{2e^3}} < b_n < 1$. Since $\sqrt[n]{\frac{1}{2e^3}} \longrightarrow 1$ then by the sandwich theorem $b_n \longrightarrow 1$.

c) $c_n = a_n^n$. Let $\frac{1}{e^3} < q < 1$. Since $a_n \longrightarrow \frac{1}{e^3}$ then there exists $N \in \mathbb{N}$ such that if $n > N$ then
$0 < a_n < q \implies 0 < c_n < q^n$. Since $q^n \longrightarrow 0$ then by the sandwich theorem $c_n \longrightarrow 0$.

3. Find the limit of the following sequences:

a) $a_n = \left(\frac{2n^2 + 3}{2n^2 + 1}\right)^n$ b) $b_n = \left(\frac{2n + 3}{3n + 4}\right)^{n+3}$

Solution. a) $a_n^n = \left(\frac{2n^2 + 3}{2n^2 + 1}\right)^{n^2} = \frac{\left(1 + \frac{3}{2n^2}\right)^{n^2}}{\left(1 + \frac{1}{2n^2}\right)^{n^2}} \longrightarrow \frac{e^{\frac{3}{2}}}{e^{\frac{1}{2}}} = e.$

Since $2 < e < 3$ then $2 < a_n^n < 3$ if n is large enough.
Then $\sqrt[n]{2} < a_n < \sqrt[n]{3}$, and since $\sqrt[n]{2} \longrightarrow 1$ and $\sqrt[n]{3} \longrightarrow 1$,
then by the sandwich theorem $a_n \longrightarrow 1$.

b) $b_n = \left(\frac{2n+3}{3n+4}\right)^{n+3} = \frac{\left(2n\left(1+\frac{3}{2n}\right)\right)^{n+3}}{\left(3n\left(1+\frac{4}{3n}\right)\right)^{n+3}} = \left(\frac{2}{3}\right)^{n+3} \cdot \frac{\left(1+\frac{3}{2n}\right)^n}{\left(1+\frac{4}{3n}\right)^n} \cdot \frac{\left(1+\frac{3}{2n}\right)^3}{\left(1+\frac{4}{3n}\right)^3} \longrightarrow 0 \cdot \frac{e^{\frac{3}{2}}}{e^{\frac{4}{3}}} \cdot \frac{1}{1} = e^{\frac{1}{6}}$

Recursive sequences

1. Let $a_1 = 2$ and $a_{n+1} = 5 - \frac{4}{a_n}$ for all $n \in \mathbb{N}$.
(Then $a_2 = 3$, $a_3 \approx 3.67, \ldots$). Prove that (a_n) is convergent and calculate its limit.

Solution. If $\exists \lim\limits_{n\to\infty} a_n = A$ then $A = 5 - \frac{4}{A} \iff A^2 - 5A + 4 = (A-1)(A-4) = 0$
$\iff A_1 = 1,\ A_2 = 4$.
Boundedness: we prove by induction that $1 < a_n < 4$ for all $n \in \mathbb{N}$.

(1) $1 < a_1 = 2 < 4$

(2) Assume that $1 < a_n < 4$

(3) Then $1 > \frac{1}{a_n} > \frac{1}{4} \implies -4 < -\frac{4}{a_n} < -1 \implies 1 < a_{n+1} = 5 - \frac{4}{a_n} < 4$

So (a_n) is bounded above.

Monotonicity: we prove by induction that (a_n) is monotonically increasing, that is, $a_n < a_{n+1}\ \forall n \in \mathbb{N}$.

(1) $a_1 = 2 < a_2 = 3$

(2) Assume that $a_n < a_{n+1}$

(3) Then $\frac{1}{a_n} > \frac{1}{a_{n+1}}$ (since $a_n > 1 > 0$) $\implies \frac{-4}{a_n} < \frac{-4}{a_{n+1}} \implies a_{n+1} = 5 - \frac{4}{a_n} < 5 - \frac{4}{a_{n+1}} = a_{n+2}$

So (a_n) is monotonically increasing.

Since (a_n) is monotonically increasing and bounded above then it is convergent.
Since $a_1 = 2$ and the sequence is monotonically increasing then $A = 1$ cannot be the limit.
So $\lim\limits_{n\to\infty} a_n = 4$.

2. Let $a_1 = 5$ and $a_{n+1} = \sqrt{10\,a_n - 21}$ for all $n \in \mathbb{N}$.
a) Prove that $3 < a_n < 7$ for all $n \in \mathbb{N}$.
b) Prove that the sequence is monotonically increasing.
c) Calculate the limit of the sequence (a_n).

Solution. a) Boundedness: we prove by induction that $3 < a_n < 7$ for all $n \in \mathbb{N}$.

(1) $3 < a_1 = 5 < 7$

(2) Assume that $3 < a_n < 7$. We need to show that this implies $3 < a_{n+1} < 7$ ($n \in \mathbb{N}$).

(3) Then $30 - 21 < 10\,a_n - 21 < 70 - 21 \implies 9 < 10\,a_n - 21 < 49 \implies 3 < \sqrt{10\,a_n - 21} < 7$

So (a_n) is bounded above.

b) Monotonicity: we prove by induction that (a_n) is monotonically increasing, that is, $a_n < a_{n+1}$ $\forall$ $n \in \mathbb{N}$.

(1) $a_1 = 5 < a_2 = \sqrt{50 - 21} = \sqrt{29}$

(2) Assume that $a_n < a_{n+1}$

(3) Then $10\,a_n - 21 < 10\,a_{n+1} - 21 \implies a_{n+1} = \sqrt{10\,a_n - 21} < \sqrt{10\,a_{n+1} - 21} = a_{n+2} \implies a_{n+1} < a_{n+2}$

So (a_n) is monotonically increasing.

c) Since (a_n) is monotonically increasing and bounded above then it is convergent.

Let $\lim\limits_{n\to\infty} a_n = A$. Then $A = \sqrt{10\,A - 21} \iff A^2 - 10\,A + 21 = (A-3)\,(A-7) = 0 \iff A_1 = 3, A_2 = 7$.

Since $a_1 = 5$ and the sequence is monotonically increasing then $A = 3$ cannot be the limit.

So $\lim\limits_{n\to\infty} a_n = 7$.

3. Let $a_1 = 3$ and $a_{n+1} = \dfrac{10}{7 - a_n}$ for all $n \in \mathbb{N}$. Prove that (a_n) is convergent and calculate its limit.

Solution. If $\exists \lim\limits_{n\to\infty} a_n = A$ then $A = \dfrac{10}{7 - A} \iff A(7 - A) - 10 = 0 \iff A^2 - 7\,A + 10 = (A-2)\,(A-5) = 0$

$\iff A_1 = 2,\ A_2 = 5$.

Boundedness: we prove by induction that $2 < a_n < 5$ for all $n \in \mathbb{N}$.

(1) $2 < a_1 = 3 < 5$

(2) Assume that $2 < a_n < 5$

(3) Then $-2 > -a_n > -5 \implies 5 > 7 - a_n > 2 \implies \dfrac{1}{5} < \dfrac{1}{7 - a_n} < \dfrac{1}{2} \implies 2 < a_{n+1} = \dfrac{10}{7 - a_n} < 5$

So (a_n) is bounded above.

Monotonicity: we prove by induction that (a_n) is monotonically decreasing, that is, $a_n > a_{n+1}$ for all $n \in \mathbb{N}$.

(1) $a_1 = 3 > a_2 = \dfrac{10}{7 - 3} = \dfrac{10}{4} = 2.5$

(2) Assume that $a_n > a_{n+1}$

(3) Then $-a_n < -a_{n+1} \implies 7 - a_n < 7 - a_{n+1}$. Since $2 < a_n < 5$ then $7 - a_n > 0$

$\implies a_{n+1} = \dfrac{10}{7 - a_n} > \dfrac{10}{7 - a_{n+1}} = a_{n+2}$

So (a_n) is monotonically decreasing.

Since (a_n) is monotonically decreasing and bounded below then it is convergent.

Since $a_1 = 3$ and the sequence is monotonically decreasing then $A = 5$ cannot be the limit.

So $\lim\limits_{n\to\infty} a_n = 2$.

Accumulation points