Calculus 1

Topics of the second midterm test

- Limits of real functions
- Types of discontinuities
- The limit $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$
- The derivative (definition, differentiation rules, equation of the tangent line)
- Elementary functions and their inverses
- L'Hospital's rule
- Analysing graphs of functions

Practice exercises for the second midterm test

(1) Limits of real functions, L'Hospital's rule

6. (10+10+10+10 points) Calculate the following limits:

a) $\lim\limits_{x\to -\infty}\left(\sqrt{x^2+x} - \sqrt{x^2+5x+3}\right)$ b) $\lim\limits_{x\to 0}\left(\dfrac{1}{e^{2x}-1} - \dfrac{1}{2x}\right)$

c) $\lim\limits_{x\to 0}\dfrac{\sin(4x^2)}{\ln(\cos(2x))}$ d) $\lim\limits_{x\to \infty}\dfrac{e^x\cosh(2x)}{\sinh(3x)}$

Solution.

a) $\lim\limits_{x\to -\infty}\left(\sqrt{x^2+x} - \sqrt{x^2+5x+3}\right) = \lim\limits_{x\to -\infty}\left(\sqrt{x^2+x} - \sqrt{x^2+5x+3}\right)\cdot\dfrac{\sqrt{x^2+x} + \sqrt{x^2+5x+3}}{\sqrt{x^2+x} + \sqrt{x^2+5x+3}}$

$= \lim\limits_{x\to -\infty}\dfrac{(x^2+x)-(x^2+5x+3)}{\sqrt{x^2+x} + \sqrt{x^2+5x+3}} = \lim\limits_{x\to -\infty}\dfrac{-4x-3}{\sqrt{x^2+x} + \sqrt{x^2+5x+3}}$ **(6p)**

$= \lim\limits_{x\to -\infty}\dfrac{x}{\sqrt{x^2}}\cdot\dfrac{-4-\frac{3}{x}}{\sqrt{1+\frac{1}{x}} + \sqrt{1+\frac{5}{x}+\frac{3}{x^2}}} = (-1)\cdot\dfrac{-4-0}{\sqrt{1+0} + \sqrt{1+0+0}} = 2$ **(4p)**

Here $\dfrac{x}{\sqrt{x^2}} = \dfrac{x}{|x|} = \dfrac{x}{-x} = -1$, since $x < 0$.

b) $\lim\limits_{x\to 0}\left(\frac{1}{e^{2x}-1}-\frac{1}{2x}\right)=\lim\limits_{x\to 0}\frac{2x-\left(e^{2x}-1\right)}{2x\left(e^{2x}-1\right)}$

The limit has the form $\frac{0}{0}$ $\Longrightarrow$ L'Hospital's rule can be applied:

$$\overset{L'H}{=}\lim_{x\to 0}\frac{2-2e^{2x}}{2\left(e^{2x}-1\right)+2x\cdot e^{2x}\cdot 2}\ \textbf{(4p)}\ \overset{L'H}{=}\lim_{x\to 0}\frac{-4e^{2x}}{4e^{2x}+4\cdot e^{2x}+4x\cdot e^{2x}\cdot 2}\ \textbf{(4p)}\ =\frac{-4}{8}=-\frac{1}{2}\ \textbf{(2p)}$$

c) The limit has the form $\frac{0}{0}$ $\Longrightarrow$ L'Hospital's rule can be applied:

$$\lim_{x\to 0}\frac{\sin\left(4x^2\right)}{\ln(\cos(2x))}\overset{L'H}{=}\lim_{x\to 0}\frac{\cos\left(4x^2\right)\cdot 8x}{\frac{1}{\cos(2X)}\cdot(-\sin(2x))\cdot 2}\ \textbf{(5p)}$$

$$=\lim_{x\to 0}\cos(2x)\cdot\cos\left(4x^2\right)\frac{2x}{\sin(2x)}\cdot(-2)=1\cdot 1\cdot 1\cdot(-2)=-2\ \textbf{(5p)}$$

d) By the definition of the functions:

$$\lim_{x\to\infty}\frac{e^x\cosh(2x)}{\sinh(3x)}=\lim_{x\to\infty}\frac{e^x\left(e^{2x}+e^{-2x}\right)}{e^{3x}-e^{-3x}}=\lim_{x\to\infty}\frac{e^{3x}+e^{-x}}{e^{3x}-e^{-3x}}\ \textbf{(4p)}\ =\lim_{x\to\infty}\frac{e^{3x}}{e^{3x}}\frac{1-e^{-4x}}{1+e^{-6x}}\ \textbf{(3p)}\ =\frac{1-0}{1+0}=1\ \textbf{(3p)}$$

4. (9+9+9 points) Calculate the following limits:

$$\text{a) }\lim_{x\to 0}\frac{\sqrt{1+8x}-e^{4x}}{x\sin 2x}\qquad\text{b) }\lim_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}}\qquad\text{c) }\lim_{x\to\infty}\frac{\sinh(2x+3)}{\cosh(2x-5)}$$

Solution. a) The limit has the form $\frac{0}{0}$ $\Longrightarrow$ L'Hospital's rule can be applied:

$$\lim_{x\to 0}\frac{\sqrt{1+8x}-e^{4x}}{x\sin 2x}\overset{L'H}{=}\lim_{x\to 0}\frac{\frac{1}{2}(1+8x)^{-\frac{1}{2}}\cdot 8-4e^{4x}}{\sin 2x+2x\cos 2x}\ (\mathbf{4}\,\boldsymbol{p})$$

$$\overset{L'H}{=}\lim_{x\to 0}\frac{-\frac{1}{4}(1+8x)^{-\frac{3}{2}}\cdot 64-16e^{4x}}{2\cos 2x+2\cos 2x+2x(-\sin 2x)\cdot 2}\ (\mathbf{3}\,\boldsymbol{p})\ =\frac{-\frac{1}{4}\cdot 64-16}{2+2+0}=-8\ \textbf{(2p)}$$

b) The limit has the form 1^∞: $(\cos x)^{\frac{1}{\sin^2 x}}=e^{\ln\left((\cos x)^{\frac{1}{\sin^2 x}}\right)}=e^{\left(\frac{1}{\sin^2 x}\ln(\cos x)\right)}$ **(3p)**

The limit of the power has the form $\frac{0}{0}$:

$$\lim_{x\to 0}\frac{\ln(\cos x)}{\sin^2 x}\overset{L'H}{=}\lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{2\sin x\cos x}=\lim_{x\to 0}\frac{1}{2\cos^2 x}=\frac{1}{2}\ \textbf{(4p)}$$

$$\Longrightarrow\ \lim_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}}=e^{\frac{1}{2}}=\sqrt{e}\ \textbf{(2p)}$$

c) By the definition of the functions:

$$\lim_{x\to\infty}\frac{\sinh(2x+3)}{\cosh(2x-5)}=\lim_{x\to\infty}\frac{e^{2x+3}-e^{-(2x+3)}}{e^{2x-5}+e^{-(2x-5)}}\ \textbf{(3p)}\ =\lim_{x\to\infty}\frac{e^{2x}}{e^{2x}}\frac{e^3-e^{-4x-3}}{e^{-5}+e^{-4x+5}}\ \textbf{(3p)}\ =\frac{e^3-0}{e^{-5}+0}=e^8\ \textbf{(3p)}$$

(2) Types of discontinuities; Elementary functions and their inverses

2. Let $f(x) = \dfrac{|x+2| \cdot \sin(x^2 - 3x)}{x^3 + x^2 - 2x}$.
Find the points of discontinuities of f. What type of discontinuities are these?

Solution. The function $f(x) = \dfrac{|x+2| \cdot \sin(x^2 - 3x)}{x(x+2)(x-1)}$ is continuous except the points $0, -2, 1$.

- $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\sin(x^2 - 3x)}{x^2 - 3x} \cdot \dfrac{(x-3)\cdot |x+2|}{(x+2)(x-1)} = 1 \cdot \dfrac{(-3)\cdot 2}{-2} = 3$

  $\Longrightarrow$ f has a removable discontinuity at $x = 0$.

- $\lim\limits_{x\to -2\pm 0} f(x) = \lim\limits_{x\to -2\pm 0} \dfrac{|x+2|}{x+2} \cdot \dfrac{\sin(x^2 - 3x)}{x(x-1)} = \pm \dfrac{\sin 10}{6}$

  $\Longrightarrow$ f has a jump discontinuity at $x = -2$

- $\lim\limits_{x\to 1\pm 0} f(x) = \lim\limits_{x\to 1\pm 0} \dfrac{|x+2| \sin(x^2 - 3x)}{x(x+2)} \cdot \dfrac{1}{x-1} = \mp\infty$

  $\Longrightarrow$ f an essential discontinuity at $x = 1$

3. (18 points) Determine the points of discontinuity of the following function. What type of discontinuities are these?

$$f(x) = \arctan\left(\frac{1}{x+2}\right) + \frac{x^2 - 1}{x^2 + 2x - 3}$$

Solution. $f(x) = \arctan\left(\dfrac{1}{x+2}\right) + \dfrac{(x-1)(x+1)}{(x-1)(x+3)}$

Since the arctan function and the polynomials are continuous and the composition and ratio of continuous functions is continuous if the denominator is not 0, then f is continuous on its domain. The points of discontinuities are $x_1 = -2, x_2 = 1, x_3 = -3$. **(3p)**

a) If $x_1 = -2$:
$$\lim_{x\to -2+0} \frac{1}{x+2} = \frac{1}{0+} = +\infty \Longrightarrow \lim_{x\to -2+} \arctan\left(\frac{1}{x+2}\right) = \frac{\pi}{2}$$
$$\lim_{x\to -2-0} \frac{1}{x+2} = \frac{1}{0-} = -\infty \Longrightarrow \lim_{x\to -2-} \arctan\left(\frac{1}{x+2}\right) = -\frac{\pi}{2} \quad \textbf{(2p)}$$

$$\Longrightarrow \lim_{x\to -2\pm 0} f(x) = \pm\frac{\pi}{2} + \lim_{x\to -2\pm 0} \frac{(x-1)(x+1)}{(x-1)(x+3)} = \pm\frac{\pi}{2} - 1 \quad \textbf{(1p)}$$

$\Longrightarrow$ f has a jump discontinuity at $x_1 = -2$. **(2p)**

b) If $x_2 = 1$: $\lim_{x \to 1 \pm 0} f(x) = \arctan \frac{1}{3} + \lim_{x \to 1 \pm 0} \frac{(x-1)(x+1)}{(x-1)(x+3)} = \arctan \frac{1}{3} + \frac{1}{2}$ **(3p)**

$\Longrightarrow$ f has a removable discontinuity at $x_2 = 1$. **(2p)**

c) If $x_3 = -3$: $\lim_{x \to -3 \pm 0} f(x) = \arctan(-1) + \lim_{x \to -3 \pm 0} \frac{(x-1)(x+1)}{(x-1)(x+3)} = -\frac{\pi}{4} + (-2) \lim_{x \to -3 \pm 0} \frac{1}{x+3} =$

$= -\frac{\pi}{4} + (-2) \cdot (\pm \infty) = \mp \infty$ **(3p)**

$\Longrightarrow$ f has an essential discontinuity at $x_3 = -3$. **(2p)**

2. (9 points) Determine the points of discontinuity of the function $f(x) = x \arctan \frac{1}{x(x+2)}$.

What type of discontinuities are these?

Solution. The arctan function is continuous and the composition and ratio of continuous functions is continuous if the denominator is not 0, so the points of discontinuities are $x_1 = 0$ and $x_2 = -2$ **(1p)**.

If $x_1 = 0$: $\lim_{x \to 0} f(x) = 0$ **(1p)**, since the arctan function is bounded **(1p)**

$\Longrightarrow$ f has a removable discontinuity at $x_1 = 0$. **(1p)**

If $x_2 = -2$: $\lim_{x \to -2+} f(x) = \lim_{x \to -2+} x \arctan \frac{1}{x(x+2)} = \lim_{y \to -\infty} (-2) \arctan y = -2 \cdot \left(-\frac{\pi}{2}\right) = \pi$

$\lim_{x \to -2-} f(x) = \lim_{x \to -2-} x \arctan \frac{1}{x(x+2)} = \lim_{y \to \infty} (-2) \arctan y = -2 \cdot \frac{\pi}{2} = -\pi$ **(4p)**

$\Longrightarrow$ f has a jump discontinuity at $x_2 = -2$ **(1p)**

(3) The derivative

3. Calculate the derivatives of the following functions. Where are these functions differentiable?

a) $f(x) = \cos\left(\frac{2 x^3 \arctan(x^2)}{2^x}\right)$ **b)** $g(x) = |x| \cdot \ln(|x| + 1)$

In case b), use the definition to calculate $g'(0)$.

Solution. a) The function f is everywhere differen-
tiable.

$$f'(x) = -\sin\left(\frac{2 x^3 \arctan(x^2)}{2^x}\right) \cdot \frac{\left(6 x^2 \arctan(x^2) + 2 x^3 \cdot \frac{2x}{x^4+1}\right) \cdot 2^x - 2 x^3 \arctan(x^2) \cdot 2^x \ln 2}{4^x}$$

b) $g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \left(\frac{|x|}{x} \cdot \ln(|x| + 1)\right) = 0$, since $\frac{|x|}{x} = \pm 1$ is bounded

and $\lim_{x \to 0} \ln(|x| + 1) = 0$.

If $x > 0$, then $g'(x) = (x \cdot \ln(x+1))' = \ln(x+1) + \frac{x}{x+1}$

If $x < 0$, then $g'(x) = (-x \cdot \ln(1-x))' = -\ln(1-x) + \frac{x}{1-x}$

The function g is everywhere differentiable.

4. (10 points) Find the values of the parameters such that the following function

be differentiable on $\mathbb{R}$: $f(x) = \begin{cases} \dfrac{x^2}{x+1} & \text{if } x \geq 1 \\ a x^2 + b & \text{if } x < 1 \end{cases}$

Solution. The function is differentiable for all a, b except $x = 1$.

If f is continuous at $x = 1$ then $\lim\limits_{x \to 1+0} f(x) = \lim\limits_{x \to 1-0} f(x) = f(1) \implies \dfrac{1}{2} = a + b$ **(3p)**

$f'(x) = \begin{cases} \dfrac{2x \cdot (x+1) - x^2}{(x+1)^2} = \dfrac{x^2 + 2x}{(x+1)^2} & \text{if } x > 1 \\ 2 a x & \text{if } x < 1 \end{cases}$ **(2p)**

If f is differentiable at $x = 1$ then $\lim\limits_{x \to 1+0} f'(x) = \lim\limits_{x \to 1-0} f'(x) \implies \dfrac{3}{4} = 2a$ **(3p)**.

The solution of the equation system is $a = \dfrac{3}{8}$, $b = \dfrac{1}{8}$. **(1p)**

5. (9 points) Find the values of the parameters such that the following function

be differentiable on $\mathbb{R}$: $f(x) = \begin{cases} \dfrac{a}{x^2+1} & \text{if } x \geq 1 \\ b x^4 + 1 & \text{if } x < 1 \end{cases}$

Solution. The function is differentiable for all a, b except $x = 1$.

If f is continuous at $x = 1$ then $\lim\limits_{x \to 1+0} f(x) = \lim\limits_{x \to 1-0} f(x) = f(1) \implies \dfrac{a}{2} = b + 1$ **(3p)**

$f'(x) = \begin{cases} -\dfrac{a}{(x^2+1)^2} \cdot 2x & \text{if } x > 1 \\ 4 b x^3 & \text{if } x < 1 \end{cases}$ **(2p)**

If f is differentiable at $x = 1$ then $\lim\limits_{x \to 1+0} f'(x) = \lim\limits_{x \to 1-0} f'(x) \implies -\dfrac{a}{2} = 4b$ **(3p)**.

The solution of the equation system is $a = \dfrac{8}{5}$, $b = -\dfrac{1}{5}$. **(1p)**

(4) Equation of the tangent line

5. (10 points) Find the equation of the tangent line to the function $f(x) = \dfrac{\cos(2x) + \ln(x+1)}{\sqrt{x^2+1}}$ at $x_0 = 0$.

Solution. $f'(x) = \dfrac{1}{x^2+1}\left(\left(-2\sin(2x) + \dfrac{1}{x+1}\right) \cdot \sqrt{x^2+1} - (\cos(2x) + \ln(x+1)) \cdot \dfrac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x\right)$ **(5p)**

$f(0) = 1$, $f'(0) = 1$ **(1p)**

The equation of the tangent line is $y = f(0) + f'(0)(x - 0)$, that is, $y = 1 + x$ **(3p)**

3. (9 points) Find the equation of the tangent line to the function $f(x) = \dfrac{e^{x^2} \ln(2x+5)}{\cos x}$ at $x_0 = 0$.

Solution. $f'(x) = \dfrac{1}{\cos^2 x}\left(\left(e^{x^2} \cdot 2x \cdot \ln(2x+5) + e^{x^2} \cdot \dfrac{1}{2x+5} \cdot 2\right) \cdot \cos x - e^{x^2} \ln(2x+5) \cdot (-\sin x)\right)$ **(5p)**

$f(0) = \ln 5$, $f'(0) = \dfrac{2}{5}$ **(1p)**

The equation of the tangent line is $y = f(0) + f'(0)(x-0)$, that is, $y = \ln 5 + \frac{2}{5}x$ **(3p)**

(5) Analysing graphs of functions

5. Analyze the following function and plot its graph: $f(x) = (x^2 - x)e^x$.

Solution.

$D_f = \mathbb{R}$; $f(x) = 0 \iff x = 0$ or $x = 1$; $\lim\limits_{x \to +\infty} f(x) = \infty$, $\lim\limits_{x \to -\infty} f(x) = 0$

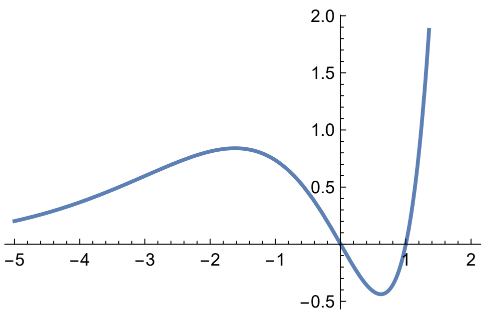
$f'(x) = e^x (x^2 + x - 1) = 0 \iff x = x_1 = \frac{1}{2}\left(-1 - \sqrt{5}\right)$ or $x = x_2 = \frac{1}{2}\left(-1 + \sqrt{5}\right)$

x	$x<x_1$	$x=x_1$	$x_1<x<x_2$	$x=x_2$	$x>x_2$
f'	+	0	−	0	+
f	↗	loc. max	↘	loc. min	↗

$(x_1 \approx -1.61803,\quad x_2 \approx 0.618034,\quad f(x_1) \approx 0.839962,\quad f(x_2) \approx -0.437971)$

$f''(x) = e^x \cdot x(x+3) = 0 \iff x_1 = -3,\ x_2 = 0$

x	$x<-3$	$x=-3$	$-3<x<0$	$x=0$	$0<x$
f''	+	0	−	0	+
f	∪	"infl.: $\frac{12}{e^3}$	∩	infl.: 0	∪



$\Longrightarrow R_f = [f(x_2), \infty) \approx [-0.437971, \infty)$

6. (18 points) Analyze the following function and plot its graph: $f(x) = \dfrac{x^2 + x + 1}{x^2 + 1}$.

Solution.

$D_f = \mathbb{R}$; $f(x) \neq 0$; $\lim\limits_{x \to \pm\infty} f(x) = 1$

$f'(x) = \dfrac{1 - x^2}{(x^2+1)^2} = 0 \iff x = \pm 1$ **(3p)**

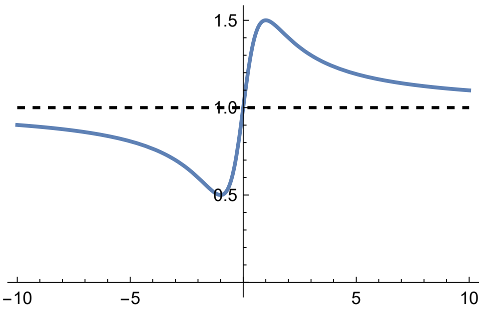
x	$x<-1$	$x=-1$	$-1<x<1$	$x=1$	$x>1$
f'	−	0	+	0	−
f	↘	min: $\frac{1}{2}$	↗	max: $\frac{3}{2}$	↘

(4p)

$f''(x) = \dfrac{2x(x^2 - 3)}{(1 + x^2)^3} = 0 \iff x = 0$ or $x = \pm\sqrt{3}$ **(3p)**

x	$x<-\sqrt{3}$	$x=-\sqrt{3}$	$-\sqrt{3}<x<0$	$x=0$	$0<x<\sqrt{3}$	$x=\sqrt{3}$	$x>\sqrt{3}$
f''	−	0	+	0	−	0	+
f	$\cap$	infl: $\frac{4-\sqrt{3}}{4}$	$\cup$	infl: 1	$\cap$	infl: $\frac{4+\sqrt{3}}{4}$	$\cup$

(4p)



$\Longrightarrow R_f = \left[\frac{1}{2}, \frac{3}{2}\right]$ **(4p)**